Fiscal Reaction Functions in an Overlapping Generations Setting:
Why "Unfair" Fiscal Stabilizations Are Most Effective

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Abstract

This paper examines the choice between alternative debt and deficit responses in an overlapping generations (OG) setting. The OG model is a convenient framework for making interest rates sensitive to the supply of public debt and for distinguishing between different types of taxes and transfers. A given deficit-reduction yields a greater reduction of future debt if the policy instrument also reduces the interest rate and raises future wages, and a lesser reduction if policy instrument does the reverse. Under benchmark assumptions, reduced outlays for retirees yield the relatively greatest debt reductions, followed by tax increases on non-savers and by cuts in public spending. Tax increases on bond-buying cohorts are least effective. Yet exempting the rich (bond-buyers) from tax increases and spending cuts seems politically unthinkable on fairness grounds.

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1. Introduction

The paper examines fiscal reaction functions in a simple overlapping generations (OG) model in which public debt crowds out capital and raises interest rates.

Most work on the intertemporal budget constraints and on the sustainability of fiscal policy has taken interest rates as given. Exogenous interest rates are a useful simplification in empirically oriented papers and they are commonly defended by reference to (approximate) Ricardian equivalence. If crowding out effects are present, however, a model with fixed interest rates and exogenous wages provides a too-benign laboratory for fiscal analysis. The dynamics of debt are less stable when interest rates rise with the level of debt than in a fixed interest rate setting. The economy is destabilized further if income taxes decline as reduced capital reduces real wages. Fiscal stabilization requires stronger policy responses. Endogenous interest rates also create interesting differences between policy instruments with otherwise equal fiscal impact.

The basic analysis is presented in three-period overlapping generations model with liquidity-constrained young. OG is a convenient framework for making interest rates sensitive to the supply of debt and for distinguishing different types of taxes and transfers. Liquidity-constrained young simplify the dynamics and create a population segment that does not participate in the bond market. Middle-aged agents are working and saving. The old are retired and dissave. Thus the model has savers, dissavers, and agents removed from the bond market. Also because the young are liquidity constrained, the basic model has the simple dynamics of Diamond (1965). Extended versions endogenize labor supply and demonstrate that the basic results generalize.

Following the literature of fiscal reaction functions, debt stabilization is captured by feedback responses from debt to taxes, transfers, and real spending (see Bohn 1998, Michel et al. 2005). The key points here are that a given deficit-reduction yields a greater reduction of future debt if the policy instrument also reduces interest rates and increases future wages. The analysis of interest rate effects yields a clear ranking of policy instruments. In a setting with inelastic labor supply, tax increases and cuts in transfers are least effective for fiscal stabilization if they fall on a cohort that saves and buys bonds (in the model: the middle aged), more effective when imposed on non-savers (the young), and most effective when imposed on cohorts that dissave (the old). Cuts in real spending are as effective as taxes-transfer changes for non-savers. (Fiscal responses to changes in the capital stock/interest rates yield a similar ranking.) If labor supply is elastic, the relative effectiveness of labor income taxes as stabilization tools depends on the level of debt and the sensitivity interest rates and wages to changes in the capital-labor ratio. Taxing an *elastic* labor supply of non-savers tends to be *more* effective than spending cuts because reduced labor supply reduces the marginal product of capital and hence reduces interest payments on the public debt.

In practice, fiscal stabilization plans often trigger political protests about hardships imposed on vulnerable population groups. From a distributional fairness perspective, one may consider taxes on middle-aged cohorts, who typically earn the highest incomes, preferable to cuts in retiree transfers and to higher taxes on the young. Taxing the middle-aged is least effective, however, because it reduces the capital stock. The relative effectiveness results may therefore explain why governments often design fiscal stabilizations that seem socially "unfair" by imposing more burdens on retirees and other non-savers than on population groups with surplus funds to save.

Section 2 presents a basic production economy with exogenous labor supply. Section 3 studies a more general model with endogenous labor. Section 4 examines the mechanisms of debt stabilization with endogenous labor in an intermediate version. Section 5 concludes.

2. The Basic Model

Consider a three-period OG economy with exogenous labor. Individuals born in period t (cohort t) have preferences

$$U_t = u(c_t^1)/\beta_0 + u(c_{t+1}^2) + \beta u(c_{t+2}^3)$$

over young-age (period 1), middle-age (period 2), and old-age consumption (period 3) consumption. Superscripts refer to age, subscripts to time; $u(\cdot)$ is increasing and strictly concave; and $\beta, \beta_0 > 0$. Cohort size N_t grows exogenously at (gross) rate $n = N_t / N_{t-1} \geq 1$. The old are retired. The middle-aged are endowed with one unit of labor (as normalization). The young are endowed with e>0 units of labor. (Empirical age-earnings profiles suggest e<1.) The effective labor force in period t is $L_t = e \cdot N_t + N_{t-1}$.

Private credit is assumed unenforceable. This is to rule out individual borrowing by the young. The parameter β_0 is assumed small enough that the young consume their after-tax labor income and never save. The middle-aged save an amount $s_t \geq 0$ for retirement.[1]

Output (Y) is produced by competitive firms with capital (K) and labor, $Y_t = F(K_t, L_t)$, where F is increasing, concave, and has constant returns to scale. To avoid separate notation for depreciation, let F include the post-depreciation value of old capital. The gross return on capital and the unit wage rate are given by

$$R_t = F_K(K_t, L_t) = F_K(K_t / L_t, 1) \equiv R(\kappa_t)$$

and $\quad w_t = F_L(K_t / L_t, 1) = w(\kappa_t)$, where $\kappa_t = K_t / L_t$

is the capital-labor ratio. Concave production implies $R'(\kappa_t) = F_{KK} \leq 0$ and $w'(\kappa_t) = F_{LK} \geq 0$.

The capital-labor ratio can be expressed as ratio $\kappa_t = k_t / \lambda$ of the per-capita capital stock $k_t = K_t / (N_t + N_{t-1} + N_{t-2})$ and the per-capita labor supply $\lambda = \frac{L_t}{N_t + N_{t-1} + N_{t-2}} = \frac{e \cdot n^2 + n}{n^2 + n + 1}$. (The distinction between k_t and κ_t becomes when labor is endogenized below.)

The government imposes taxes (θ_t^1, θ_t^2) on young and middle-aged workers. Taxes finance real spending, G_t, transfers to the old, denoted γ_t, and payments on the public debt. The government budget equation is

$$D_{t+1} = G_t + R_t D_t - N_t \theta_t^1 - N_{t-1} \theta_t^2 + N_{t-2} \gamma_t,$$

where D_{t+1} denote the end-of period-t debt that is passed on to period t+1. Bonds have to pay the same return as capital in this deterministic economy. The budget equation in per-capita terms is

$$\sigma^1 \cdot \theta_t^1 + \sigma^2 \cdot \theta_t^2 + n \cdot d_{t+1} = g_t + R_t \cdot d_t + \sigma^3 \cdot \gamma_t,$$

where $\quad \sigma^1 = \frac{n^2}{1+n+n^2}$, $\sigma^2 = \frac{n}{1+n+n^2}$, and $\sigma^3 = \frac{1}{1+n+n^2}$

are the population shares of young, middle-aged, and old.

[1] The parameter β_0 has no other function the model. The lack of saving by the young is consistent with empirical work by Poterba (2004) suggesting that individuals start to save for retirement around age 40.

The individual budget equations are $c_t^1 = e \cdot w_t - \theta_t^1$ for the young (invoking the no savings assumption), $c_t^2 = w_t - \theta_t^2 - s_t$ for the middle-aged, where s_t is savings, and $c_t^3 = R_t \cdot s_{t-1} + \gamma_t$. Capital market equilibrium requires $K_{t+1} + D_{t+1} = N_{t-1} s_t$.

Though the paper's focus is on propagation, disturbances must have a source. To be specific, let government spending and wages be subject to additive i.i.d. shocks $(\hat{g}_t, \hat{w}_t)$.[2] They have a natural interpretation as fiscal shocks and business cycle disturbances. This completes the model.

Now consider policy. There are four independent policy instruments: taxes on the young θ_t^1, taxes on the middle-aged θ_t^2, transfers to retirees γ_t, and government spending. Spending is decomposed into a "discretionary" component $\tilde{g}_t$ and an exogenous component $\hat{g}_t$, $g_t = \tilde{g}_t + \hat{g}_t$. Taxes on young and middle aged can also be expressed in terms of tax rates on wage income (τ_t^1, τ_t^2). Revenues are naturally bounded by wage incomes, $0 \le \theta_t^1 = e \cdot w_t \cdot \tau_t^1 \le e \cdot w_t$ and $0 \le \theta_t^2 = w_t \cdot \tau_t^2 \le w_t$. Limits on transfers, spending and debt will have to be determined. Spending and transfers are generally non-negative.[3]

Fiscal reaction functions describe how policy responds to variations in public debt, and perhaps to other relevant variables. The model's minimal state vector consists of the per-capita capital stock, the per-capita public debt, and the wage and spending shocks. Adding extraneous, policy-induced state variables would merely distract. The policy instruments are therefore assumed time-invariant functions of (at most) the state vector $(\kappa_t, d_t, \hat{w}_t, \hat{g}_t)$. To focus on responses to debt and capital, I generally assume zero responses to current shocks.[4] Retiree transfers in particular are assumed predetermined, which will simplify savings decisions.

[2] To avoid clutter, dependence on shocks is often suppressed. For example, the more elaborate notation for output $Y_t = F(K_t, L_t) + \hat{w}_t \cdot L_t$ and for wages $w_t = w(\kappa_t) + \hat{w}_t$ is used only when needed. In reality, war spending tends to have positive mean; it is subsumed below into discretionary spending without loss of generality.

[3] Transfers to the young and middle-aged could easily be added. Because labor is fixed, taxes on the young and middle agent can be interpreted as wage taxes ,or equivalently, lump sum taxes. Taxes on the old (negative transfers) are ruled out (except for one special argument detailed below) to avoid introducing taxes that induce purchases of government bonds "enforced" by otherwise prohibitive lump-sum taxes. A more general policy dependence on current shocks could also be added, but would distract from propagation issues.

[4] Immediate responses to shocks may well be desirable, but this would distract from propagation issues. Responses to current shocks are formally allowed to avoid existence issue when shocks arise while debt is near an upper limit. Whenever debt would otherwise breach a limits, one may assume (without loss of generality) an immediate spending response.

The budgetary effects of the four policy instruments is measured by their impact on the government's primary surplus π_t in period t:

$$\pi_t = \sigma^1 \cdot \theta_t^1 + \sigma^2 \cdot \theta_t^2 - \sigma^3 \cdot \gamma_t - g_t \tag{1}$$

Let $\tilde{\pi}_t = \pi_t - \hat{g}_t$ denote surplus without the exogenous spending shock, the controlled, discretionary component. Then debt accumulation can then be written as

$$d_{t+1} = \frac{1}{n}[R(\kappa_t) \cdot d_t + \hat{g}_t - \tilde{\pi}_t] \tag{2}$$

a sum of endogenous propagation, a shock, and policy responses.[5] Note that if tax rates were used as policy instruments, $\pi_t = [\sigma^1 \cdot e \cdot \tau_t^1 + \sigma^2 \cdot \tau_t^2] \cdot w_t - \sigma^3 \cdot \gamma_t - g_t$ would depend on the capital-labor ratio through wages. The ramifications will be examined below.

The model's dynamics depend critically on savings behavior, as in the Diamond (1965) economy. Here the middle-aged play the role of Diamond's young. They choose savings to maximize

$$u(w_t - \theta_t^2 - s_t) + \beta u(R(\kappa_{t+1}) \cdot s_t + \gamma_{t+1}),$$

The first-order condition is

$$u'(w_t - \theta_t^2 - s_t) = R_{t+1} \cdot \beta u'(R_{t+1} \cdot s_t + \gamma_{t+1}) + \vartheta \tag{3}$$

where $\vartheta \geq 0$ is the Kuhn-Tucker multiplier on $s_t \geq 0$. The non-negativity constraint might bind if transfers are sufficiently generous to destroy savings incentives. But then capital would be zero or be owned by the government ($k_t = -d_t > 0$), an uninteresting scenario. Hence I assume $s_t > 0$ and hence $\vartheta = 0$. Then (3) implies a savings function

$$s_t = s(w_t - \theta_t^2 + \frac{\gamma_{t+1}}{R_{t+1}}, R_{t+1}) - \frac{\gamma_{t+1}}{R_{t+1}}.$$

The term $w_t - \theta_t^2 + \gamma_{t+1}/R_{t+1}$ can be interpreted as net wealth (W). Let s_W and s_R denote the partial derivatives of $s(\cdot)$. Time-separable utility implies $0 < s_W < 1$. Savings are therefore increasing in $w_t - \theta_t^2$ and decreasing in γ_{t+1}. Capital market equilibrium requires

$$\lambda \cdot \kappa_{t+1} + d_{t+1} = \frac{\sigma^2}{n} s_t = \sigma^3 \cdot [s\left(w_t - \theta_t^2 + \frac{\gamma_{t+1}}{R(\kappa_{t+1})}, R(\kappa_{t+1})\right) - \frac{\gamma_{t+1}}{R(\kappa_{t+1})}]. \tag{4}$$

using $\sigma^2 / n = \sigma^3$. Equations (2) and (4) implicitly define a vector Markov process for the capital-labor ratio and for debt, $\kappa_{t+1} = \tilde{\kappa}(\kappa_t, d_t, \hat{w}_t, \hat{g}_t)$ and $d_{t+1} = \tilde{d}(\kappa_t, d_t, \hat{w}_t, \hat{g}_t)$. A deterministic (zero

[5] State-contingent debt could serve as an alternative or additional stabilization device (see Lucas-Stokey 1983; Bohn 1990). This is well known and would be distracting here, because responses of the primary surplus to shocks would still be required, except in extreme cases where-contingent debt absorbs all shocks. Hence I treat debt management as exogenous.

shocks) steady state is defined by a vector of exogenous policy parameters $(\theta^1, \theta^2, \gamma, g)$, an implied primary surplus

$$\pi = \sigma^1 \cdot \theta^1 + \sigma^2 \cdot \theta^2 - \sigma^3 \cdot \gamma - g \tag{5}$$

and associated steady state values (κ, d). The latter must satisfy

$$\lambda\kappa + d = \sigma^3 \cdot [s\left(w(\kappa) - \theta^2 + \tfrac{\gamma}{R(\kappa)}, R(\kappa)\right) - \tfrac{\gamma}{R(\kappa)}] \tag{6}$$

and $$d \cdot [R(\kappa) - n] = \pi\,. \tag{7}$$

The model analysis proceeds in three steps. Step 1 examines debt-dynamics with exogenous factor prices, the special case of linear production. Step 2 examines the model without debt to establish benchmark conditions for stability and capital-dynamics. Step 3 combines capital and debt to demonstrate the destabilizing effects of their interaction. In each case, local stability around a steady can evaluated by linearizing (2) and/or (4) around a steady state.

2.1. Exogenous Factor Prices

Much of the literature on intertemporal budget constraints and fiscal sustainability assumes constant or exogenous interest rates. To bridge the general OG model and the literature on fiscal constraints with exogenous interest rates, consider the special, limiting case of linear production: $Y_t = R \cdot K_t + w \cdot L_t$. It implies fixed returns to capital and to labor.

With exogenous factor prices, next period's capital is given by

$$\kappa_{t+1} \cdot \lambda = \sigma^3 \cdot [s\left(w - \theta_t^2 + \tfrac{\gamma_{t+1}}{R}, R\right) - \tfrac{\gamma_{t+1}}{R}] - d_{t+1}\,. \tag{4'}$$

Importantly, the capital-labor ratio does not depend on its own lag. Government debt is thus the only essential state variable. Dependence on κ_t could be introduced through policy, but such policy responses would merely add unnecessary noise.

Debt is limited by the availability of savings. If transfers, taxes, and capital are non-negative, as assumed above, (4') implies $d_{t+1} \le s(\overline{w}, \overline{R}) = \overline{d}$.[6] Linearizing (1) and (2), one finds that the stochastic process for debt has characteristic root

[6] This bound could be relaxed if one introduced lump-sum taxes on retirees and transfers to the middle-aged (Buiter and Kletzer, 1992). But this would be an exercise in "labeling" (in the sense of Kotlikoff 1993), because the government's lump-sum tax claim is effectively offsetting the debt. The non-negativity of transfers makes debt an economically non-trivial concept.

$$\mu_d = \frac{\partial \tilde{d}_{t+1}}{\partial d_t} = \frac{R}{n} - \frac{1}{n}\tilde{\pi}_d \text{, where}$$

$$\tilde{\pi}_d = \sigma^1 \cdot \theta_d^1 + \sigma^2 \cdot \theta_d^2 - \sigma^3 \cdot \gamma_d - \tilde{g}_d . \qquad (8)$$

The process is stable if $|\mu_d| < 1$. The convergence is monotone if $\mu_d \in [0,1)$, or equivalently $\tilde{\pi}_d \in (R-n, R-n+1]$. The process would display convergence with oscillations if $\mu_d \in (-1,0)$. Oscillations in debt are not empirically relevant and they would imply implausibly high response coefficients $\tilde{\pi}_d$. The practically relevant issue is whether or not, and how much, policy makers must adjust taxes and spending to ensure convergence. To avoid distracting and irrelevant cases, the paper will focus on conditions for *monotone* convergence, and not merely convergence, and particularly on the *minimal policy responses* required for stability. With linear production, the critical issue for stability is that the response coefficient $\tilde{\pi}_d$ is strictly above the lower bound $R-n$.[7]

The interpretation of $\tilde{\pi}_d > R-n$ depends on the relationship between interest rate and population growth. If the economy is dynamically efficient, $R \geq n$, a stable process for debt requires a strictly positive response of the primary surplus to debt—either increased taxes, or reduced transfers, or reduced real spending, or a combination. In the Golden Rule case, $R = n$, an arbitrarily small response suffices to tip the dynamics into the stable region. If $R > n$, the linear combination of the responses in (8) must add up to a value $\tilde{\pi}_d$ that exceeds $R-n$.

The stability arguments above depend only on the response coefficient $\tilde{\pi}_d$. The different policy instruments—taxes, transfers, and spending—have the same weight in the response coefficient $\tilde{\pi}_d$ as they have in the primary surplus itself. In this sense, all policy instruments are equally effective for stabilizing the debt. For reference below, relative effectiveness is best defined explicitly:

<u>Definition 1</u>: Policy tools are called <u>equally effective</u> (for some purpose) if their relative impact is proportional to their contribution to the primary surplus. A policy tool is called <u>more effective</u> than another if the policy tool's relative impact exceeds the relative weights in the primary surplus.

In this section, all policy instruments are equally effective for stabilizing the public debt.

[7] Note that conditions for stability differ from the conditions required for an intertemporal budget constraint (IBC). The IBC can be satisfied even if per-capita debt grows explosively, provided the growth rate is less than the interest rate (see Bohn 1998). This paper focuses on stability.

If the economy is dynamically inefficient, $R < n$, public debt is stationary even if policy does not respond to fluctuations (meaning $\theta_d^1 = \theta_d^2 = \gamma_d = \tilde{g}_d = 0$ so $\tilde{\pi}_d = 0$). For reference below, a scenario with unresponsive policy is worth defining:

<u>Definition 2</u>*: Public debt is called* <u>*self-stabilizing*</u>*, if the economy is locally stable around a steady state with non-zero debt even if the components of the primary surplus do not respond to the economy's state variables.*

With exogenous factor prices, public debt is self-stabilizing for $R < n$ but not for $R \geq n$.

In summary, one obtains the following simple benchmark results:

1. Debt-stabilization requires $\tilde{\pi}_d > R - n$.

2. Debt is self-stabilizing for $R < n$.

3. If debt-stabilization is needed, all policy instruments are equally effective.

The next section will establish a second benchmark by modeling endogenous factor prices in an economy without debt.

2.2. Capital Accumulation without Public Debt

In general, the linkage between wages and capital introduces a propagation mechanism through the capital-labor ratio. To isolate the private-sector dynamics, consider for this section a policy scenario with zero debt, balanced budget, and constant taxes, transfers, and real spending.

When discussion general production, it is convenient to exclude the linear case discussed above, to avoid repeated case distinctions. Assume therefore that F is strictly concave so $w'(\kappa) > 0$ and $R'(\kappa) < 0$ are non-zero. Also assume that there is a maximum sustainable capital-labor ratio $\overline{\kappa}$ such that $F(\overline{\kappa},1) < \overline{\kappa}$ for all $\kappa > \overline{\kappa}$.

With zero debt, the dynamics of capital in (4) reduces to

$$\kappa_{t+1} = \tfrac{\sigma^3}{\lambda} \cdot [s\Big(w(\kappa_t) + \hat{w}_t - \theta_t^2 + \tfrac{\gamma_{t+1}}{R(\kappa_{t+1})}, R(\kappa_{t+1})\Big) - \tfrac{\gamma_{t+1}}{R(\kappa_{t+1})}], \qquad (4'')$$

which defines an implicit function $\kappa_{t+1} = \tilde{\kappa}(\kappa_t, \hat{w}_t)$ linking current to future capital. Stability depends on the characteristic root $\mu_{0\kappa} = \partial\kappa_{t+1} / \partial\kappa_t = \omega_0 / \psi_0$ where

$$\omega_0 = \sigma^3 s_W \cdot w'(\kappa) > 0 \text{ and } \psi_0 = \lambda + \sigma^3 (s_R + (1 - s_W) \tfrac{\gamma}{R^2}) \cdot \{-R'(\kappa)\}.$$

Local stability and monotone convergence require $0 \le \omega_0 / \psi_0 < 1$. This is satisfied if (w', R') are sufficiently small, because $\omega_0 \to 0$ and $\psi_1 \to \lambda > 0$ as $(w', R') \to (0,0)$.[8]

Instabilities in capital accumulation *per se* would distract from the paper's focus on fiscal issues. Hence I will impose

<u>Assumption 1</u>: $0 \le \omega_0 < \psi_0$ *holds for all steady states.*

Note that the steady state is a function of (θ^2, γ), the policy variables appearing in (4"). Because savings are decreasing functions of θ_t^2 and of γ_{t+1}, variations in the steady state parameters (θ^2, γ) yield a range of steady state values $\kappa = \kappa^*(\theta^2, \gamma)$. For small (θ^2, γ), these steady states may be dynamically inefficient, i.e., $R[\kappa^*(\theta^2, \gamma)] < n$.

This concludes the special cases. Now consider the interaction of debt and capital.

2.3. Debt and Capital with Endogenous Factor Prices

The main issue is the interaction of fiscal dynamics and the capital stock. The interaction goes both ways. Capital accumulation depends on debt through crowding out and on taxes and transfers through their respective impacts on savings. Debt accumulation depends on capital through the interest rate on debt and potentially through revenues from wage income taxes.

To characterize the dynamic structure, one may linearize (2) and (4) around a steady state. This yields a first-order autoregressive system:

$$\begin{pmatrix} 1 & 0 \\ 1+\frac{1-s_W}{R}\sigma^3\gamma_d & \psi_0 + \frac{1-s_W}{R}\sigma^3\gamma_\kappa \end{pmatrix} \begin{pmatrix} d_{t+1}-d \\ \kappa_{t+1}-\kappa \end{pmatrix} = \begin{pmatrix} \frac{R-\tilde{\pi}_d}{n} & \frac{R'(\kappa)d-\tilde{\pi}_\kappa}{n} \\ -s_W\sigma^3\theta_d^2 & \omega_0 - s_W\sigma^3\theta_\kappa^2 \end{pmatrix} \begin{pmatrix} d_t-d \\ \kappa_t-\kappa \end{pmatrix} + \begin{pmatrix} \frac{1}{n} & -\frac{\tilde{\pi}_{\hat{w}}}{n} \\ 0 & s_W\sigma^3 \end{pmatrix} \begin{pmatrix} \hat{g}_t \\ \hat{w}_t \end{pmatrix} \tag{9}$$

The spending shock $\hat{g}_t$ serves as disturbance to debt accumulation. The wage shock $\hat{w}_t$ serves as disturbance to capital accumulation. For reference below, define $x_t = (d_t - d, \kappa_t - \kappa)$ and let the matrices in (9) be denoted by Ψ, Ω, Z, respectively, so $\Psi \cdot x_{t+1} = \Omega \cdot x_t + Z \cdot (\hat{g}_t, \hat{w}_t)'$. Define $\Psi = \begin{pmatrix} 1 & 0 \\ \psi_{kd} & \psi_{kk} \end{pmatrix}$ and $\Omega = \begin{pmatrix} \omega_{dd} & \omega_{d\kappa} \\ \omega_{kd} & \omega_{kk} \end{pmatrix}$ to denote matrix elements.

[8] Note that $s_R \ge 0$ is sufficient for $\psi_0 \ge \lambda > 0$. The stability properties (and problems) of this mapping have been much discussed since Diamond (1965); hence further discussion is probably more distracting than helpful.

The first line of (9) describes the dynamics of debt. Note that $\omega_{dd} = (R - \tilde{\pi}_d)/n$, the impact of current on future debt, equals the characteristic root μ_d of the debt-dynamics in Section 2.1. The $R'(\kappa)$-term in $\omega_{d\kappa} = (R'(\kappa)\cdot d - \tilde{\pi}_\kappa)/n$ captures the impact of capital on debt accumulation through endogenous interest rates.

The second line of (9) describes the dynamics of the capital-labor ratio. These dynamics are evidently influenced by the responses of transfers and middle-age taxes to changes in debt and capital, $(\gamma_\kappa, \gamma_d)$ and $(\theta_\kappa^2, \theta_d^2)$. To isolate the impact of policy responses, note that the model's "uncontrolled" dynamics without policy responses reduce to

$$\begin{pmatrix} 1 & 0 \\ 1 & \psi_0 \end{pmatrix}\begin{pmatrix} d_{t+1} - d \\ \kappa_{t+1} - \kappa \end{pmatrix} = \begin{pmatrix} R/n & R'(\kappa)d/n \\ 0 & \omega_0 \end{pmatrix}\begin{pmatrix} d_t - d \\ \kappa_t - \kappa \end{pmatrix} + \begin{pmatrix} 1/n & 0 \\ 0 & \sigma^3 s_W \end{pmatrix}\begin{pmatrix} \hat{g}_t \\ \hat{w}_t \end{pmatrix} \tag{9'}$$

Without policy responses, the impact of capital on future capital depends on the same coefficients (ψ_0, ω_0) as in Section 2.2.

The basic interaction between capital and debt is captured by the two off-diagonal terms in (9'): $\psi_{kd} = 1$ captures crowding out; and $\omega_{d\kappa} = R'(\kappa)/n \cdot d < 0$ captures the impact of capital on debt accumulation through endogenous interest rates. These two interactions provide the intuition to why endogenous interest rates are destabilizing: <u>Debt crowds out capital, and lower capital reinforces debt accumulation through rising interest rates.</u> It is straightforward to verify (see below) that (9') has a characteristic root strictly greater than $\max(\mu_d, \omega_0/\psi_0)$. Endogenous interest rates thus destabilize the economy.

A somewhat different "uncontrolled" dynamics is obtained if tax policy is parameterized by tax rates. Constant tax rates are equivalent to assuming endogenous revenues with

$$\theta_\kappa^2 = w'(\kappa)\cdot\tau^2 > 0,\ \pi_k = (\sigma^1\tau^1 e + \sigma^2\tau^2)\cdot w'(\kappa) \geq 0, \text{ and } \pi_{\hat{w}} = \sigma^1\tau^1 e + \sigma^2\tau^2 \geq 0.$$

This implies

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 1 & \psi_0 \end{pmatrix}\begin{pmatrix} d_{t+1} - d \\ \kappa_{t+1} - \kappa \end{pmatrix} &= \begin{pmatrix} R/n & \frac{1}{n}[R'(\kappa)d - (\sigma^1\tau^1 e + \sigma^2\tau^2)\cdot w'(\kappa)] \\ 0 & \omega_0\cdot(1-\tau^2) \end{pmatrix}\begin{pmatrix} d_t - d \\ \kappa_t - \kappa \end{pmatrix} \\ &+ \begin{pmatrix} 1/n & -(\sigma^1\tau^1 e + \sigma^2\tau^2) \\ 0 & \sigma^3 s_W \end{pmatrix}\begin{pmatrix} \hat{g}_t \\ \hat{w}_t \end{pmatrix} \end{aligned} \tag{9''}$$

The reduced coefficient $\omega_{\kappa\kappa} = \omega_0 \cdot (1-\tau^2)$ captures the well-known automatic stabilizer role of income taxes for the private sector. However, because $w'(\kappa) > 0$, fixed tax rates magnify the impact of lower capital on debt (increase $|\omega_{dk}|$), and this turns out to be destabilizing.

In general, the stability and convergence properties of (9), (9'), and (9") depend on the roots on the characteristic polynomial (rescaled by $|\Psi|$ for convenience)

$$P(\mu) = \tfrac{1}{|\Psi|} \cdot |\Omega - \Psi \cdot \mu| = \mu^2 - (\omega_{dd} + \tfrac{\omega_{\kappa\kappa}}{\psi_{\kappa\kappa}} - \tfrac{\omega_{d\kappa}\psi_{\kappa d}}{\psi_{\kappa\kappa}}) \cdot \mu + (\tfrac{\omega_{\kappa\kappa}}{\psi_{\kappa\kappa}}\omega_{dd} - \tfrac{\omega_{d\kappa}\omega_{\kappa d}}{\psi_{\kappa\kappa}}).$$

The roots are

$$\mu_{1,2} = \tfrac{1}{2}(\omega_{dd} + \tfrac{\omega_{\kappa\kappa}}{\psi_{\kappa\kappa}} - \tfrac{\omega_{d\kappa}\psi_{\kappa d}}{\psi_{\kappa\kappa}}) \pm \sqrt{\tfrac{1}{4}(\omega_{dd} + \tfrac{\omega_{\kappa\kappa}}{\psi_{\kappa\kappa}} - \tfrac{\omega_{d\kappa}\psi_{\kappa d}}{\psi_{\kappa\kappa}})^2 - (\tfrac{\omega_{\kappa\kappa}}{\psi_{\kappa\kappa}}\omega_{dd} - \tfrac{\omega_{d\kappa}\omega_{\kappa d}}{\psi_{\kappa\kappa}})}$$

Under weak conditions (assumed throughout to avoid uninteresting special cases), the discriminant is non-negative, implying real roots, and both roots are strictly positive.[9]

The key question is under what conditions both roots are less than one. One finds that because $P(\mu)$ is quadratic, $0 < \mu_1 < \mu_2 < 1$ if and only if $P(1) > 0$, which is equivalent to

$$\omega_{dd} < 1 - \tfrac{(-\omega_{d\kappa})}{\psi_{\kappa\kappa} - \omega_{\kappa\kappa}}(\psi_{\kappa d} - \omega_{\kappa d}) \equiv \overline{\omega}_{dd} \qquad (10)$$

The uncontrolled dynamics in (9') and (9") imply $\omega_{d\kappa} < 0$, $\psi_{\kappa d} - \omega_{\kappa d} > 0$, $\psi_{\kappa\kappa} - \omega_{\kappa\kappa} > 0$, and hence $\overline{\omega}_{dd} < 1$. The same applies for (9) with sufficiently "small" policy coefficients. Stability thus requires that the response of future to current debt is bounded away from one, a more stringent requirement than in the case of exogenous factor returns. More specifically:

Proposition 1:

a. If taxes are lump-sum taxes, debt is self-stabilizing if and only if $R < n - \Lambda_\theta$,

where $\Lambda_\theta = \frac{-R'(\kappa)}{\psi_0 - \omega_0} \cdot d > 0.$

b. If taxes are wage taxes, debt is self-stabilizing if and only if $R < n - \Lambda_\tau$,

where $\Lambda_\tau = \frac{-R'(\kappa)}{\psi_0 - \omega_0 \cdot (1-\tau^2)} \cdot d + \frac{w'(\kappa) \cdot (\sigma^1 \tau^1 e + \sigma^2 \tau^2)}{\psi_0 - \omega_0 \cdot (1-\tau^2)} > 0.$

c. Dynamic inefficiency does not ensure a self-stabilizing debt.

Proof: (a) and (b) follow directly from (10) and either (9') or (9"), respectively. Specifically, (10) reduces to

[9] Sufficient for real roots are $\psi_{\kappa d} - \omega_{\kappa d} \geq 0$ and $\omega_{d\kappa} \leq 0$. Strictly positive roots are implied by $P(0) = |\Omega| / |\Psi| > 0$. All these conditions hold for reasonable small policy coefficients in (9).

$\frac{R}{n} < 1 + \frac{-R'(\kappa)d/n}{\psi_0 - \omega_0}$ for $\theta_\kappa^2 = \gamma_\kappa = \pi_\kappa = 0$ and $\theta_d^2 = \gamma_d = \pi_d = 0$, which implies (a). (c) follows directly from (a) and (b). QED.

For interest rates in the range $R \in [n - \Lambda_\theta, n)$ with lump-sum taxes, and for $R \in [n - \Lambda_\tau, n)$ with income taxes, a marginal increase in debt raises interest rates enough that the additional interest cost places debt on an explosive path; and marginal decrease in debt would reduce interest rates enough to put debt on an accelerating downward path.

Note that $\Lambda_\tau > \Lambda_\theta$ is possible, and it hold unambiguously for small values of debt. Then income taxes are *destabilizing* as compared to lump-sum taxes. Their destabilizing impact on public debt outweighs their automatic stabilizer role for the private sector.

Next, consider policies that respond flexibly to debt, but no differently to the capital-labor ratio. With either tax system, (10) can be written as

$$\sigma^1 \theta_d^1 + \sigma^2 \left[1 + \Lambda_i \cdot \frac{\sigma^2}{n} s_W\right] \cdot \theta_d^2 - \sigma^3 \left[1 - \Lambda_i \cdot \frac{1 - s_W}{R}\right] \cdot \gamma_d - \tilde{g}_d > R - n + \Lambda_i \qquad (11)$$

where $\Delta_i = \Delta_\theta, \Delta_\tau > 0$, depending on the tax system. Transfers γ_d and middle-age taxes θ_d^2 evidently enter (11) with different weights than in the primary surplus. The factors $\Phi^2 = 1 - \Lambda_i \cdot \frac{\sigma^2}{n} s_W < 1$ and $\Phi^3 = 1 + \Lambda_i \cdot \frac{1 - s_W}{R} > 1$ can be interpreted as measures of relative effectiveness.

From Prop.1, debt-stabilizing policy responses are needed whenever $R - n + \Lambda_i \geq 0$. The R.H.S of (11) reveals how the various responses combine. If only spending is reduced in response to high debt, the response $-\tilde{g}_d$ must exceed $R - n + \Lambda_i$. If only taxes on the young are increased, the response θ_d^1 must be such that the deficit-reduction $\sigma^1 \theta_d^1$ exceeds the same value $R - n + \Lambda_i$. In this sense, spending cuts and tax increases on the young taxes are equally effective. This applies for lump-sum taxes and for wage taxes. If retiree transfers are cut instead, it suffices if the resulting deficit-reduction $-\sigma^3 \gamma_d$ exceeds $(R - n + \Lambda_i)/\Phi^3$. Because $\Phi^3 > 1$ this is a smaller value, implying greater effectiveness. Conversely, $\Phi^2 < 1$ implies that period-2 tax increases are less effective at best (if $0 < \Phi^2 < 1$, the deficit-reduction $\sigma^2 \theta_d^2$ must exceed $(R - n + \Lambda_i)/\Phi^2 > (R - n + \Lambda_i)$, and possibly counterproductive (if $\Phi^2 < 0$).

In summary, one obtains a clear ranking of effectiveness:

Proposition 2:

a. Reduced transfers are most effective for stabilizing the public debt.

b. Spending and taxes on the young are equally effective for stabilizing the public debt, but less effective than reduced transfers.

c. Higher middle-age taxes are least effective for stabilizing the public debt, and they may be counterproductive.

Proof: Follows from $\Phi^3 > 1$ and $\Phi^2 < 1$ by comparing like terms in (8) and (11). QED.

The intuition for this relative ranking is that higher middle-age taxes reduce savings, which raises interest rates, and thus reinforces debt accumulation in a destabilizing direction. An expectation that high debt triggers reduced transfers, in contrast, provides savings incentives that reduce interest rates and dampen debt accumulation.

For completeness, one may consider general policy responses to the capital-labor ratio. Recall that the destabilizing interaction of debt and capital goes through the off-diagonal $\omega_{dk} = (R'(\kappa)d - \tilde{\pi}_\kappa)/n$ in (9). The interaction can be reduced by making $|R'(\kappa)d - \tilde{\pi}_\kappa|$ small. By setting $\tilde{\pi}_k = R'(\kappa)d$, debt accumulation can be decoupled from capital. This requires policy responses that fully offset all fluctuations in interest payment on the debt. Because $R'(\kappa) < 0$, attempts to reduce $|R'(\kappa)d - \tilde{\pi}_\kappa|$ require a *negative* response of the primary surplus to higher capital. The policy instruments are not equally effective in this context, despite the symmetry suggested by $\tilde{\pi}_\kappa$. This is because the tax and transfer responses θ_κ^2 and γ_κ have a separate impact on capital accumulation. If $\theta_\kappa^2 < 0$, as needed to reduce $|\omega_{dk}|$, $\omega_{\kappa\kappa}$ is increased; if $\gamma_\kappa > 0$, which also reduces $|\omega_{dk}|$, $\psi_{\kappa\kappa}$ is increased. Setting $\theta_\kappa^2 < 0$ thus leads to more persistent and potentially destabilizing fluctuations in capital, whereas $\gamma_k > 0$ implies less persistence. (Note that with income taxes, setting $\theta_\kappa^2 \leq 0$ requires tax rate responses to capital that more than offset the wage movements.) Endogenous transfers are again most effective for stabilization, whereas taxes-responses on middle-age savers are least effective.

Because direct responses of primary surplus to capital tend to obscure the interaction of debt and capital, I will omit such responses in the following. To avoid multiple cases, the extended model will treat taxes on young and middle cohorts as wage income taxes.

3. An Extended Model With Endogenous Labor Supply

This section adds endogenous labor supply to the basic model. The main objective is to show that the interaction of debt and capital is again destabilizing, and that the different policy instruments can be ranked. The analytical strategy is again to derive individual demand and supply functions, then to examine the special cases of linear production and zero debt, and finally to examine the interaction of endogenous factor prices and non-zero debt.

Let (l_t^1, l_{t+1}^2) be the labor supplies of the young and middle-aged, respectively, both limited by unit time. Labor supply per-capita is then

$$\lambda_t = \sigma^1 e \cdot l_t^1 + \sigma^2 \cdot l_t^2; \tag{12}$$

the capital-labor ratio is $\kappa_t = k_t / \lambda_t$. While capital is predetermined, labor supply is not—a major complication compared to the basic model.

With endogenous labor, and assuming taxes on the young and middle-aged are wage income taxes, tax revenues are $\theta_t^1 = e w_t l_t^1 \tau_t^1$ and $\theta_t^2 = w_t l_t^2 \tau_t^2$. An important implication is that at given tax rates, the primary surplus depends on endogenous wages and on endogenous labor supplies:

$$\pi_t = \sigma^1 \cdot e w_t l_t^1 \cdot \tau_t^1 + \sigma^2 \cdot w_t l_t^2 \cdot \tau_t^2 - \sigma^3 \cdot \gamma_t - \tilde{g}_t - \hat{g}_t. \tag{13}$$

Let preferences be

$$U_t = v_1(c_t^1, 1 - l_t^1)/\beta_0 + v_2(c_{t+1}^2, 1 - l_{t+1}^2) + \beta u(c_{t+2}^3).$$

The period utilities (v_1, v_2) are increasing and concave and allowed to differ, to accommodate age-dependent labor supply elasticities; β_0 is again assumed small enough that the young are liquidity constrained.

For the young, optimal labor supply satisfies the static Euler equation

$$e \cdot [w_t(1 - \tau_t^1)] \cdot v_{1,c}(c_t^1, 1 - l_t^1) = v_{1,1-l}(c_t^1, 1 - l_t^1),$$

where $c_t^1 = e \cdot w_t \cdot l_t^1 - \theta_t^1 = e \cdot w_t \cdot (1-\tau_t^1) \cdot l_t^1$. Let $l_t^1 = l^1[e \cdot w_t(1-\tau_t^1)]$ denote the resulting labor supply function and let $\varepsilon_w^1 = l_w^1 \cdot e \cdot w(1-\tau^1)/l^1$ denote the (uncompensated) labor supply elasticity.

For the middle-aged, optimal capital-accumulation and labor supply decisions are characterized by the Euler equations

$$(w(\kappa_t) + \hat{w}_t)(1-\tau_t^2) \cdot v_{2,c}(c_t^2, 1-l_t^2) = v_{2,1-l}(c_t^2, 1-l_t^2)$$

and $$v_{2,c}(c_t^2, 1-l_t^2) = \beta E_t[u'(c_{t+1}^3) R(\kappa_{t+1})]$$

The return to capital is stochastic because period-(t+1) labor supply is uncertain when period-t savings decisions are made.

Real bond returns (R^b) are generally linked to the return on capital by the arbitrage equation $E_t[u(c_{t+1}^3) R_{t+1}^b] = E_t[u(c_{t+1}^3) R(\kappa_{t+1})]$. Because this paper is neither about the equity premium nor about debt management, I simply assume that the bond return $R_{t+1}^b = R(\kappa_{t+1})$ equals the return to capital in all states of nature.[10] This maintains the key link between bond returns to returns to capital.

The middle cohort's problem is modeled as perturbation of deterministic work and savings decisions.[11] Under perfect foresight, the Euler equations and budget constraints would yield labor supply and savings as functions of three variables: the after-tax wage $w_t(1-\tau_t^2)$, the return to capital R_{t+1}, and retiree transfers γ_{t+1}. Let $l_t^2 = l^2[w_t(1-\tau_t^2), R_{t+1}, \gamma_{t+1}]$ denote the resulting labor supply function, let $(l_w^2, l_R^2, l_\gamma^2)$ denote the partial derivatives, and define the elasticities $\varepsilon_w^2 = l_w^2 \cdot w(1-\tau^2)/l^2$ and $\varepsilon_\gamma^2 = l_\gamma^2 \cdot \gamma / l^2$. Conditional on labor supply, savings can be written as

$$s_t = s\left(w_t(1-\tau_t^2) \cdot l_t^2 + \tfrac{\gamma_{t+1}}{R_{t+1}}, R_{t+1}\right) - \tfrac{\gamma_{t+1}}{R_{t+1}}.$$

Equilibrium on capital markets requires

$$\kappa_{t+1} \cdot \lambda_{t+1} + d_{t+1} = \sigma^3 \cdot [s\left((w(\kappa_{t+1}) + \hat{w}_t)(1-\tau_t^2) \cdot l_t^2 + \tfrac{\gamma_{t+1}}{R(\kappa_{t+1})}, R(\kappa_{t+1})\right) - \tfrac{\gamma_{t+1}}{R(\kappa_{t+1})}] \tag{14}$$

Steady state vectors (d, κ, λ) of debt, capital/labor, and labor are characterized by (14), (12), and (2), using (13) for π. The propagation of fluctuation around a steady state are similarly characterized by linearizing (14), (12), and (2), using (13) for π_t.

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[10] This could be implemented in practice with nominal bonds and appropriately state-contingent inflation. All linearized results remain valid if one instead assumed safe debt. Section 4 presents a version of the model that avoids decisions under uncertainty.

[11] The standard macro approach would be to cast the middle cohort's problem as choice under uncertainty in a Markovian state space. But this would obscure the Diamond-intuition about fiscal policy in OG models.

For modeling fluctuation around a steady state, individual behavior can be characterized by the parameter vector $(\varepsilon_w^1, \varepsilon_w^2, \varepsilon_\gamma^2, l_R^2, s_W, s_R)$ evaluated at the steady state. Intuitively, ε_w^1 and ε_w^2 characterize labor supply; ε_γ^2 captures the negative income effect of higher transfers; l_R^2 captures the income and incentive effects of higher interest rates on labor supply; and (s_W, s_R) again reflect savings behavior. Parameters $(\varepsilon_w^1, \varepsilon_w^2, l_R^2, s_R)$ may be positive or negative, depending on the interaction of income and substitution effects. Assuming consumption and leisure are normal goods, ε_γ^2 is bounded by $0 > \varepsilon_\gamma^2 > -\frac{\gamma/R}{w(1-\tau^2)l+\gamma/R} > -1$, and $0 < s_W < 1$.

To avoid a multitude of distracting case distinctions, my interpretation will focus on non-negative values of (l_R^2, s_R), on tax rates below the peaks of the "Laffer curves" (i.e., $\frac{\tau^1}{1-\tau^1}\varepsilon_w^1 < 1$ and $\frac{\tau^2}{1-\tau^2}\varepsilon_w^2 < 1$), and on parameter combinations such that capital accumulation in the no-debt economy is saddle-path stable and displays monotone convergence. The main analytical complication is the link between current labor supply and future interest rates, which are linked to future capital and future labor supply through the capital-labor ratio. While next period's capital is predetermined, labor supply is stochastic, and in turn affected by the following period's interest rate. Hence current middle-aged must form expectations and in effect solve for a rational expectations equilibrium.

Consider first the special case of linear production. Then debt is the only state variable. The dynamics are modified from the basic model, however, because debt stabilization through taxes and reduced transfers triggers labor supply responses. Linearizing (2) and (13), one finds

$$\mu_d = \frac{R - \sigma^1 ew(1-\frac{\tau^1}{1-\tau^1}\varepsilon_{1,w})\cdot\tau_d^1 - \sigma^2 w(1-\frac{\tau^2}{1-\tau^2}\varepsilon_{2,w})\cdot\tau_d^2 + \sigma^3\cdot\gamma_d + \tilde{g}_d}{n + \sigma^2\theta^2/\gamma\cdot\varepsilon_\gamma^2\cdot\gamma_d} \tag{15}$$

With zero policy responses, this reduces to $\mu_d = R/n$, so self-stabilization again holds if and only if the steady state is dynamically inefficient. If not, $\mu_d < 1$ requires

$$(1-\tfrac{\tau^1}{1-\tau^1}\varepsilon_{1,w})\cdot\sigma^1 ew\tau_d^1 + (1-\tfrac{\tau^2}{1-\tau^2}\varepsilon_{2,w})\cdot\sigma^2 w\tau_d^2 - \sigma^3\cdot\gamma_d\cdot(1-n\cdot\theta^2/\gamma\cdot\varepsilon_\gamma^2) - \tilde{g}_d > R-n$$

which implies relative effectiveness measures

$$\Phi^1 = 1-\tfrac{\tau^1}{1-\tau^1}\varepsilon_{1,w},\ \Phi^2 = 1-\tfrac{\tau^2}{1-\tau^2}\varepsilon_{2,w},\ \text{and}\ \Phi^3 = 1-\tfrac{n\cdot\theta^2}{\gamma}\cdot\varepsilon_\gamma^2 > 1.$$

Positive labor supply elasticities make tax responses less effective for debt stabilization than spending cuts. Reduction in transfers are always more effective than spending cuts because the expectation of reduced transfers increases labor supply and hence tax revenues.

Consider next the special case of no debt. To be specific, assume fixed tax rates and fixed transfers, assume spending responds to shocks to maintain budget balance. Linearizing (14) and (13), respectively, one finds

$$\begin{pmatrix} \psi_0 + \sigma^3 s_W \cdot w(1-\tau^2) l_R^2 \cdot [-R'(\kappa)] & \kappa \\ \sigma^2 l_R^2 \cdot R'(\kappa) & 0 \end{pmatrix} \begin{pmatrix} \kappa_{t+1} - \kappa \\ \lambda_{t+1} - \lambda \end{pmatrix} = \begin{pmatrix} \omega_0 (1-\tau^2)(1+\varepsilon_l^2) & 0 \\ -\lambda_w \cdot w'(\kappa) & 1 \end{pmatrix} \begin{pmatrix} \kappa_t - \kappa \\ \lambda_t - \lambda \end{pmatrix} + \Sigma_0 \cdot \hat{w}_t$$

where $\lambda_w = \sigma^1 \frac{ew(1-\tau^1)}{l^1} \cdot \varepsilon_w^1 + \sigma^2 \frac{w(1-\tau^2)}{l^1} \cdot \varepsilon_w^2$ is a combination of labor supply elasticities. Σ is uninteresting for examining propagation—a generator of disturbances—and hence left unspecified. Let $\Psi_0 = \begin{pmatrix} \psi_0 + \sigma^3 s_W \cdot w(1-\tau^2) l_R^2 \cdot [-R'(\kappa)] & \kappa \\ \sigma^2 l_R^2 \cdot R'(\kappa) & 0 \end{pmatrix}$ and $\Omega_0 = \begin{pmatrix} \omega_0 (1-\tau^2)(1+\varepsilon_l^2) & 0 \\ -\lambda_w \cdot w'(\kappa) & 1 \end{pmatrix}$ be the matrices characterizing the propagation of disturbances to $x_{0,t} = (\kappa_t - \kappa, \lambda_t - \lambda)'$, and let

$$P_0(\mu) = |\Omega_0 - \mu \Psi_0| = \mu^2 \cdot \kappa \cdot \sigma^2 l_R^2 \cdot (-R') \\ -\mu[\psi_0 + \sigma^3 s_W \cdot w(1-\tau^2) l_R^2 \cdot [-R'(\kappa)] + \kappa \cdot \lambda_w \cdot w'(\kappa)] + \omega_0 \cdot (1-\tau^2)(1+\varepsilon_l^2)$$

denote the characteristic polynomial. (Subscript-0 is used to denote the private sector dynamics.) Saddle-path stability and monotone convergence require that $P_0(\mu)$ has exactly one root inside the unit interval. Note that $P_0(0) = \omega_0 \cdot (1-\tau^2)(1+\varepsilon_l^2) > 0$ and

$$P_0(1) = (-R') \cdot \{\kappa \cdot n - s_W \cdot w(1-\tau^2)\} \sigma^3 \cdot l_R^2 + \omega_0 \cdot (1-\tau^2)(1+\varepsilon_l^2) - [\psi_0 + \kappa \cdot \lambda_w \cdot w'(\kappa)].$$

In the special case $l_R^2 = 0$, labor supply is static and $P_0(\mu)$ has a single positive root $\mu_{00} = \frac{\omega_0 \cdot (1-\tau^2)(1+\varepsilon_l^2)}{\psi_0 + \kappa \cdot \lambda_w \cdot w'(\kappa)}$. Stability requires $\mu_{00} < 1$, which is implied by Assumption 1 for sufficiently small ε_l^2 and/or sufficiently high tax rate τ^2. If $l_R^2 > 0$, then $\kappa \cdot \sigma^2 l_R^2 \cdot (-R') > 0$, so $P_0(\mu)$ has one root in $(0,1)$ and a root in $(1,\infty)$ if and only if $P_0(1) < 0$. Because $\omega_0 \cdot (1-\tau^2)(1+\varepsilon_l^2) - [\psi_0 + \kappa \cdot \lambda_w \cdot w'(\kappa)] < 0$ for $\mu_{00} < 1$, this is satisfied for sufficiently small l_R^2. If $l_R^2 < 0$, $P_0(\mu)$ has one root in (0,1) and a negative root outside [-1,1] if and only if $P_0(1) < 0$ and $P_0(-1) > 0$; this is satisfied if $|l_R^2|$ is small.

In summary, the dynamics of capital and labor are saddle-path stable under modest restrictions on middle-age labor supply, notably small ε_l^2 and small $|l_R^2|$. Because labor supply of prime-age worker is empirically inelastic, these are reasonable assumptions. For reference below, let the roots of $P_0(\mu)$ be denoted $\mu_{01} \in (0,1)$ and μ_{02}, where $\mu_{02} \in (1,\infty)$ for $l_R^2 > 0$ and $\mu_{02} \in (-\infty,-1)$ for $l_R^2 < 0$.

Turning to the general analysis with endogenous factor prices and positive debt, the linearized dynamics can be written as $\Gamma \cdot x_{t+1} = \Omega \cdot x_t + \Sigma \cdot (\hat{w}_t, \hat{g}_t)$, where Ψ and Ω are 3x3 matrices, Σ is a 3x2 matrix, and $x_t = (d_t - d, \kappa_t - \kappa, \lambda_t - \lambda)'$ is the vector of endogenous variables. Let the linearizations of (2), (14), (12) be in rows 1-3, in this order, and let Ψ_{ij} and Ω_{ij} denote elements of Ψ and Ω, where i,j=1,2,3 indicates row and column positions. From (2), one obtains $\Psi_{11} = 1$ and $\Omega_{11} = \mu_d$, as in the exogenous factor price case. From (14) and (12), 2x2 block in rows/columns #2-3 is identical to Ψ_0 and Ω_0. This correspondence makes the general dynamics comparable to the special cases.

The remaining matrix elements are: $\Psi_{12} = \sigma^2 w \tau^2 l_R^2 \cdot R'(\kappa)$, which is negative iff $l_R^2 > 0$; $\Omega_{12} = R'(\kappa) \cdot d - \pi_w \cdot w'(\kappa) < 0$, where $\pi_w = \sigma^1 e \tau^1 (1 + \varepsilon_{1,w}) + \sigma^2 \tau^2 (1 + \varepsilon_{2,w}) > 0$ captures the revenue effects of higher wages; $\Psi_{21} = 1 + \sigma^3 (\frac{1-s_W}{R} - s_W w^2 (1 - \tau^2) l_\gamma^2) \gamma_d$ and $\Omega_{21} = -\sigma^3 s_W w^2 l^2 (1 + \varepsilon_l^2) \tau_d^2$, which capture policy responses; $\Psi_{13} = \Omega_{13} = 0$; $\Psi_{31} = \sigma^2 l_\gamma^2 \gamma_d$; and $\Omega_{31} = \sigma^1 l^1 \varepsilon_l^1 \cdot \tau_d^1 + \sigma^2 l^2 \varepsilon_l^2 \cdot \tau_d^2$, which also captures policy responses. One may interpret Ω_{12}, Ψ_{21} and Ω_{21} as generalizations of $\omega_{d\kappa}$, $\psi_{\kappa d}$, and $\omega_{\kappa d}$ from the basic model.

Let $P(\mu) = |\Omega - \mu\Psi|$ denote the characteristic polynomial of the 3x3 system, with roots denoted μ_1, μ_2, μ_3. It can be written as $P(\mu) = (\mu_d - \mu) \cdot P_0(\mu) + P^*(\mu)$, where

$$P^*(\mu) = -(\Omega_{12} - \mu\Psi_{12}) \cdot \begin{vmatrix} \Omega_{21} - \mu\Psi_{21} & \Omega_{23} - \mu\Psi_{23} \\ \Omega_{31} - \mu\Psi_{31} & \Omega_{33} - \mu\Psi_{33} \end{vmatrix},$$

captures interaction effects between debt and capital-labor dynamics. (This exploits $\Psi_{13} = \Omega_{13} = 0$.) From the zero-debt case, $\Omega_{23} - \mu\Psi_{23} = -\mu \cdot \kappa$ and $\Omega_{33} - \mu\Psi_{33} = 1$. Hence

$$P^*(\mu) = -(\Omega_{12} - \mu\Psi_{12}) \cdot \{(\Omega_{21} - \mu\Psi_{21}) + \mu \cdot \kappa \cdot (\Omega_{31} - \mu\Psi_{31})\}$$

Note that $(\mu_d - \mu) \cdot P_0(\mu)$ is a cubic polynomial with lead term $-\kappa \cdot \sigma^2 l_R^2 \cdot (-R')$, intercept $\mu_d \cdot P_0(0) > 0$ and roots μ_d, $\mu_{01} \in (0,1)$ and μ_{02}. To avoid distracting case distinctions, suppose $l_R^2 > 0$ and $\mu_{01} < \mu_d$. Then the cubic term is negative and the sign of $(\mu_d - \mu) \cdot P_0(\mu)$ oscillates between roots, starting positive and ending negative.

Without policy responses, $P^*(\mu)$ simplifies further to $P^*(\mu) = \mu \cdot (\Omega_{12} - \mu\Psi_{12})$, a quadratic polynomial with zero intercept. Hence $P(0) = \mu_d \cdot P_0(0) > 0$ is unaffected by P^*. Recall that $\Omega_{12} = R'(\kappa) \cdot d - \pi_w \cdot w'(\kappa) < 0$ captures the main interaction effect from the basic model, whereas Ψ_{12} is (negatively) proportional to l_R^2 and captures the complications due to endogenous middle-age labor supply. If $l_R^2 > 0$, $P^*(\mu) < 0$ applies for $\mu \in (0, \Omega_{12}/\Psi_{12})$; assuming $\Psi_{12} < |\Omega_{12}|$, this includes $\mu \in (0,1]$. If $l_R^2 < 0$, $P^*(\mu) < 0$ applies for all $\mu > 0$. In both cases, $P(\mu)$ has as many roots in the unit interval as $(\mu_d - \mu) \cdot P_0(\mu)$. Moreover, $\mu_2 > \min(\mu_d, 1)$, $0 < \mu_1 < \mu_{01}$, and $\mu_3 > 1$. Stability depends on the second root, which is increased. As in the basic model, the interaction of debt and capital is thus destabilizing. Dynamic inefficiency does not ensure self-stabilization.

With policy responses, assume again $\Omega_{12} - \mu\Psi_{12} < 0$ on $\mu \in (0,1]$. Then the sign of $P^*(\mu)$ on $\mu \in (0,1]$ is determined by $\{(\Omega_{21} - \mu\Psi_{21}) + \mu \cdot \kappa \cdot (\Omega_{31} - \mu\Psi_{31})\}$. Apart from the crowding out effect captured above (the unit term in Ψ_{21}), this expression reflects multiple policy responses. One can show (with more tedious algebra), that $dP^*/d\tau_d^1 > 0$ iff $\varepsilon_l^1 > 0$; $dP^*/d\gamma_d < 0$; and $dP^*/d\tau_d^2$ has ambiguous sign—negative if $|\varepsilon_l^2|$ is small, but potentially positive if ε_l^2 is large and positive. Hence tax increases on the young and cuts in transfers are stabilizing (in the sense of reducing μ_2) whereas tax increases on the middle cohort are (for small $|\varepsilon_l^2|$) destabilizing, all relative to their impact through the surplus. The intuition for transfers and for middle-age taxes increases is analogous to the basic model.

A new result is the stabilizing interaction effect of tax responses when their labor supply is elastic—always for the young, at high ε_l^2 also for the middle-aged. The intuition is that ceteris paribus, a decline in labor supply raises the capital-labor ratio, hence reduces interest rates and dampens debt accumulation. This intuition is clearly counter to the usual of elasticity intuition of public-finance which, to avoid tax distortions, discourages taxes on elastic supplies.

Overall, the extended model shows that the basic results about self-stabilization and relative effectiveness generalize, and it suggests the counterintuitive result that taxing elastic labor supply may be stabilizing.

The full model is unfortunately complicated, and readers may be suspicious about the proliferation of auxiliary assumptions and distracted by the tedious algebra. The complications are almost entirely due to the endogenous labor supply of the middle-aged. This is modified in the next section.

4. Inspecting the Mechanism

This section examines an intermediate version with variable young-age labor supply but inelastic middle-age supply. This specification helps to refine the intuition for the general results. It is also empirically plausible because the labor supply of the young is substantially more elastic than the labor supply of "prime-age" workers.

For this section, let preferences be

$$U_t = v_1(c_t^1, 1-l_t^1)/\beta_0 + u(c_{t+1}^2) + \beta u(c_{t+2}^3).$$

where $v(\cdot)$ is increasing and concave. The young cohort's labor supply l_t^1 is endogenous; $l_t^2 = 1$ is exogenous. The parameter β_0 is again assumed small enough that the young are liquidity constrained.

This setting is particularly insightful if one makes judicious assumption about timing. Assume the labor market clears before the period's stochastic shocks are revealed, whereas savings decisions are made afterwards. Then period-t savers can perfectly anticipate the next period's labor supply, and hence the capital-labor ratio and the return on savings.

The young cohort maximizes expected utility, taking the tax rate $\tau_t^1 = \tau^1(k_t, d_t)$ and the state-contingent profile of wages $w(k_t/\lambda_t) + \hat{w}_t$ as given. Optimal labor supply is characterized by the first order condition

$$E_{\hat{w}_t}\left[e \cdot (w(k_t/\lambda_t) + \hat{w}_t)(1-\tau_t^1) \cdot v_c(c_t^1(l_t^1), 1-l_t^1) - v_{1-l}(c_t^1(l_t^1), 1-l_t^1)\right] = 0, \qquad (16)$$

evaluated at the equilibrium per-capital labor supply $\lambda_t = \sigma^1 e \cdot l_t^1 + \sigma^2$. Equation (16) implicitly defines a labor supply function $l_t^1 = l^*(k_t, 1-\tau_t^1)$.

For comparison to Section 2, tax policy is most conveniently parameterized by revenues. Revenues θ_t^1 are increasing in the tax rate, assuming again a tax rate below the peak of the Laffer curve. With some tedious algebra, one can express the capital-labor ratio as function of a revenue target θ_t^1 and of per-capita capital, $\kappa_t = \kappa(k_t, \theta_t^1)$. One can show that $\partial\kappa_t / \partial\theta_t^1 = \kappa_\theta$ is positive (negative) if and only if young agents' labor supply elasticity is positive (negative), i.e., if the incentive effects are greater (smaller) than the income effects.

Government debt and capital market equilibrium are still described by conditions (2) and (4), except that λ_t is variable. For this section, capital is a more convenient state variable rather than the capital-labor ratio. The linearized dynamics around a steady state have the same Markov structure as in Section 2, but with modified coefficients. Consider $w_t = w(\kappa(k_t, \theta_t^1))$ and $R_{t+1} = R(\kappa(k_{t+1}, \theta_{t+1}^1))$ functions of capital and revenues, redefine

$$\omega_0 = \sigma^3 s_W \cdot w'(\kappa)\kappa_k \text{ and } \psi_0 = 1 + \sigma^3 (s_R + (1 - s_W)\tfrac{\gamma}{R^2})(-R'(\kappa)\kappa_k,$$

and assume $\psi_0 > \omega_0 > 0$. Then the dynamics of (d,k) are qualitatively identical to the dynamics of (d,κ) in (9), but with modified coefficients. Specifically, $x_t = (d_t - d, k_t - k)$ satisfies $\Psi \cdot x_{t+1} = \Omega \cdot x_t + \Sigma \cdot (\hat{g}_t, \hat{w}_t)'$ with

$$\omega_{dd} = (R - \tilde{\pi}_d - R'(\kappa)d \cdot \kappa_\theta \theta_d^1)/n,\ \omega_{dk} = [R'(\kappa)d \cdot (\kappa_k + \kappa_\theta \theta_k^1) - \tilde{\pi}_k]/n$$

$$\psi_{kk} = \psi_0 + \tfrac{1-s_W}{R}\sigma^2 \gamma_k + \sigma^2 (s_R + (1 - s_W)\tfrac{\gamma}{R^2})(-R')\kappa_\theta \theta_k^1$$

$$\psi_{kd} = 1 + \tfrac{1-s_W}{R}\sigma^3 \gamma_d - \sigma^3 (s_R + (1 - s_W)\tfrac{\gamma}{R^2})(-R')\kappa_\theta \theta_d^1$$

$$\omega_{kd} = s_W \sigma^3 (-\theta_d^2 + w'(\kappa)\kappa_\theta \theta_d^1)$$

and $$\omega_{kk} = \omega_0 + s_W \sigma^3 (-\theta_k^2 + w'(\kappa)\kappa_\theta \theta_k^1).$$

The main new elements are the terms involving (θ_d^1, θ_k^1), which describe the response of taxes on the young to debt and capital.

The conditions for self-stabilizing debt remain substantively unchanged. With policy responses to debt but not to capital, the stability condition (10) implies

$$\Phi^1 \cdot \sigma^1 \cdot \theta_d^1 + \Phi^2 \cdot \sigma^2 \cdot \theta_d^2 - \Phi^3 \cdot \sigma^3 \cdot \gamma_d - \tilde{g}_d > R - n + \tfrac{(-R')d \cdot \kappa_k}{\psi_0 - \omega_0} \tag{14}$$

where $\Phi^1 = 1 + \frac{1}{\sigma^1}\frac{(-R')d \cdot \kappa_k}{\psi_0 - \omega_0}\kappa_\theta$, $\Phi^2 = 1 - \frac{(-R')d \cdot \kappa_k}{\psi_0 - \omega_0} s_W < 1$, and $\Phi^3 = 1 + \frac{(-R')d \cdot \kappa_k}{\psi_0 - \omega_0}\frac{1 - s_W}{R} > 1$.

The key difference to the basic model is that for $\kappa_\theta \neq 0$, θ_d^1 enters with a weighting factor: $\Phi^1 > 1$ if labor supply is elastic so $\kappa_\theta > 0$, whereas $\Phi^1 < 1$ if the young have a "backward bending" labor supply curve.

Inspecting the weights, one finds:

Proposition 3: *With a variable labor supply of young agents:*

a. The effectiveness of taxes on the young for stabilizing debt is increasing in the labor supply elasticity.

b. If young-age labor supply elasticity has positive elasticity (negative), taxes on the young are more (less) effective than spending cuts.

c. If the labor supply elasticity is high enough that $\kappa_\theta > \sigma^1(1-s_W)/R\cdot\kappa_k$*, then taxes on the young are more effective that cuts in retiree transfers (*$\Phi^1 > \Phi^3 > 1$*).*

d. If the labor supply elasticity is low enough that $\kappa_\theta < -\sigma^2 s_W \kappa_k$*, then taxes on the young are less effective that taxes on the middle aged (*$\Phi^1 < \Phi^2 < 1$*).*

The intuition is that a reduced labor supply raises the capital-labor ratio, reduces the interest rate, and therefore reduces debt accumulation. If higher taxes are needed because of high debt, an elastically declining labor supply provides additional stabilization through lower interest rates, whereas a negative elasticity would trigger a destabilizing interest rate increase.

Standard theories of second-best taxation suggest that taxing a commodity with elastic supply has a higher welfare cost (excess burden). From a welfare perspective, taxing the inelastic labor supply of the middle-aged should be superior. The effectiveness ranking here is thus strikingly at variance with standard efficiency arguments.

5. Conclusions

This paper examines the choice between alternative debt-deficit responses in an overlapping generations (OG) setting. The OG model is a convenient framework for making interest rates sensitive to the supply of public debt and for distinguishing between different types of taxes and transfers. A given deficit-reduction yields a greater reduction of future debt if the policy instrument also reduces the interest rate, and a lesser reduction if policy instrument raises the interest rate. Reduced outlays for retirees yield the relatively greatest debt reductions, followed by tax increases on non-savers and by cuts in public spending. Tax increases on bond-buying cohorts rank last.

These results on the relative effectiveness of taxes increases, reduced transfers, and spending cuts are a challenge for social policy. In practice, the middle-aged are earning higher incomes than retirees and young workers. The middle-aged are also less likely to be liquidity constrained than young worker, and—because they can vary their labor supply—more able to respond to shocks than retirees. From a distributional and social perspective, a fiscal stabilization through taxes on the middle-aged is therefore preferable to reduced retirees transfers and to higher taxes on the young. The relative effectiveness ranking goes in the opposite direction. The relative effectiveness results are consistent with, and they may provide a political economy explanation for, the observation that fiscal stabilizations often impose considerable hardship on vulnerable population groups.

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