

Note on Linearized Solutions to the Optimal Growth Model

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This note reviews the linearized dynamics of the optimal growth model and derives log-linearized solutions.

General Problem: Linearization

Linearization is a common approach in macroeconomics to obtain approximate solutions. The basic principle is the Taylor series approximation.

Consider first the univariate case. Suppose $y = h(x)$ is a twice differentiable function that you want to linearize around point $\bar{x}$. Taylor's Theorem says that

$$h(x) = h(\bar{x}) + h'(\bar{x}) \cdot (x - \bar{x}) + \frac{1}{2} h''(\xi) \cdot (x - \bar{x})^2$$

where ξ is between x and $\bar{x}$. The last term is known as the error term, and it is often written as $O[(x - \bar{x})^2]$ to highlight that the error grows with the square of the distance between x and $\bar{x}$. A linear approximation is obtained by omitting the error term:

$$h(x) \approx h(\bar{x}) + h'(\bar{x}) \cdot (x - \bar{x})$$

Note that this relationship is not an equation, but an approximation.

Taylor's theorem applies similarly to multivariate functions. Suppose $y = h(x_1, \dots, x_n)$ is a twice differentiable function in $\mathbb{R}^n$ that you want to linearize at a point $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$. Taylor's theorem says

$$h(x_1, \dots, x_n) = h(\bar{x}) + \sum_{i=1}^n \frac{\partial h}{\partial x_i}(\bar{x}) \cdot (x_i - \bar{x}_i) + O(\|x - \bar{x}\|)$$

where the last term is an unspecified error that is a bounded multiple of the Euclidean distance between $(x_1, \dots, x_n)$ and $\bar{x}$ (that is, it has the same order of magnitude "O"). As an approximation, this is written as

$$h(x_1, \dots, x_n) \approx h(\bar{x}) + \sum_{i=1}^n \frac{\partial h}{\partial x_i}(\bar{x}) \cdot (x_i - \bar{x}_i)$$

Application to Optimal Growth

In the optimal growth model, the non-linear differential equations for k and c are

$$\begin{aligned}\dot{k} &= f(k) - c - (n + g + \delta)k \\ \dot{c} &= \frac{1}{\theta} [f'(k) - \delta - \rho - \theta g] \cdot c\end{aligned}$$

The steady state conditions are

$$f'(k^*) = \delta + \rho + \theta g$$

and

$$c^* = f(k^*) - (n + g + \delta)k^*$$

In logarithms, one can write

$$\begin{aligned}\dot{k}/k &= \frac{d \ln(k)}{dt} = f(k)/k - c/k - (n + g + \delta) \\ &= f(e^{\ln(k)})e^{-\ln(k)} - e^{\ln(c)-\ln(k)} - (n + g + \delta) = h_1(\ln(k), \ln(c)) \\ \dot{c}/c &= \frac{d \ln(c)}{dt} = \frac{1}{\theta} f'(e^{\ln(k)}) - \frac{1}{\theta} (\delta - \rho - \theta g) = h_2(\ln(k))\end{aligned}$$

The Taylor series approximation at $(c, k) = (c^*, k^*)$ simplifies because

$$h_1(\ln(k^*), \ln(c^*)) = 0 \text{ and } h_2(\ln(k^*)) = 0.$$

The relevant partial derivatives are:

$$\begin{aligned}\frac{\partial h_1}{\partial \ln(k)} &= \frac{\partial}{\partial \ln(k)} [f(e^{\ln(k)})e^{-\ln(k)} - e^{\ln(c)-\ln(k)}] = f' \cdot e^{\ln(k)} e^{-\ln(k)} - f \cdot e^{-\ln(k)} + e^{\ln(c)-\ln(k)} \\ &= f' - \frac{f}{k} + \frac{c}{k} \\ \frac{\partial h_1}{\partial \ln(c)} &= \frac{\partial}{\partial \ln(c)} [-e^{\ln(c)-\ln(k)}] = -e^{\ln(c)-\ln(k)} = -\frac{c}{k}, \text{ and} \\ \frac{\partial h_2}{\partial \ln(k)} &= \frac{\partial}{\partial \ln(k)} [\frac{1}{\theta} f'(e^{\ln(k)})] = \frac{1}{\theta} f''(e^{\ln(k)})e^{\ln(k)} = \frac{1}{\theta} f''(k)k\end{aligned}$$

Evaluating them at (c^*, k^*) one obtains

$$\frac{\dot{k}}{k} \approx [f' - \frac{f}{k^*} + \frac{c^*}{k^*}] \cdot \ln(k/k^*) - \frac{c^*}{k^*} \ln(c/c^*) = \beta \cdot \ln(k/k^*) - \frac{c^*}{k^*} \ln(c/c^*)$$

and

$$\frac{\dot{c}}{c} \approx \frac{f''(k^*)k^*}{\theta} \cdot \ln(k/k^*).$$

Collecting c - and k -terms, the log-linearized equations can be written in matrix form as:

$$\begin{pmatrix} \dot{k}/k \\ \dot{c}/c \end{pmatrix} = \begin{pmatrix} \beta & -\frac{c^*}{k^*} \\ -\kappa & 0 \end{pmatrix} \cdot \begin{pmatrix} \ln k - \ln k^* \\ \ln c - \ln c^* \end{pmatrix} = \tilde{A} \cdot \begin{pmatrix} \ln(k/k^*) \\ \ln(c/c^*) \end{pmatrix} \text{ for } \tilde{A} = \begin{pmatrix} \beta & -\frac{c^*}{k^*} \\ -\kappa & 0 \end{pmatrix}$$

where $\kappa = -\frac{1}{\theta} f''(k^*)k^* > 0$. This is a system of homogenous linear differential equations with constant coefficients.

General Problem: Solving Systems of Linear Differential Equations

In general, consider a system of n linear differential equations written in vector notation as

$$\dot{y} = A \cdot y.$$

Here y denotes an n -dimensional vectors of variables $(y_1, \dots, y_n)$, y -dot the corresponding vector of time derivatives, and A is an n -by- n matrix of fixed coefficients. One can show that the general solution for each of the variables is a linear combination of exponential functions that involve the eigenvalues and eigenvectors of the matrix A . (The proof would require too much linear algebra to be worth doing. See the appendix of Barro/Sala-i-Martin.)

Definitions: An eigenvector v of a matrix A is a non-zero vector such that Av is proportional to v with some proportionality factor μ , which is a scalar number. The value μ is called the eigenvalue associated with the eigenvector v .

In algebraic terms, $A \cdot v = \mu \cdot v$ implies $(A - \mu \cdot I) \cdot v = 0$, which implies that the n -by- n matrix $A - \mu \cdot I$ must have a zero determinant. This determinant is an n -th order polynomial in μ , which has n roots (real or complex, perhaps some identical ones). For this reason, an n -dimensional matrix generally has n eigenvalues and eigenvectors. The eigenvalues are commonly obtained by solving the characteristic equation $|A - \mu \cdot I| = 0$.

Math Result: If the eigenvalues are real and distinct (denoted $\mu_1, \dots, \mu_n$), the general solution for each variable y_j is a sum of exponentials $y_j(t) = \sum_{i=1}^n v_{ij} \cdot e^{\mu_i t}$, where the coefficients v_{ij} depend on boundary conditions.

Sketch of the Proof: The key to the proof is to transform the vector system into a set of separate, univariate linear differential equations, each of which is solved by an exponential function. That is, assemble the eigenvectors of A in a matrix $V = (v_1, \dots, v_n)$ and assemble the associated eigenvalues into a diagonal matrix M . Because $A \cdot v_i = \mu_i \cdot v_i = v_i \cdot \mu_i$ [Read: 1-by- n vector $A \cdot v_i$ = 1-by- n vector $v_i \cdot \mu_i$] for all i , one obtains the matrix equation $A \cdot V = V \cdot M$. [Read: n -by- n matrix $A \cdot V$ = n -by- n matrix $V \cdot M$. Hint: Write out what the elements are in the example.] Let me assert without proof that V is invertible if the eigenvalues are distinct. Then $V^{-1} \cdot A \cdot V = M$; also, $V^{-1}V = V \cdot V^{-1} = I$.

Define the vector $z = V^{-1} \cdot y$, which is a set of n linear combinations of y -variables, and consider the original linear system $\frac{dy}{dt} = A \cdot y$. Pre-multiply the system by V^{-1} and insert $V \cdot V^{-1}$ as needed, one finds (with brackets inserted for clarity)

$$\frac{dz}{dt} = V^{-1} \frac{dy}{dt} = V^{-1} \cdot A \cdot [V \cdot V^{-1}] \cdot y = V^{-1} [A \cdot V] \cdot [V^{-1} y] = V^{-1} [V \cdot M] \cdot z = V^{-1} V \cdot M \cdot z = M \cdot z$$

Recall that M is a diagonal matrix. Hence $\frac{dz}{dt} = M \cdot z$ is simply a list of n separate differential equations $\dot{z}_i = \mu_i \cdot z_i$. The solutions are $z_i(t) = z_i(0) \cdot e^{\mu_i \cdot t}$. Transforming back, $z = V^{-1} \cdot y$ implies $y = V \cdot z$, which means that each y -variable is a linear combination of z -variables, i.e., a linear combination of exponential functions.

Application to Optimal Growth

In the growth model, we have $n=2$, $y_1 = \ln(k) - \ln(k^*)$, $y_2 = \ln(c) - \ln(c^*)$, and $A = \tilde{A} = (a_{ij})$, where $a_{11} = \beta > 0$, $a_{12} = -c^*/k^* < 0$, $a_{21} = -\kappa < 0$, and $a_{22} = 0$.

The condition $|A - \mu \cdot I| = 0$ yields a quadratic equation for the eigenvalues μ_i , $i=1,2$:

$$(a_{11} - \mu)(a_{22} - \mu) - a_{21}a_{12} = \mu^2 - a_{11}\mu - a_{21}a_{12} = 0$$

The solutions are:

$$\mu_{1,2} = \frac{a_{11}}{2} \pm \sqrt{a_{11}^2 + a_{21}a_{12}} = -\frac{\beta}{2} \pm \sqrt{\left(\frac{\beta}{2}\right)^2 + \kappa \frac{c^*}{k^*}}$$

Because $\kappa > 0$, the root is greater than $\beta/2$ in absolute value. Hence $\mu_1 < 0$ and $\mu_2 > \beta$. The general solutions can be written as

$$y_1(t) = \varphi_{11} \cdot e^{\mu_1 t} + \varphi_{12} \cdot e^{\mu_2 t} \quad \text{and} \quad y_2(t) = \varphi_{21} \cdot e^{\mu_1 t} + \varphi_{22} \cdot e^{\mu_2 t}$$

with undetermined coefficients $\varphi_{i,j}$ that are determined by the problem's boundary conditions.

Because $e^{\mu_2 t}$ diverges to infinity with $\mu_2 > 0$, convergence to a steady state immediately implies zero coefficients on the second exponential term, i.e., $\varphi_{12} = \varphi_{22} = 0$. The given initial value $k(0) = k_0$ determines $y_1(0) = \ln(k_0) - \ln(k^*) = \varphi_{11} \cdot e^{\mu_1 \cdot 0} = \varphi_{11}$, hence $y_1(t) = [\ln(k_0) - \ln(k^*)] \cdot e^{\mu_1 t}$ for all t .

The remaining coefficient, φ_{21} , can be determined by combining the solution for capital, which implies $\frac{\dot{k}}{k} = \mu_1 \cdot \ln(k/k^*)$ with the original differential equation for capital:

$$\begin{aligned} \frac{\dot{k}}{k} &= \beta \cdot \ln(k/k^*) - \frac{c^*}{k^*} \ln(c/c^*) = \mu_1 \cdot \ln(k/k^*) \\ \Rightarrow \quad \ln(c/c^*) &= \frac{k^*}{c^*} (\beta - \mu_1) \cdot \ln(k/k^*) \end{aligned}$$

At $t=0$, one obtains $\varphi_{21} = \ln(c(0)) - \ln(c^*) = \frac{k^*}{c^*} (\beta - \mu_1) \cdot [\ln(k(0)) - \ln(k^*)] = \frac{k^*}{c^*} (\beta - \mu_1) \cdot \varphi_{11}$.

Because $\beta - \mu_1 = \mu_2$, this can be written more concisely as $\varphi_{21} / \varphi_{11} = \frac{k^*}{c^*} \mu_2$. This ratio can also be interpreted as the slope of the saddle path.

More Intuition: What are the transformed variables in the Growth Model application?

[This section is supplemental—intended for students who don't find matrix equations insightful. You may skip this section if you don't find it insightful.]

In the linearized growth model, the question answered by the z -variables is the following: Can we find linear combinations of $y_1 = \ln(k) - \ln(k^*)$ and $y_2 = \ln(c) - \ln(c^*)$ such that the time-derivative of each linear combination depends only on its own level and not on the other variable?

To explore if this is possible, let's define variables $z_i = y_1 + \xi_i \cdot y_2$ for some yet-undetermined coefficients ξ_i , $i=1,2$. We are looking for values ξ_i so that dz_i/dt is a function of z_i only. If we can such values, then each z_i is a single-variable differential equation that we can solve. To find such linear combinations, we use the *method of undetermined coefficients*; that is, we postulate that linear combinations with unknown coefficients exist, and then compute what they are. (This approach is a commonly-used general method, here introduced by example. Drop subscript i in the following when a generic z is considered.)

In the application, we know that $\dot{y}_1 = a_{11}y_1 + a_{12}y_2$ and $\dot{y}_2 = a_{21}y_1$.

Hence

$$\begin{aligned}\dot{z} &= \dot{y}_1 + \xi \dot{y}_2 = (a_{11} + \xi a_{21})y_1 + a_{12}y_2 = (a_{11} + \xi a_{21})(z - \xi y_2) + a_{12}y_2 \\ &= (a_{11} + \xi a_{21})z + [a_{12} - (a_{11} + \xi a_{21})\xi] \cdot y_2\end{aligned}$$

The variable y_2 drops out, if and only if $(a_{11} + \xi a_{21})\xi - a_{12} = 0$. This condition is a quadratic equation in ξ , which has two solutions:

$$\xi_{1,2} = -\frac{a_{11}}{2a_{21}} \pm \sqrt{\left(\frac{a_{11}}{2a_{21}}\right)^2 + \frac{a_{12}}{a_{21}}} = \frac{\beta}{2\kappa} \pm \sqrt{\left(\frac{\beta}{2\kappa}\right)^2 + \frac{c^*/k^*}{\kappa}}.$$

Define the variables $z_1 = \ln(k) + \xi_1 \cdot \ln(c)$ and $z_2 = \ln(k) + \xi_2 \cdot \ln(c)$ as the linear combinations of $\ln(k)$ and $\ln(c)$ with these specific ξ -values. Define $\mu_i = a_{11} + \xi_i \cdot a_{21}$, $i=1,2$. Then z_1 and z_2 satisfy separate differential equations: $\dot{z}_1 = \mu_1 \cdot z_1$ and $\dot{z}_2 = \mu_2 \cdot z_2$. Note that the coefficients μ_1 and μ_2 can also be expressed as $\mu_i = a_{12}/\xi_i$ and that

$$\mu_{1,2} = a_{11} + \xi_{1,2}a_{21} = \frac{a_{11}}{2} \pm \sqrt{a_{11}^2 + a_{21}a_{12}} = -\frac{\beta}{2} \pm \sqrt{\left(\frac{\beta}{2}\right)^2 + \kappa \frac{c^*}{k^*}}$$

The two equations $\dot{z}_1 = \mu_1 \cdot z_1$ and $\dot{z}_2 = \mu_2 \cdot z_2$ are each homogeneous linear differential equations with fixed coefficients. Their solutions have the form

$$z_1(t) = z_1(0) \cdot e^{\mu_1 \cdot t} \text{ and } z_2(t) = z_2(0) \cdot e^{\mu_2 \cdot t}.$$

Thus we have shown that two different linear combinations of $\ln(k)$ and $\ln(c)$ have exponential solutions.

Now we can transform back to the original variables: Because $z_1 = y_1 + \xi_1 \cdot y_2$ and $z_2 = y_1 + \xi_2 \cdot y_2$, one obtains

$$y_2 = \frac{z_2 - z_1}{\xi_2 - \xi_1} \text{ and } y_1 = z_1 - \xi_1 \cdot y_2 = \frac{\xi_2}{\xi_2 - \xi_1} z_1 - \frac{\xi_1}{\xi_2 - \xi_1} z_2.$$

$$\text{Hence } y_1(t) = \ln(k) - \ln(k^*) = \frac{\xi_2}{\xi_2 - \xi_1} z_1(0) \cdot e^{\mu_1 t} - \frac{\xi_1}{\xi_2 - \xi_1} z_2(0) \cdot e^{\mu_2 t} \quad (1)$$

$$\text{and } y_2(t) = \ln(c) - \ln(c^*) = -\frac{1}{\xi_2 - \xi_1} z_1(0) \cdot e^{\mu_1 t} + \frac{1}{\xi_2 - \xi_1} z_2(0) \cdot e^{\mu_2 t}. \quad (2)$$

This confirms that the solutions for capital and consumption are indeed linear combinations of exponential functions. Moreover, one can see how the 'weights' on the exponentials are related to

initial conditions. Finally, one may confirm that μ_1 and μ_2 are the eigenvalues of A. To see this, write out the equation for eigenvalues:

$$(*) \quad 0 = |A - \mu \cdot I| = \begin{vmatrix} a_{11} - \mu & a_{12} \\ a_{21} & -\mu \end{vmatrix} = \mu^2 - a_{11}\mu - a_{12}a_{21}.$$

This quadratic equation is indeed solved by the values computed above. (Check for yourself.) Also note that the product of the eigenvalues equals the determinant of A, here $|A| = -a_{12}a_{21} = \mu_1\mu_2$.

Back to matrix analysis: When we defined z_1, z_2 in terms of some unknowns ξ_1, ξ_2 , we were in effect assuming that V^{-1} could be written as a list of row-vectors $(1, \xi_i)$. For $V^{-1} = \begin{pmatrix} 1 & \xi_1 \\ 1 & \xi_2 \end{pmatrix}$, matrix inversion yields $V = \frac{1}{\xi_2 - \xi_1} \begin{pmatrix} \xi_2 & -\xi_1 \\ -1 & 1 \end{pmatrix}$. Looking back to the solutions (1) and (2), one can see that the weights on

the exponential functions in y_1 and y_2 can be interpreted as the elements of V. To confirm that V is indeed the matrix of eigenvectors, let's compute $V^{-1} \cdot A \cdot V = M$ and confirm that the result is a diagonal matrix with eigenvalues; one finds

$$V^{-1} \cdot A = \begin{pmatrix} 1 & \xi_1 \\ 1 & \xi_2 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 0 \end{pmatrix} = \begin{pmatrix} a_{11} + \xi_1 a_{21} & a_{12} \\ a_{11} + \xi_2 a_{21} & a_{12} \end{pmatrix} = \begin{pmatrix} a_{12}/\xi_1 & a_{12} \\ a_{12}/\xi_2 & a_{12} \end{pmatrix}$$

using $(a_{11} + \xi_i a_{21}) - a_{12} = a_{12}/\xi_i$. Moreover,

$$V^{-1} A \cdot V = \frac{1}{\xi_2 - \xi_1} \cdot \begin{pmatrix} a_{12}/\xi_1 & a_{12} \\ a_{12}/\xi_2 & a_{12} \end{pmatrix} \cdot \begin{pmatrix} \xi_2 & -\xi_1 \\ -1 & 1 \end{pmatrix} = \frac{a_{12}}{\xi_2 - \xi_1} \cdot \begin{pmatrix} \xi_2/\xi_1 - 1 & 0 \\ 0 & 1 - \xi_1/\xi_2 \end{pmatrix} = \begin{pmatrix} a_{12}/\xi_1 & 0 \\ 0 & a_{12}/\xi_2 \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}$$

using $a_{12}/\xi_i = \mu_i$. In the 'direct' calculations, we in effect ended up computing the eigenvectors through the method of undetermined coefficients. Going through these calculations one should suffice to obtain the intuition. The standard approach to solving systems of differential equations is to calculate the eigenvalues of A from the characteristic equation $|A - \mu \cdot I| = 0$ and then assert immediately that the solutions have the form $y_j(t) = \sum_{i=1}^n v_{ij} \cdot e^{\mu_i \cdot t}$. This avoids the tedious calculations required to compute V.

The last step is to exploit the boundary conditions. In the growth model, $\mu_1 < 0$ and $\mu_2 > 0$. Hence convergence to a steady state requires $z_2(0) = 0$. The initial capital stock implies

$$y_1(0) = \ln(k(0)) - \ln(k^*) = \frac{\xi_2}{\xi_2 - \xi_1} z_1(0).$$

There are two alternative ways to compute y_2 . If one has calculated V and already knows (ξ_1, ξ_2) , it's easy to calculate $y_2(0) = \ln(c(0)) - \ln(c^*) = -\frac{1}{\xi_2 - \xi_1} z_1(0)$. Indeed, note that

$$y_2(t) = -\frac{1}{\xi_2} \cdot y_1(t) = -\frac{1}{\xi_2} \cdot y_1(t) = \frac{\mu_2}{\kappa} \cdot y_1(t)$$

holds for all t. This provides the slope of the log-linearized saddle-path. If V has not been computed (most common in applications), one computes $y_2(0)$ by combining the solution to the other variable with one of the original differential equations. For example, $\frac{dy_2}{dt} = a_{21} \cdot y_1(t) = \mu_1 \cdot y_2(t)$ also implies $y_2(t) = \frac{a_{21}}{\mu_1} \cdot y_1(t) = \frac{\mu_2}{-a_{21}} \cdot y_1(t) = \frac{\mu_2}{\kappa} \cdot y_1(t)$, using $\mu_1\mu_2 = -a_{12}a_{21}$.

Conclusions

In summary, the application demonstrates how a system of linear differential equations can be solved by transforming it into a list of distinct single differential equations. The example also demonstrates that the solutions involve exponential functions with the eigenvalues of A in the exponents. Both findings are true in general. In most applications, it is much easier to solve the *characteristic equation* (*) than to find all the z -variables and ξ -coefficients. This motivates the standard approach:

- (1) Compute the eigenvalues from the characteristic equation.
- (2) Write the solutions as linear combination of exponentials with eigenvalues.
- (3) Determine the weights on the exponential terms from the boundary conditions.