ENDOGENOUS GOVERNMENT SPENDING AND RICARDIAN EQUIVALENCE

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9-90
(revision of 13-88)

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First Draft: April 1988
Revised: March 1990

Abstract

Many analyses of debt policy assume exogenous government expenditures. Instead, I use an optimizing model in which the government endogenously selects values of taxes, spending, and debt to maximize welfare. If demand for publicly provided goods is elastic, a debt-financed tax cut increases consumption, because individuals rationally expect some reduced government spending in the future. Even though future taxes rise, they do not offset the expansionary effect of the current tax cut on consumption. Depending on preferences, the marginal propensity to consume out of tax cuts can take any value between zero and the marginal propensity out of ordinary income.

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1. Introduction

The main implication of the Ricardian approach to budget deficits (Barro (1974) and (1988))[1] is the prediction that consumers do not spend more when they receive a deficit-financed tax cut. The argument is that rational forward-looking individuals save the additional disposable income, anticipating that the increased government debt is financed by future tax increases. Holding government spending fixed, the current tax cut is exactly offset in present value terms by higher taxes in the future. Permanent disposable income and therefore consumption remain unchanged.

Two steps are critical in this argument:

(a) Consumer behavior depends only on the present value of taxes, or equivalently, on the present value of government spending plus initial debt.

(b) Current and future government spending is held constant, i.e., exogenous.

The present value argument, (a), has been debated extensively,[2] but there has been little discussion about the assumptions on government behavior implicit in condition (b). But tax cuts should affect consumption if they signal changes in future government spending, as Feldstein (1982) pointed out. Predictions about government behavior are therefore crucial, if one wants to apply the Ricardian approach as "the benchmark model for assessing fiscal policy" (Barro (1989)).[3]

A problem in deriving such predictions is the lack of a consensus model of government behavior. The key questions are: Is the government following arbitrary rules or is it optimizing?[4] If optimizing, is it maximizing welfare or something else?[5] Ricardian equivalence, (a), combined with an arbitrary rule, namely some rule of exogenous spending, (b), obviously yields the striking implication that consumers should not care about budget deficits. In

contrast, this paper shows that optimizing government behavior inevitably links taxes to future government spending, violating condition (b). Ricardian equivalence cannot be applied. The issue here is not whether Ricardian equivalence characterizes private behavior, but whether there will ever be a Ricardian experiment of a tax cut combined with expected tax increases of equal present value.

To generate alternative predictions about government policy and its implications for consumption, the paper develops a model with welfare-maximizing government and forward looking, "Ricardian" consumers. Welfare-maximization is assumed because Ricardian theorists do not seem to object to this assumption (see Barro (1979)). But since the results are based on a simple cost-benefit tradeoff (see below), they should not be sensitive to changes in the objective function, as long as government behavior is systematic.[6] The implications for co-movements of consumption and deficits differ dramatically from those of the Ricardian approach: The marginal propensity to consume income obtained from a tax cut can take any value between zero and the marginal propensity to consume out of ordinary income.

The basic argument why consumers react to tax cuts is as follows. A government that cares about welfare trades off marginal increases in spending on publicly provided goods against marginal reductions in private consumption. Assuming distortionary taxes (which would not cause a "first-order" violation of Ricardian equivalence under exogenous spending), higher taxes increase the marginal cost of publicly provided goods. If consumers observe a tax-cut in this environment, they know - following the Ricardian logic - that either future taxes will have to be raised or future government spending be cut. But because of the marginal cost argument, they rationally anticipate that higher taxes will generally be accompanied by spending reduc-

tions. Thus, the Ricardian argument that lower taxes today are exactly offset by higher future taxes does not hold. Consumption increases.

The implications are twofold. On the one hand, one should be cautious in simply assuming exogenous spending in deriving policy implications from Ricardian equivalence. The results are sensitive to this assumption and exogenous spending is not the result of optimizing behavior.[7] On the other hand, a positive correlation between consumption and deficits provides no evidence against forward looking consumer behavior.

The paper is organized as follows. Section 2 describes the model. In Section 3, the effect of fiscal policy on consumption is analyzed. Section 4 demonstrates the link between deficits and subsequent spending policy decisions. The conclusions are reviewed in Section 5.

2. The Model

Government activity and its implications for rational private behavior will be analyzed in a simple infinite horizon, representative agent economy. To shift the focus away from questions of individual behavior, all private-sector complications that would lead to non-Ricardian results in a model with exogenous government spending are omitted.[8] That is, if government spending were exogenous, the model would replicate the results of Barro (1979).

The economy consists of identical, infinitely lived individuals and a government. Each individual has preferences over two goods, c_t and g_t. In period t, individuals maximize the utility function

$$E_t \sum_{i=0}^{\infty} \delta^i \cdot \left[u(c_{t+i}) + v(g_{t+i}, \gamma_{t+i}) \right] , \tag{1}$$

where $u(\cdot)$ and $v(\cdot)$ are concave in c and g, respectively, γ_t is a random shock to preferences, and $0 < \delta < 1$ is the rate of time preference. Individuals buy

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the good c_t privately, but good g_t is provided by the government in equal quantity to all individuals.[9] The shock γ_t is intended to reflect changing preferences for the publicly provided good; therefore, assume $v_{g\gamma} \neq 0$.

Individuals take government decisions about taxation and spending as given. They have initial wealth (debt and capital) and earn after-tax income y_t (defined below) to pay for the private good and to invest in capital, K_t, and government bonds, D_t.

Each unit of capital K_t yields $R \equiv 1 + r > 1$ units of goods in period $t+1$. To simplify, assume that the interest rate r is constant, that the rate of time discount equals the interest rate, i.e., $\delta R = 1$, and that the government can commit itself not to tax capital (to eliminate time consistency issues). Then government bonds and capital are perfect substitutes. The individual budget constraint in period t is

$$c_t + K_t + D_t = R \cdot (K_{t-1} + D_{t-1}) + y_t \qquad (2)$$

The government uses taxation and debt financing to pay for the publicly provided good g_t. It chooses spending and taxes to maximize welfare, i.e., the same utility function (1).[10] Without loss of generality, units are defined so that c_t and g_t have identical production cost per capita and that prices are normalized to one. Given initial government debt, the budget constraint in period t is

$$g_t + R \cdot D_{t-1} = T_t + D_t , \qquad (3)$$

where T_t denotes tax revenues.

Since lump-sum taxes are not widely used in practice, taxes are allowed to have distortionary effects. (This feature in itself does not cause real effects of tax policy; see Corollary 2 below.) Denote the exogenous,

stochastic income that would be realized if there were no taxes by Y_t. If the government raises revenue T_t, it imposes a burden on individuals that exceeds T_t by $h(T_t, Y_t, \epsilon_t)$, so that disposable income is

$$y_t = y(T_t, Y_t, \epsilon_t) = Y_t - T_t - h(T_t, Y_t, \epsilon_t) \ . \tag{4}$$

It is assumed that the excess burden $h(\cdot)$ is an increasing convex function of tax revenue, and that there is a stochastic shock ϵ_t, which increases the marginal cost of raising revenue, i.e., $h_T > 0$, $h_{TT} > 0$, and $h_{T\epsilon} \neq 0$. These assumptions on excess burden closely follow Barro (1979). I show in the appendix that they may be viewed as a reduced form solution of the individuals' problem in a model with taxes on labor income.[11]

An advantage of summarizing excess burden in this form is that the function $h(\cdot)$ captures all aspects of self-interested private behavior that may reduce social welfare, such as efforts to minimize individual tax payments. Therefore, the equilibrium allocation can be obtained by solving a social planner's problem in which the government is allowed to choose consumption in addition to its policy variables, but takes the loss function $h(T_t, Y_t, \epsilon_t)$ as given.

Unfortunately, a lengthy and complicated stochastic dynamic programming approach is required to characterize the outcomes in general. But the stochastic shocks, γ_t, Y_t, and ϵ_t, are only needed to generate movements in macroeconomic variables that can be interpreted as unexpected policy changes. The intuition comes out most clearly if one concentrates on policy changes caused by stochastic shocks in one particular period, say $t = 0$, and abstracts from later stochastic disturbances. Therefore, the main text will state and discuss all results under the simplifying assumptions

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$$\gamma_t = 0, \quad Y_t = \bar{Y}, \text{ and } \varepsilon_t = 0 \text{ for all } t \geq 1 , \tag{5}$$

where $\bar{Y}$ is a constant. The general results, which are qualitatively identical, are stated and proven in the appendix. The exposition here will concentrate on the simpler case where (5) is assumed. To simplify notation, also assume that initial values of debt and capital are $D_{-1} = K_{-1} = 0$. Shocks in period $t = 0$ generate alternative policies that will be compared. From then on, the game evolves deterministically and can be solved for any combination of period-0 shocks.

The social planner's problem in period zero is then to maximize utility (1) subject to constraints (2)-(4). The resulting first order conditions are

$$u_c(c_0) = u_c(c_t) \tag{6a}$$

$$v_g(g_0, \gamma_0) = v_g(g_t) \tag{6b}$$

$$v_g(g_0, \gamma_0) - (1 + h_T(T_0, Y_0, \varepsilon_0)) \cdot u_c(c_0) = 0 \tag{6c}$$

$$v_g(g_t) - (1 + h_T(T_t)) \cdot u_c(c_t) = 0 \tag{6d}$$

for all periods $t \geq 1$, where constant arguments (due to (5)) are omitted. Since equation (6a) is the first order condition for individuals maximizing (1) subject to (2), the social planning problem solves the game between government and individuals. Equations (6a, b) determine the intertemporal allocation of consumption and government spending. Equations (6c, d) characterized the tradeoff between private consumption and publicly provided goods. Since equation (6a) implies constant consumption for all periods and equations (6b, d) imply constant government spending and taxes for all periods $t \geq 1$, define $c \equiv c_t = c_0$, $g \equiv g_t$, and $T \equiv T_t$ for all $t \geq 1$. Then the budget constraints (2) and (3) simplify to

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$$(1 + r) \cdot c = r \cdot [Y_0 - T_0 - h(T_0, Y_0, \epsilon_0)] + [\bar{Y} - T - h(T)], \qquad (7a)$$

$$r \cdot g_0 + g = r \cdot T_0 + T, \qquad (7b)$$

respectively, which provides an explicit solution for c. Links between government policy and consumption in this setting are analyzed in the next section.

3. Fiscal Policy and Consumption

Equations (6) and (7) determine a unique solution for (c, g_0, T_0, g, T) for each possible set of realizations of $(Y_0, \epsilon_0, \gamma_0)$. An econometrician may draw different realizations of shocks $(Y_0, \epsilon_0, \gamma_0)$ to study the join movements of consumption and government policy.[12] Since the expected values of $(Y_0, \epsilon_0, \gamma_0)$ are $(\bar{Y}, 0, 0)$, deviations of government spending and taxes from their values under $(\bar{Y}, 0, 0)$ can be interpreted as unexpected changes in policy. In this way, the shocks provide a justification to do essentially comparative static analysis in a model with optimal policy.

In order to allow variations in the three key variables c, g_0, and T_0, at least three sources of variability are necessary. Here, each of the three shocks $(Y_0, \epsilon_0, \gamma_0)$ affects all endogenous variables, but not equally strongly. Since the main effect of ϵ_0 is on T_0, observed variations in T_0 can be interpreted as largely caused by changes in ϵ_0. Similarly, changes Y_0 move c and changes in γ_0 strongly affect g_0. The main issue is whether - and if yes, how much - consumers react to debt-financed tax cuts, i.e., how consumption varies with a set of shocks that reduce taxes T_0 and just leave government spending g_0 unchanged. Such a casual interpretation of co-movements between consumption and policy variables is possible, because changes in consumption can be decomposed into changes directly due to shocks and changes due to tax policy, as follows.

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Equation (7a) shows that consumption depends only on the present value of disposable income (recall the definition (4)), where period-0 disposable income is a function of shocks and taxes and future disposable income depends only on future taxes, T. If one defines "before-tax income" $y^* = Y_0 - h(\bar{T}, Y_0, \varepsilon_0)$, for some initial value $\bar{T}$, marginal changes in disposable income, Y_0, can be decomposed into changes caused by taxes and changes due to shocks that enter through y^*:

$$dy_0 = (1 + h_T) \cdot dT_0 + dy^* . \tag{8}$$

Thus, if consumption varies with taxes T_0 after controlling for changes in y^*, one may conclude that policy affects consumption. If the effect remains even with unchanged current government spending, changes in current taxes are not offset in present value terms by future changes in taxes, i.e., they must signal future changes in government spending.

The magnitude of policy-induced consumption changes can be evaluated in comparison to a change in disposable income that is not accompanied by policy changes. Therefore, define the (ordinary) marginal propensity to consume (MPC) as

$$MPC_Y = \left. \frac{dc_0}{dy_0} \right|\; dg_0 = 0,\; dT_0 = 0 ,$$

and define the marginal propensity to consume income obtained from a tax cut as

$$MPC_T = \left. \frac{dc_0}{dy_0} \right|\; dg_0 = 0,\; dy^* = 0 .$$

Both MPC's have a regression interpretation. MPC_T and MPC_Y would be the coefficients on taxes and before-tax income, y^*, respectively, in a regression of consumption on these two variables, a constant, and government spending. Very similar regressions have been run by Feldstein (1982) and others (see the

survey by Bernheim (1987)). The following result relates the two MPC's to the fundamental parameters of the models:

Theorem 1: Define $\Delta = (1 + r) \cdot (h_{TT} \cdot u_c + (-v_{gg})) + (1 + h_T)^2 \cdot (-u_{cc}) > 0$. Then,

$$MPC_Y = r/\Delta \cdot (h_{TT} \cdot u_c + (-v_{gg}))$$

$$MPC_T = r/\Delta \cdot (h_{TT} \cdot u_c)$$

Proof: Marginal changes in c_0, T_0, y^* are obtained by taking the total differential of (6), (7), and (8). The MPC's are ratios of pairs of these changes. See also the more general proof in the appendix.

Corollary 1 (General Case): If h_{TT}, $(-v_{gg})$, and $(-u_{cc})$ are strictly positive and finite, $0 < MPC_T < MPC_Y < r/(1 + r)$.

Corollary 2 (A "Ricardian" Result): As $(-v_{gg}(g)) \to \infty$ at some value of g, consumption does not vary with taxes: $MPC_T \to 0$.

Corollary 3: As $(-v_{gg}(g)) \to 0$, consumers do not distinguish receipts from tax-cuts from other income: $MPC_T \to MPC_Y$.

I show in the appendix that all qualitative results, but not the exact formulas in Theorem 1, generalize to environments without assumption (5) and without separability between goods c and g. Thus, the reaction of consumption to tax cuts is, in general, positive but less than the reaction to other changes in disposable income. The key equation for understanding the results is (6d), which requires that the marginal rate of substitution between privately and publicly purchased goods equals their relative price:

$$\frac{v_g(g)}{u_c(c)} = 1 + h_T(T) \ . \tag{9}$$

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The relative price of publicly provided goods is unity (production cost) plus the marginal cost of distortions h_T. Higher taxes increase the marginal cost of g and induce substitution from good g to the private good c. Because of this negative link between government spending and taxes, the budgetary adjustment following a period-0 tax cut generally involves both higher taxes and lower government spending. Consumers rationally anticipate this adjustment. Because of lower anticipated government spending, a tax cut unambigously increases consumption. But because of higher future taxes, they react less to a tax cut than to higher income from other sources.

If $(-v_{gg}) \to \infty$ at some value of g (Corollary 2), government spending is fixed at that value. Then a tax cut does not change the present value of taxes and therefore leaves consumption unchanged. Since the government behaves as the Ricardian approach assumes, Ricardian equivalence emerges. Interestingly, this holds even though taxes are distortionary.[13]

If $(-v_{gg}) = 0$ (Corollary 3), demand for publicly provided goods is infinitely elastic to changes in relative price. Different levels of period-0 debt do not lead to differences in taxes later on. Therefore, individuals correctly view any period-zero tax cut as increase in net wealth and spend it like any other income.[14]

In general, consumer reactions are in between those of Corollaries 2 and 3.[15] If demand for the good g is very sensitive to cost (v_{gg} small in absolute value), higher debt will not lead to future tax increases but rather to future reductions in spending. Then the result is far from the Ricardian prediction. On the other hand, Ricardian equivalence will be a good benchmark, if government spending is very inelastic ($(-v_{gg})$ large).

Another feature of the optimizing model is interesting: When aggregate income increases, consumption rises by less than $r/(1 + r)$, the MPC in the

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standard permanent income model. The reason for the lower MPC_y is that increased aggregate income is optimally allocated to both goods. Therefore, whenever income increases on aggregate, a fraction $\left(1 - (1 + r)/r \cdot MPC_y\right) > 0$ must be set aside for future taxes.[16]

4. A Net Wealth Interpretation

To understand consumer behavior in the welfare-maximizing model, it is useful to reexamine Barro's (1974) famous question of what constitutes net wealth. Period-0 events influence the future through debt, D_0, and capital, K_0, which determine initial conditions at $t = 1$. One can show:

Theorem 2: Taxes in periods $t \geq 1$ are a function of initial debt and capital, $T = T(D_0, K_0)$, with derivatives

$$T_K = r \cdot (1 + h_T) \cdot (-u_{cc})/A ,$$

$$T_D = r\left(-v_{gg} - (1 + h_T)u_{cc}\right)/A ,$$

where $A = -v_{gg} - (1 + h_T)^2 u_{cc} + h_{TT}u_c > 0$.

Proof: By taking the total differential of (6d) or (9) subject to the budget constraints. ∎

For given capital, higher government debt that is held by individuals leads to higher taxes. But since $T_D < r$, taxes do not increase enough to pay off the debt at constant government spending. In this model, the answer to Barro's question is that the fraction $(1 - T_D/r)$ of government bonds are indeed net wealth. In addition, higher aggregate capital, K_0, leads to somewhat higher taxes (since $0 < T_K < T_D < r$), which are used to increase the availability of good g. Thus, not all of aggregate capital is net wealth either, which motivates the result $MPC_y < r/(1 + r)$ above.

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5. Conclusions

The paper shows that an optimizing model of government behavior has implications that can differ radically from those of the Ricardian approach, even though both approaches assume rational, forward-looking individuals. Both agree that a tax cut has a smaller effect on consumption than other increases in disposable income because some increase in future taxes is expected. But the optimizing model predicts also lower future government spending which reduces the need for higher taxes. The magnitude of the spending change depends on how sensitive the demand for public goods is to cost. With high elasticity, individuals will spend almost as much income from tax cuts as they spend from other disposable income. With zero elasticity, Ricardian equivalence emerges.

The policy implications are wide-ranging. Unless the assumption of exogenous government spending is true, correlation between consumption and deficits provides no evidence against forward looking consumer bahavior. Research on the determinants of government spending is needed to determine how rational consumers should react to tax cuts. Perhaps the way the current "deficit-problem" in the United States is resolved by spending cuts and/or tax increases will be a revealing experiment.

Footnotes

[1] The label "Ricardian" is used without suggesting an interpretation of Ricardo's writings; see O'Driscoll (1977).

[2] See, e.g., Abel (1986), Barro (1988), and Bernheim (1987).

[3] In light of the debate between Feldstein (1982), Aschauer (1985) and (1988), and Bernheim (1987) on how to test Ricardian equivalence in a world with uncertain government spending, I should emphasize that the paper is not a critique of Ricardian equivalence when interpreted as a statement about rational consumer behavior. In fact, the model is set up in a way that none of the standard criticisms (e.g., involving finite lives or imperfect capital markets) applies. The question is whether its conditions will ever be satisfied under reasonable assumptions about government behavior. See Buchanan (1976) for similar doubts about the relevance of pure financial changes and see Bohn (1989) for empirical evidence.

[4] See Lucas (1976), Sargent (1984), Sims (1986).

[5] See, e.g., Buchanan (1958), Barro (1979), Lucas and Stokey (1983).

[6] For example, biases in spending decisions motivated by public-choice arguments could be accommodated (see footnote 10).

[7] Therefore, Lucas' (1976) critique would suggest caution even if government spending were found to be exogenous historically.

[8] Examples of such complications are capital market imperfections, finite lives and different bequest motives or uncertainty about the incidence of future taxes. See, e.g., Abel (1986), Barro (1979), (1981), (1989), Barsky, Mankiw, Zeldes (1986), Carmichael (1982), Chan (1983), and the recent surveys by Aschauer (1988) and Bernheim (1987). To obtain unique policies, distortionary taxes are allowed, but they are introduced in a way that would yield Ricardian results in the presence of exogenous spending, as in Barro (1979); see Corollary 2 below.

[9] This good may be a public good in the sense that no person can be excluded from consuming it. Then taxation arises naturally as solution of the externality problem. But it may be any other good or service that is financed by taxes. Separability is only imposed to eliminate inessential interaction between consumption and government spending choices. The proofs in the appendix cover even the more general case of non-separable preferences, which yields similar results as the theorems below, except that the mixed partial derivative of the utility function also enters into the marginal propensities to consume.

[10] Readers who have different beliefs about the government objective function (cf. Buchanan (1958)) may interpret the $v(\cdot)$-part of (1) as whatever determines government's choice of public spending.

[11]A key restriction implied by this specification is that taxation causes distortions within a period, but not intertermporally. It is well known that tax induced distortions may take a much more complicated form and they may provide alternative justifications for real effects of deficits (see Judd (1985a, b), (1987a, b)). But such complications would be distracting here.

[12]Because shocks occur only once, an observer would need a cross section of economies to draw inferences. In the general case with repeated shocks, a time series on one economy would be sufficient, provided changing initial conditions are taken into account. The appendix shows that the theoretical results stated below under the simplifying assumption (5) generalize to the case of repeated shocks.

[13]The reason is that optimal policy equalizes excess burden over time, $h_T(T_0, Y_0, \epsilon_0) = h_T(T_t)$ for all t, as in Barro (1979). Barro's (1989) claim that distortionary taxes induce only second-order effects is confirmed in this model. Of course, distortionary taxes may make a difference in other models; see Judd (1985a, b), (1987a, b) for detailed analyses.

[14]Notice, though, that the MPC is less than $r/(1 + r)$, because any increase in net wealth, is allocated to both goods, leading to somewhat higher taxes later. In contrast, the Ricardian case has $MPC_Y = r/(1 + r)$, because spending on public goods does not increase even if private wealth rises.

[15]Since the focus of this paper is on exogeneity of government spending, only two limiting cases are explored. Other limiting results can be derived easily, e.g., Ricardian equivalence for $h_{TT}(\cdot) \to 0$, which replicates Barro (1974).

[16]This argument does not involve taxes on capital that would distort individual saving decisions. If one introduced idiosyncratic shocks to income, individuals would indeed consume $r/(1 + r)$ of such increases.

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Appendix

A.1. A Model of Taxation and Labor Supply

The precise specification of tax distortions has been deferred to the appendix because only the reduced form h() matters for the model. Here I will demonstrate that there is a model of taxation that yields this reduced form. But the main model would apply for any model of taxation with similar features.

A specific set of assumptions that yields tax distortions of the form h() is as follows. Suppose individuals are endowed with Y_t units of raw products (ℓ) that they inelastically supply to two production processes, denoted by $\ell_{1t} + \ell_{2t} = Y_t$. We assume that Y_t is exogenously given and equal to $Y_t = 1 + \psi_t$, where ψ_t is a random shock. Process #1 generates one unit of output per unit input, $y_{1t} = \ell_{1t}$. Process #2 generates $y_{2t} = F(\ell_{2t}, \epsilon_t)$ units of output, where ϵ_t is another stochastic shock and $0 \leq \partial F/\partial \ell_{2t} = F_\ell \leq 1$. To obtain the reduced form variables used in the main text, define revenue as $T_t = \ell_{1t} \cdot \tau_t$ and after tax income as $y_t = y_{1t} + y_{2t} - T_t = (1 - \tau_t) \cdot \ell_{1t} + F(Y_t - \ell_{1t}, \epsilon_t)$.

The idea is that process #2 is less efficient than process #1, but that the government cannot observe how much a certain individual uses process #2, i.e., that it cannot tax it. For example, process #1 may be regular work while #2 is work at home. (Alternatively, the same setup could be interpreted in terms of a labor-leisure choice, if one replaced y_t in budget constraint (2) by $y_{1t} - T_t$ and replaced preferences by $\bar{u}(c_t, \ell_{2t}, g_t, Y_t) = u(c_t + F\ell_{2t}, \epsilon_t), g_t, Y_t).)$

Taxation is generally distortionary, because it can only be levied on process #1 and therefore induces individuals to shift resources to the less

efficient process #2. Lump-sum taxes are obtained in the special case when $F_\ell(0, \epsilon_t)$ converges to zero.

The main task is now to impose sufficient conditions on the basic production process to make sure that the reduced form derivatives have the correct sign. The only significant complication is that too strong substitution between the production processes may generate a Laffer curve effect. This possibility makes the assumptions complicated. In detail, I assume that the production process $y_{2t} = F(\ell_{2t}, \epsilon_t)$ is continuous, has continuous partial derivatives, and satisfies the following properties:

Assumption: Given the initial value of debt D_{t-1} and given any values of ϵ_t, ψ_t on the support of their distributions, there is a value ℓ^*, $0 < \ell^* \leq Y_t = 1 + \psi_t$, such that

(1a) $F_{\ell\ell} < 0$, (1b) $F_{\ell\epsilon} > 0$, (2a) $F_{\ell\ell\ell} \geq 0$, (2b) $F_{\ell\ell\epsilon} \geq 0$,

(3a) $\alpha = (-F_{\ell\ell}) \cdot (Y_t - \ell_{2t}) - (1 - F_\ell) > 0$

for all $\ell^* \geq 0$; moreover

(3b) $\left(1 - F_\ell(\ell^*, \epsilon_t) \cdot (Y_t - \ell^*)\right) > \max(R \cdot D_{t-1}, 0)$

(3c) $\lim_{\ell_{2t} \to \ell^*} \alpha = 0$.

The assumption looks complicated because of a Laffer curve effect. Tax revenue is supposed to rise initially when the tax rate and ℓ_{2t} are increased. This is condition (3a). But revenue is zero at $\ell_{2t} = Y_t$. Condition (3c) defines ℓ^* as the level of nontaxed activity that maximizes tax revenue. Hence we are only concerned with the properties of $F()$ for $\ell_{2t} \leq \ell^*$. Conditions (1a, b) guarantee that a shock ϵ_t raises ℓ_{2t}, (3b) guarantees that it is feasible to finance the debt, and (2a, b) assure that the derivatives of the loss function $h()$ have the signs that intuition suggests. Notice that (3b) just requires that initial debt is low enough. There are functions that

satisfy the assumptions, e.g., $F(\ell, \epsilon) = \ell - a \cdot (\ell - \epsilon)^2$, $\ell^* = (1 + \psi_t - \epsilon_t)/2$, provided $|\psi_t| \leq 1/8$ and $|\epsilon_t| \leq 1/8$ are bounded and $R \cdot D_{t-1}$ is small enough (e.g., zero).

Suppose tax rates are in the interval $0 \leq \tau_t \leq F_\ell(\ell^*, \epsilon_t)$. Each individual maximizes (1) subject to (2) where

$$y_t - T_t = \ell_{1t} \cdot (1 - \tau_t) + F(\ell_{2t}, \epsilon_t), \quad \ell_{1t} + \ell_{2t} \leq Y_t, \quad \ell_{1t}, \ell_{2t} \geq 0 .$$

The Kuhn-Tucker conditions for optimal labor allocation are (using the facts that $\ell_{2t} \leq \ell^* < Y_t$ and $F_\ell > 0$ for $\ell_{2t} \leq \ell^*$)

$$1 - \tau_t - F_\ell(\ell_{2t}, \epsilon_t) + \lambda_t = 0 , \quad \lambda_t \geq 0 , \quad \lambda_t \cdot \ell_{2t} = 0 .$$

The solution is $\ell_{2t} = 0$, if $\tau_t < 1 - F_\ell(0, \epsilon_t)$ and $\ell_{2t} = F_\ell^{-1}(1 - \tau_t, \epsilon_t)$, if $1 - F_\ell(0, \epsilon_t) \leq \tau_t \leq 1 - F_\ell(\ell^*, \epsilon_t)$. Notice that the solution involves only τ_t and ϵ_t, i.e., it is independent of the choice of consumption or any other variable in the individual's problem.

Next, one can express the individual's solution in terms of the reduced form variables. The case $\tau_t < 1 - F_\ell(0, \epsilon_t)$ is the situation with nondistortionary taxes. We clearly have $T_t = \tau_t \cdot Y_t$, $y_t = Y_t$, and $h(\cdot) = 0$. Then marginal changes in ϵ_t have no effect on the allocation. The more interesting case is $F_\ell(0, \epsilon_t) \leq \tau_t \leq 1 - F_\ell(\ell^*, \epsilon_t)$. Then $\ell_{1t} = Y_t = F_\ell^{-1}(1 - \tau_t, \epsilon_t)$, $T_t = \tau_t(Y_t - F_\ell^{-1}(1 - \tau_t, \epsilon_t)) = T(Y_t, \epsilon_t, \tau_t)$, and $y_t = Y_t - H(\tau_t, \epsilon_t)$, where $H(\tau_t, \epsilon_t) = F_\ell^{-1}(1 - \tau_t, \epsilon_t) + F(F_\ell^{-1}(1 - \tau_t, \epsilon_t), \epsilon_t)$. Notice that $\partial T_t/\partial \tau = Y_t - \ell_{2t} + \tau_t/F_{\ell\ell} = 1/(-F_{\ell\ell}) \cdot [(-F_{\ell\ell}) \cdot (Y_t - \ell_{2t}) - (1 - F_\ell)] > 0$, hence we can write $\tau_t = \tau(T_t, \psi_t, \epsilon_t)$. Then $h(T_t, Y_t, \epsilon_t) = H(\tau(T_t, Y_t - 1, \epsilon_t), \epsilon_t)$ is the loss in output as the result of distortionary taxation. The partial derivatives of $h(\cdot)$ are

$$h_T = \tau_t/\alpha > 0 \; , \quad h_\varepsilon = h_T \cdot \ell_{1t} \cdot F_{\ell\varepsilon} - F_\varepsilon \; , \quad h_Y = -\tau_t^2/\alpha < 0$$

$$h_{TT} = -F_{\ell\ell}/\alpha + \tau_t/\alpha^3(-2 \cdot F_{\ell\ell} + \ell_{1t} \cdot F_{\ell\ell\ell}) > 0$$

$$h_{T\varepsilon} = T_t \cdot F_{\ell\varepsilon}/\alpha^3(-2 \cdot F_{\ell\ell} + \ell_{1t} \cdot F_{\ell\ell\ell}) + T_t/\alpha^2 \cdot F_{\ell\ell\varepsilon} > 0$$

$$h_{TY} = -T_t/\alpha^3(2 \cdot F_{\ell\ell} + \tau_t \cdot F_{\ell\ell\ell}) < 0 \; .$$

For the general theorems in Appendix A.2 one needs $h_\varepsilon \le 0$, which can be assured by assuming that F_ε is large enough. The intuition that ε_t largely causes incentive effects and that ψ_t is largely an income change corresponding to a situation in which h_ε, $h_{T\psi}$, and h_ψ are small in absolute value. Overall, theis model of taxation yields tax distortions of the type assumed in the text.

A.2. Results without Simplifying Assumptions

In this appendix, I will show that the qualitiative implications of Theorems 1 generalize easily to a setting where new shocks generate repeated policy changes and where consumption c and government spending g are not necessarily separable.

Since the empirical literature (e.g. Aschauer (1985)) suggests that private and publically provided goods are non-separable, I will generalize utility (1) and assume that individual utility is a function $u(c_t, g_t, \gamma_t)$ that is increasing and concave in c and g with $u_{c\gamma} = 0$ and $u_{c\gamma} > 0$. However, even though I want to allow for substitution between c and g, I do not want this issue to dominate the analysis. To limit substitution effects it is sufficient to assume that goods c_t and g_t are not complements in utility and that none of the goods is inferior. Formally, this means $u_{cg} \le 0$, $u_{cc} - (u_c/u_g)u_{cg} < 0$ and $u_{gg} - (u_g/u_c)u_{gg} < 0$.

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As example, consider utility functions of the form $v(c + \alpha g)$ with $0 \leq \alpha < 1$, which are popular in the empirical literature (e.g., Aschauer (1985)). They imply that government spending is a partial substitute for private consumption. But functions of this type do not provide a motivation for government spending (the optimal level of g would be zero). If we supplement such a function by some additional preference for the good g that motivates its existence, e.g., $v_1(c + \alpha g) + v_2(g, \gamma)$, the complete utility function satisfies my assumptions.

With these assumptions, the social planner's problem of finding equilibrium consumption, government spending, taxes, capital, and debt can be solved by dynamic programming. State variables are initial capital and debt and the random shocks. Let

$$V(K_t, D_t) = \max E_{t+1}\left[\sum_{i=1}^{\infty} \delta^i \cdot u(c_{t+i}, g_{t+i}, \gamma_{t+i})\right] \qquad (5)$$

be the value function. It indicates the maximum welfare from period $t + 1$ on, where the maximum is over choices of consumption, government spending and taxes subject to budget constraints (2) and (3) and the condition that the function for output (4) is satisfied in each period. The problem in period t is to solve

$$V(K_{t-1}, D_{t-1}) = \max u(c_t, g_t, \gamma_t) + \delta \cdot E_t V(K_t, D_t) \qquad (6)$$

subject to

$$K_t = R \cdot K_{t-1} + Y_t - c_t - g_t - h(T_t, Y_t, \epsilon_t) \qquad (7)$$

$$D_t = g_t + R \cdot D_{t-1} - T_t . \qquad (8)$$

The first order conditions for optimal consumption, government spending and taxation are

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$$u_c(c_t, g_t, \gamma_t) - \delta \cdot E_t V_K(D_t, K_t) = 0 , \qquad (9)$$

$$u_g(c_t, g_t, \gamma_t) + \delta \cdot E_t[V_D(D_t, K_t) - V_K(D_t, K_t)] = 0 , \qquad (10)$$

$$\delta \cdot E_t[-h_T(T_t, Y_t, \epsilon_t) \cdot V_K(D_t, K_t) - V_D(D_t, K_t)] = 0 . \qquad (11)$$

The solution to this problem is a mapping Γ from the state variables $(K_{t-1}, D_{t-1}, Y_t, \epsilon_t, \gamma_t)$ to optimal values $(c_t, K_t, g_t, T_t, D_t) = \Gamma(K_{t-1}, D_{t-1}, Y_t, \epsilon_t, \gamma_t)$.

Note that the system Γ is highly nonlinear so that random shocks may have complicated effects through expectations and risk attitudes. Since the only function of random shocks is to provide a motivation for some noise in the macroeconomy, I will assume that the distribution of shocks is sufficiently tight, whenever derivatives could not be signed otherwise. In addition, I assume that a solution Γ exists and that the value function is strictly concave at the solution. (An existence problem might arise if financing the debt is not feasible; i.e., we assume that initial debt is sufficiently small. The value function is always weakly concave and it is strictly concave if the game ends after a finite number of periods. We exclude the degenerate case that $V_{KK}V_{DD} - V_{DK}^2$ converges to zero in the transition from a finite to an infinite-period economy.)

Generalizations of Theorem 1 and its corollaries are based on a characterization of the mapping Γ. The general theorems are:

Theorem A1: The mapping Γ has derivatives as indicated in Table A1.

Theorem A2: Consider all realizations of $(K_{t-1}, D_{t-1}, Y_t, \epsilon_t, \gamma_t)$ that satisfy $\Delta g_t = 0$ and

$$Y_t - \bar{Y} = h(T^*, Y_t, \epsilon_t) - h(T^*, 0, 0). \qquad (*)$$

Then $\Delta c_t > 0$ if and only if $\Delta D_t = -\Delta T_t > 0$.

Theorem A3: $MPC_Y > MPC_T > 0$. $(MPC_T/MPC_Y) \to 1$ as $u_{gg} \to 0$.

The condition (*) on Y_t in Theorem A2 is a generalization of condition (8) to arbitrary rather than only infinitesimal changes in taxes. Theorem A2 shows that consumption and taxes always move in opposite directions as long as government spending and before-tax income are held constant.

Proof of Theorem A1: I prove the results in Table A1 first for a truncated economy by induction and then take the limit. The steps are straighforward but extremely tedious. Essentially, one has to solve 3 first order conditions and two constraints to determine the 5 endogenous variables as functions of the 5 state variables.

It is necessary to define some notation. For the infinite horizon economy, let $U_{ij} = \delta E_t V_{ij}$; $i, j = D, K$. Let $f_T = 1 + h_T$, and

$$U = \begin{pmatrix} U_{KK} & U_{KD} \\ U_{DK} & U_{DD} \end{pmatrix}, \qquad u = \begin{pmatrix} u_{cc} & u_{cg} \\ u_{gc} & u_{gg} \end{pmatrix},$$

$$\bar{h} = (h_T, 1), \quad \bar{f} = (f_T, 1),$$

$$\Omega = \begin{pmatrix} u_{cc} + U_{KK} & u_{cg} + U_{KK} - U_{KD} & +h_T U_{KK} + U_{KD} \\ -u_{cc} + u_{cg} - U_{KD} & u_{gg} - i_{cg} + u_{DD} - U_{KD} & -h_T U_{KD} - U_{DD} \\ -f_T u_{cc} + u_{cg} & u_{gg} - f_T u_{cg} & -h_{TT} u_c \end{pmatrix}$$

$$B = \begin{pmatrix} R \cdot U_{DK} & R \cdot U_{KK} & -h_\epsilon \cdot U_{KK} & 0 & U_{KK} \cdot (1 - h_\psi) \\ -R \cdot U_{DD} & -R \cdot U_{DK} & h_\epsilon \cdot U_{DK} & -u_{gY} & -U_{DK} \cdot (1 - h_\psi) \\ 0 & 0 & h_{T\epsilon} \cdot u_c & -u_{gY} & h_{TY} \cdot u_c \end{pmatrix}$$

where the jth columns of Ω and B are denoted by Ω_j, B_j, respectively.

Suppose our game is ended after a finite number of periods, T. Terminal values $D_T = K_T = 0$ are given. Define a value function $V^{t-1}(D_{t-1}, K_{t-1})$ by $V^{T-1} = \max u(c_T, g_T, \gamma_T)$, $V^t = \max u(c_t, g_t, \gamma_t) + \delta E_t V^{t+1}$, $t < T$, where the maximum is taken subject to the budget constraints.

For the truncated economy, define Ω^t and B^t analogous to Ω and B with U_{ij} replaced by $U^t_{ij} = \delta E_t V^t_{ij}$. Let c^t_K, c^t_D, g^t_K, g^t_D, T^t_D, and T^t_K be the derivatives of c_t, g_t, and T_t with respect to changes in state variables D_{t-1} and K_{t-1}. For the induction, it is useful to define the expressions

$$X^t_1 = (u^t_{gg} - u^t_{cg})U^t_{KK} + u^t_{cg}U^t_{DK}, \quad X^t_2 = u^t_{cc}U^t_{DD} - (u^t_{cc} - u^t_{cg})U^t_{DK} ,$$

$$X^t_3 = (u^t_{cc} - u^t_{cg})U^t_{KK} - u^t_{cc}U^t_{DK}, \quad X^t_4 = U^t_{DK} + h^t_T U^t_{KK}, \text{ and } X^t_5 = U^t_{DD} + h^t_T U^t_{DK}$$

(where superscripts indicate the period). Note that

$$|\Omega^t| = -u_c \cdot h_{TT} \cdot \left(|u^t| + |U^t| + X^t_1 + X^t_2 + X^t_3\right) + |u^t| \cdot \tilde{h}U\tilde{h}' + |U| \cdot \tilde{f}u\tilde{f}' .$$

Then we can obtain the following results.

Lemma 1:

For the final period $t = T$, the derivatives of the endogenous variables with respect to state variables satisfy

$$c^t_K > 0 , \quad g^t_K > 0, \quad g^t_D < 0 , \tag{A1}$$

and

$$T^t_D > 0 , \quad T^t_K > 0 . \tag{A2}$$

Proof: We have to maximize

$$U\left[R \cdot (K_{T-1} + D_{T-1}) + Y_T - T_T - h(T_T, \epsilon_T, Y_T), T_T - R \cdot D_{T-1}, \gamma_t\right] ,$$

which leads to the first order condition $u_g - (1 - h_T) \cdot u_c = 0$. Taking the total differential implies

$$T_K = R/A \cdot \left[u_{cg} - (1 + h_T)u_{cc} \right] > 0$$

$$T_D = R/A \cdot \left(u_{cg} - u_{gg} + (1 + h_T)(u_{cg} - u_{cc}) \right) > 0$$

$$g_D = T_D - R = -R/A \cdot \left[h_{TT}u_c - h_T \cdot (u_{cc}(1 + h_T) - u_{cg}) \right] < 0$$

$$g_K = T_K > 0$$

$$c_D = R = (1 + h_T)T_K = R/A \cdot \left[h_{TT}u_c - u_{gg} + (1 + h_T)u_{cg} \right] > 0$$

$$d_D = R = (1 + h_T)T_D = R/A \cdot \left[h_{TT}u_c + h_T(u_{gg} - (1 + h_T)u_{cg} \right] \gtrless 0$$

where $A = h_{TT}u_c - u_{gg} + 2(1 + h_T)u_{cg} - (1 + h_T)^2 u_{cc} > 0$.   QED.

Lemma 2:

In period $t - T - 1$ we have

$$U_{KK}^t < 0, \quad |U^t| > 0, \quad |\Omega^t| < 0 , \tag{A3}$$

$$x_1^t > 0, \quad x_2^t > 0, \quad x_3^t > 0 . \tag{A4}$$

$$x_4^t < 0, \quad x_5^t < 0 . \tag{A5}$$

Proof: The envelope conditions are $V_K^{t-1} = R\, E_t u_c^t$ and $V_D^{t-1} = R\, E_t(u_c^t - u_g^t)$. Therefore,

$$U_{KK}^{t-1} = R\, \delta\, E_{t-1}(u_{cc}^t c_K^t + u_{cg}^t g_K^t)$$

$$U_{KD}^{t-1} = U_{DK}^{t-1} = R\delta\, E_{t-1}(u_{cc}^t c_D^t + u_{cg}^t g_D^t)$$

$$= R\delta\, E_{t-1}\left[(u_{cc}^t - u_{cg}^t)c_K^t + (u_{cg}^t - u_{gg}^t)g_K^t) \right]$$

$$U_{DD}^{t-1} = R\delta\, E_{t-1}\left[(u_{cc}^t - u_{cg}^t)c_D^t + (u_{cg}^t - u_{gg}^t)g_D^t \right]$$

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Using the results of Lemma 1, we obtain for t = T

$$U_{KK}^{t-1} = \delta^1 R^2 E_{t-1}[|u^t|/A \cdot (h_{TT}u_c^t u_{cc}^t - |u^t|)] \ .$$

We know that $|u^t| > 0$ and $u_{cc}^t < 0$ by concavity of u(); also A > 0, $u_c > 0$, and $h_{TT} > 0$. But we do not know the conditional covariances between these expressions. Here we need the assumption that shocks are sufficiently small. The conditional covariances are of the order of magnitude of the variances in the shocks; hence for small shocks $U_{KK}^{t-1} < 0$ follows from $E_{t-1}|u^t| > 0$, $E_{t-1}u_{cc}^t < 0$, $E_{t-1}A > 0$, $E_{t-1}u_c > 0$, and $E_{t-1}h_{TT} > 0$. From here on we will use the assumption of small shocks to sign expressions without mentioning it explicitly. Note that we do compute the exact solution to the dynamic programming problem; the assumption of small shocks is only considered as a sufficient condition for the characterization of the exact solution. Next,

$$|U^{t-1}| = U_{KK}^{t-1}U_{DD}^{t-1} - (U_{KD}^{t-1})^2 = \delta^2 R^2 E_{t-1}|u^t|(c_D^t \cdot g_K^t - c_K^t \cdot g_D^t)$$

$$= \delta^2 R^4 E_{t-1}|u^t|/E_{t-1}A \cdot E_{t-1}(h_{TT}u_c^t) > 0 \ ,$$

$$X_1^{t-1} = \delta R E_{t-1}[((u_{gg}^{t-1} = u_{cg}^{t-1})u_{cc}^t + (u_{cc}^{t-1} - u_{cg}^{t-1})u_{cg}^t)c_K^t$$

$$+ ((u_{gg}^{t-1} - u_{cg}^{t-1})u_{cg}^t + (u_{cc}^{t-1} - u_{cg}^{t-1})u_{gg}^t)g_K^t]$$

$$= \delta R(u_{gg}^{t-1}u_{cc}^{t-1} - u_{cg}^{t-1}u_{cg}^{t-1})E_{t-1}c_K^t > 0$$

where we approximated (using the first order condition $u_g^{t-1} = \delta R E_{t-1}(u_g^t)$ and the assumption of small shocks)

$$E_{t-1}(u_{gg}^{t-1}u_{cc}^t - u_{cg}^{t-1}u_{cg}^t) \approx \delta R(u_{gg}^{t-1}u_{cc}^{t-1} - u_{cg}^{t-1}u_{cg}^{t-1}) > 0 \ ,$$

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$$E_{t-1}(u_{gg}^{t-1}u_{cg}^t - u_{cg}^{t-1}u_{gg}^t) \approx 0 .$$

Using similar approximations, we get

$$X_2^{t-1} = \delta R E_{t-1}[((u_{cc}^t - u_{cg}^t)u_{cc}^{t-1} - (u_{cc}^{t-1} - u_{cg}^{t-1})u_{cc}^t)c_D^t$$

$$+ ((u_{cg}^t - u_{gg}^t)u_{cc}^{t-1} + (u_{cc}^{t-1} - u_{cg}^{t-1})u_{cg}^t)g_D^t]$$

$$\approx -\delta R |u^{t-1}| E_{t-1} g_D^t > 0$$

$$X_3^{t-1} = \delta R E_{t-1}[((u_{cc}^{t-1} - u_{cg}^{t-1})u_{cc}^t - (u_{cc}^t - c_{cg}^t)u_{cc}^{t-1})c_K^t$$

$$+ ((u_{gg}^{t-1} - u_{cg}^{t-1})u_{cg}^t - (u_{cg}^t u_{gg}^t)u_{cc}^{t-1})g_K^t]$$

$$\approx \delta R |u^{t-1}| E_{t-1} g_K^t > 0 .$$

Using these results, $|\Omega^{t-1}| < 0$ is immediate. Moreover, $V_D^{t-1} = RE_t(u_c^t - u_g^t) = RE_t h_T^t u_c^t$ implies

$$U_{KD}^{t-1} = R\delta E_{t-1}[-h_T^t(u_{cc}^t \cdot c_K^t + u_{cg}^t \cdot g_K^t) - h_{TT}^t u_c^t T_K^t]$$

$$U_{DD}^{t-1} = R\delta E_{t-1}[-h_T^t(u_{cc}^t \cdot c_D^t + u_{cg}^t \cdot g_D^t) - h_{TT}^t u_c^t T_D^t] .$$

Using these alternative expressions,

$$X_4^{t-1} = \delta R E_{t-1}[((h_T^{t-1} - h_T^t)u_c^t)((u_{cc}^t \cdot c_K^t + u_{cg}^t \cdot g_K^t)/u_c^t) - h_{TT}^t u_c^t T_K^t]$$

Note that $E_{t-1}(h_T^{t-1} - h_T^t)u_c^t = u_g^{t-1} - u_c^{t-1} + \delta E_{t-1} V_D^t = 0$, hence

$$X_4^{t-1} \approx \delta R E_{t-1}[-h_{TT}^t u_c^t T_K^t] < 0 ,$$

provided shocks are small. Similarly,

$$X_5^{t-1} = \delta R E_{t-1}[((h_T^{t-1} - h_T^t)u_c^t)((u_{cc}^t \cdot c_D^t + u_{cg}^t \cdot g_D^t)/u_c^t) - h_{TT}^t u_c^t T_D^t]$$

$$= \delta R E_{t-1}[-h_{TT}^t u_c^t T_D^t] < 0 .$$

QED.

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Lemma 3:

Suppose (A3) and (A4) hold for some period $t = i + 1$, $i + 1 \leq T$. Then (A3) and (A4) also hold for period $t = i$.

Proof: Let $t = i + 1$. The endogenous variables satisfy the first order conditions (9)-(11).

The first order conditions (10) and (11) can be simplified to

$$u_g(u_t, g_t, \gamma_t) - u_c(c_t, g_t, \gamma_t) + \delta \cdot E_t[V_D^t(D_t, K_t)] = 0 \ ,$$

$$u_g(u_t, g_t, \gamma_t) - (1 + h_T(T_t, Y_t, \varepsilon_t)) \cdot u_c(c_t, g_t, \gamma_t) = 0 \ .$$

Taking the total differential of (9) and these two equations, we obtain

$$\Omega^t \cdot (dc_t \ dg_t \ dT_t)' = B^t \cdot (dD_{t-1} \ dK_{t-1} \ d\varepsilon_t \ d\gamma_t \ dY_t)' \ .$$

Then Cramer's rule implies

$$c_K^t = \frac{\partial c_t}{\partial K_{t-1}} = \frac{-R}{|\Omega^t|} \cdot [h_{TT}u_c(X_1^t + |U^t|) - (u_{gg} - f_T^t u_{cg}) \cdot |U^t|] > 0$$

$$c_D^t = \frac{\partial c_t}{\partial D_{t-1}} = \frac{-R}{|\Omega^t|} \cdot [h_{TT}u_c(u_{gg}U_{DK}^t - u_{cg}(U_{DK}^t - U_{DD}^t) + |U^t|)$$

$$+ h_T(u_{gg} - f_T u_{cg}) \cdot |U^t|]$$

$$g_K^t = \frac{\partial g_t}{\partial K_{t-1}} = \frac{-R}{|\Omega^t|} \cdot [h_{TT}u_c \cdot X_3^t + |U^t| \cdot (u_{cg} - f_T \cdot u_{cc})] > 0$$

$$g_D^t = \frac{\partial g_t}{\partial D_{t-1}} = \frac{-R}{|\Omega^t|} \cdot [-h_{TT}u_c \cdot (|U^t| + X_2^t) - |U^t| \cdot h_T \cdot (u_{cg} - f_T \cdot u_{cc})] < 0$$

$$T_K^t = \frac{\partial T_t}{\partial K_{t-1}} = \frac{-R}{|\Omega^t|} \cdot [-|u| \cdot X_4^t + |U^t| \cdot (u_{cg} f_T \cdot u_{cc})] > 0$$

$$T_D^t = \frac{\partial T_t}{\partial D_{t-1}} = \frac{-R}{|\Omega^t|} \cdot [-|u|X_5^t + |U^t| \cdot (u_{cg} - f_T u_{cc})] > 0 \ .$$

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In determining the signs, we also use the fact that $f_T u_{cc} - u_{cg} = f_T(u_{cc} - (u_c/u_g)u_{cg}) < 0$ and $u_{gg} - f_T u_{cg} = u_{gg} - (u_g/u_c)u_{cg} < 0$. Analogous to the Lemma 2 we get

$$U_{KK}^{t-1} = R\delta E_{t-1}(u_{cc}^t c_K^t + u_{cg}^t g_K^t)$$

$$= -R^2 \delta E_{t-1}[|U^t|(h_{TT} u_c u_{cc}^t - |u^t|) + U_{KK}^t |u^t|]/|\Omega^t| < 0$$

given that (A3) and (A4) hold for period $t = i + 1$.

$$|u^{t-1}| = \delta^2 R^2 E_{t-1}|u^t|(c_D^t g_K^t - c_K^t g_D^t)$$

$$= \delta^2 R^4 E_{t-1}[(h_{TT} u_c^t)^2 (|u^t|^2 + |u^t|(x_1^t + x_2^t + x_3^t) + x_1^t x_2^t$$

$$+ x_3^t(u_{gg}^t U_{DK}^t - u_{cg}^t(U_{DK}^t - U_{DD}^t))]$$

$$- h_{TT} u_c^t |u^t|(fu^t f |u^t| + hu^t h |u^t|))]$$

$$\approx -\delta^2 R^4 E_{t-1}|u^t|/E_{t-1}|\Omega^t| \cdot E_{t-1}(h_{TT} u_c^t |u^t|) > 0 .$$

The arguments for x_1^{t-1}, x_2^{t-1}, $x_3^{t-1} > 0$ and $|\Omega^{t-1}| < 0$ are identical to the arguments in Lemma 2. QED.

Notice that Barro's (1974) case, $h(\cdot) = 0$, implies that $\Omega_2 = 0$. Hence $|\Omega| = 0$ and the proof fails. This reflects the fact that then the mapping Γ would have no unique solution.

Lemma 4:

Suppose (A3) and (A5) hold for some period $t = i + 1$, $i + 1 \leq T$. Then (A5) also holds for period $t = i$.

Proof: Use the same arguments as in Lemma 2.

Corollary: (A1) - (A5) hold for all periods $t \leq T$.

Proof: Statements (A3) - (A5) follow by induction from Lemmas 2 and 3. Inspection of the derivatives in the proof of Lemma 3 shows that they imply (A1) and (A2).

Lemma 5:

Take the limit $T \to \infty$. Then (A1) - (A5) still hold as weak inequalities. Under the assumptions $|U| > 0$, all other inequalities in (A1) - (A5) are strict.

Proof: As $T \to \infty$, Lemmas 3 and 4 immediately imply $U_{KK}^{t-1} \leq 0$, $|U^{t-1}| \geq 0$, $|\Omega^{t-1}| \leq 0$, $X_1^{t-1} \geq 0$, $X_2^{t-1} \geq 0$, $X_3^{t-1} \geq 0$, $X_4^{t-1} \leq 0$, $X_5^{t-1} \leq 0$. But the recursion formulas show that the inequalities are strict in period $t-1$, provided $|U^t| > 0$ holds in period t.   QED.

The other derivatives in Table 1 can be computed directly from

$$\Omega \cdot (dc_t \quad dg_t \quad dT_t)' = (B_3 \quad B_4 \quad B_5) \cdot (d\varepsilon_{t-1} \quad d\gamma_{t-1} \quad dY_{t-1})'$$

and the budget constraints (for dD_t and DK_t). Using the results in Lemma 3, we obtain

$$\frac{\partial D_t}{\partial K_{t-1}} = \frac{-R}{|\Omega|} \cdot [h_{TT}u_c X_3^t + |u| \cdot X_4^t]$$

$$\frac{\partial D_t}{\partial D_{t-1}} = \frac{-R}{|\Omega|} \cdot [h_{TT}u_c(|u| + X_1^t + X_3^t) - |U|(u_{gg} - f_T u_{cg}) - |u|h_T X_4^t] > 0$$

$$\frac{\partial K_t}{\partial K_{t-1}} = \frac{-R}{|\Omega|} \cdot [h_{TT}u_c(|u| + X_2^t) - |u|X_5^t] > 0$$

$$\frac{\partial K_t}{\partial D_{t-1}} = \frac{-R}{|\Omega|} \cdot [h_{TT}u_c(u_{cc} - u_{cg}) \cdot (U_{DD} - U_{DK}) + h_T X_5^t|u| - h_T|U|(u_{gg} - f_T u_{cg})]$$

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To compute the effects of y_t and ε_t, define

$$A_1 = u_c(f_T|U| + (u_{gg} - u_{cg})X_4^t + u_{cg}X_5^t)/(-|\Omega|) > 0$$

$$A_2 = u_c(h_T X_3^t - X_2^t - |U|)/(-|\Omega|)$$

$$A_3 = u_c(|U| + |u| + X_1^t + X_2^t + X_3^t)/|\Omega| < 0 \ .$$

Then

$$\frac{\partial c_t}{\partial Y_t} = h_{TY} \cdot A_1 + \frac{1 - h_Y}{R} \cdot \frac{\partial c_t}{\partial K_{t-1}} \ , \qquad \frac{\partial c_t}{\partial \varepsilon_t} - h_{T\varepsilon} \cdot A_1 + \frac{h_\varepsilon}{R} \frac{\partial c_t}{\partial K_{t-1}}$$

$$\frac{\partial g_t}{\partial Y_t} = h_{TY} \cdot A_2 + \frac{1 - h_Y}{R} \cdot \frac{\partial g_t}{\partial K_{t-1}} \ , \qquad \frac{\partial g_t}{\partial \varepsilon_t} = h_{T\varepsilon} \cdot A_2 + \frac{h_\varepsilon}{R} \frac{\partial g_t}{\partial K_{t-1}}$$

$$\frac{\partial T_t}{\partial Y_t} = h_{TY} \cdot A_3 + \frac{1 - h_Y}{R} \cdot \frac{\partial T_t}{\partial K_{t-1}} \ , \qquad \frac{\partial T_t}{\partial \varepsilon_t} = h_{T\varepsilon} \cdot A_3 + \frac{h_\varepsilon}{R} \frac{\partial T_t}{\partial K_{t-1}}$$

$$\frac{\partial D_t}{\partial Y_t} = h_{TY} \cdot (A_2 - A_3) + \frac{1 - h_Y}{R} \cdot \frac{\partial D_t}{\partial K_{t-1}} \ , \qquad \frac{\partial D_t}{\partial \varepsilon_t} = h_{T\varepsilon} \cdot (A_2 - A_3) + \frac{h_\varepsilon}{R} \frac{\partial D_t}{\partial K_{t-1}}$$

$$\frac{\partial K_t}{\partial Y_t} = -h_{TY} \cdot (A_1 + A_2 + h_T A_3) + \frac{1 - h_Y}{R} \cdot \frac{\partial K_t}{\partial K_{t-1}} \ ,$$

$$\frac{\partial K_t}{\partial \varepsilon_t} = -h_{T\varepsilon}(A_1 + A_2 + h_T A_3) + \frac{h_\varepsilon}{R} \cdot \frac{\partial K_t}{\partial K_{t-1}} \ .$$

Note that $A_2 - A_3 = u_c(X_1^t + f_T X_3^t + |u|)/(-|\Omega|) > 0$ and $A_1 + A_2 + h_T A_3 = u_c/|\Omega| \cdot (h_t|u| + f_T X_2^t - u_{gg}U_{DK} + u_{cg}(U_{DK} - U_{DD}))$, which may be positive or negative.

Then the signs in Table A1 are obtained under the assumption that h_{TY} is small enough so that the expressions with $(1 - h_Y)$ determine the sign of the effects of Y_t and that h_ε is small enough that expressions with $h_{T\varepsilon}$ determine the direction of the effects of ε_t. Finally,

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$$\frac{\partial c_t}{\partial \gamma_t} = \frac{-u_{gY}}{|\Omega|} \cdot \left[h_{TT}u_c(u_{cg} + U_{KK} - U_{KD}) - f_T|U| - u_{cg}\tilde{h}U\tilde{h} \right]$$

$$\frac{\partial g_t}{\partial \gamma_t} = \frac{-u_{gY}}{|\Omega|} \cdot \left[-h_{TT}u_c(u_{cc} + U_{KK}) + u_{cc} \cdot \tilde{h}U\tilde{h}' + |U| \right] > 0$$

$$\frac{\partial T_t}{\partial \gamma_t} = \frac{-u_{gY}}{|\Omega|} \cdot \left[|U| + X_2^t - h_T X_3^t \right]$$

$$\frac{\partial d_t}{\partial \gamma_t} = \frac{-u_{gY}}{|\Omega|} \cdot \left[-h_{TT}u_c(u_{cc} + U_{KK}) + (f_T u_{cc} - u_{cg}) \cdot X_4^t \right] > 0$$

$$\frac{\partial K_t}{\partial \gamma_t} = \frac{-u_{gY}}{\Omega} \cdot \left[h_{TT}u_c(u_{cc} - u_{cg} + U_{KD}) - u_{cc}f_T X_5^t \right] \cdot$$

Proof of Theorem A2: The key observation for the proof is that the assumptions select a one-dimensional subspace, a curve, out of the three-dimensional space of realizations of $(Y_t, \epsilon_t, \gamma_t)$. An innovation like Δc_t is a function of the actual realization and can be determined by integrating the partial derivatives of the variable along this curve from $(\bar{Y}, 0, 0)$ to the realization.

The endogenous variables c_t, g_t, and T_t are differentiable functions of $(D_{t-1}, K_{t-1}, Y_t, \epsilon_t, \gamma_t)$. Denote them by $c(Y_t, \epsilon_t, \gamma_t)$, $g(Y_t, \epsilon_t, \gamma_t)$, and $T(Y_t, \epsilon_t, \gamma_t)$, respectively, at the given values of D_{t-1}, K_{t-1}. Let innovations be denoted by Δ's, e.g., $\Delta c(Y_t, \epsilon_t, \gamma_t) = c(Y_t, \epsilon_t, \gamma_t) - c(\bar{Y}, 0, 0)$, and derivatives by subscripts, e.g., $c_Y = \Delta c_Y = \partial c(Y_t, \epsilon_t, \gamma_t)/\partial Y_t$.

A debt-financed tax cut as specified in Theorem A2 is a realization of $(Y_t, \epsilon_t, \gamma_t)$ in R^3 in the subspace $\{(Y_t, \epsilon_t, \gamma_t)|\Delta g = 0, (*) \text{ holds}\}$. Equation (*) implies

$$dY_t - h_Y dY_t - h_\epsilon d\epsilon_t = 0 . \tag{A6}$$

Since $g_\gamma > 0$ and $h_Y \leq 0$, the system of equations $\Delta g = 0$ and (*) defines implicit functions $\gamma_t = \gamma^*(\epsilon_t)$ and $Y_t = Y^*(\epsilon_t)$ with derivatives $\gamma_\epsilon^*(\epsilon_t) = -g_\epsilon/g_\gamma - g_Y/g_\gamma \cdot h_\epsilon/(1 - h_Y)$ and $Y_\epsilon^*(\epsilon_t) = h_\epsilon/(1 - h_Y)$; the subspace is the

curve $(Y^*(x), x, \gamma^*(x))$ in R^3, where the index x is any number in the support of ϵ_t.

Consider a specific realization $(Y_t, \epsilon_t, \gamma_t) = (Y^*(\epsilon_t), \epsilon_t, \tau^*(\epsilon_t))$ on this curve. We have $\Delta c_t = \Delta c(Y^*(\epsilon_t), \epsilon_t, \gamma^*(\epsilon_t)) = \int_0^{\epsilon_t} c_x(x) dx$ and $\Delta T_t = \int_0^{\epsilon_t} T_x(x)$, where $c_x(x) = c_\epsilon + c_Y Y^*(x) + c_\gamma \gamma_\epsilon^*(x)$, $T_x(x) = T_\epsilon + T_Y Y_\epsilon^*(x) + T_\gamma \gamma_\epsilon^*(x)$.

We have to prove that $c_x > 0$ and $T_x < 0$. Using $dg_t = 0$ and $dY_t = h_\epsilon/(1 - h_Y) d\epsilon_t$ in the total differential

$$\Omega (dc_t \quad dg_t \quad dT_t)' = (B_3 \quad B_4 \quad B_5)(d\epsilon_t \quad d\gamma_t \quad dY_t)'$$

we get

$$(\Omega_1 - B_4 \quad \Omega_3)(c_x \quad \gamma_\epsilon^* \quad T_x)' = B_3 + h_\epsilon/(1 - h_Y)B_5$$

which is a system of three equations for $(c_x \quad \gamma_\epsilon^* \quad T_x)$. Note that $B_3 + h_\epsilon/(1 - h_Y)B_5 = [0 \quad 0 \quad z]'$ where $z = u_c(h_{T\epsilon}(1 - h_Y) + h_\epsilon h_{TY})/(1 - h_Y) > 0$. By Cramer's rule, we get

$$c_x = \begin{vmatrix} 0 & 0 & X_4^t \\ 0 & 1 & X_5^t \\ 1 & 1 & h_{TT} u_c \end{vmatrix} \cdot \frac{u_{g\gamma} z}{A} = u_{g\gamma}(-X_4^t)z/A < 0$$

and

$$T_x = u_{g\gamma}(u_{cc} + U_{KK})z/A < 0 \qquad\qquad \text{(QED)}.$$

Proof of Theorem A3: To derive MPC_Y, we have to consider realizations $(Y_t, \epsilon_t, \gamma_t)$ with the property $\Delta g_t = \Delta T_t = 0$. Noting that $dy_t = dY_t - h_\epsilon d\epsilon_t - h_Y dY_t$, the total differential

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$$\Omega(dc_t \quad dgt \quad DT_t)' = (B_3 \quad B_4 \quad B_5)(d\epsilon_t \quad d\gamma_t \quad dY_t)'$$

in the proof of Lemma 5 reduces to

$$\Omega_1 \, dc_t = \left(B_3 + h_\epsilon/(1 - h_Y)B_5\right)d\epsilon_t + B_4 d\gamma_t + B_5/(1 - h_Y)dy_t \ ,$$

$$\left(\Omega_1, \ -B_4, \ -B_3-h_\epsilon/(1 - h_Y)B_5\right))dc_t \quad d\epsilon_t \quad d\gamma_y)' = B_5/(1 -h_Y)dy_t \ ,$$

which implies

$$MPC_Y = |\left(B_5/(1 - h_Y), \ -B_4, \ -B_3 h_\epsilon/(1 - h_Y)B_5\right)|/|\left(\Omega_1, \ - B_4, \ -B_3-h_\epsilon/(1 - h_Y)B_5\right)|$$

$$= U_{KK}/(u_{cc} + U_{KK}) > 0 \ .$$

I showed above that $MPC_T = -c_x/T_x(1 + h_T) = X_4^t/(u_{cc}^t + u_{KK}^t)/(1 + h_T) > 0$, hence $\theta = MPC_T/MPC_Y > 0$. To show that $\theta < 1$, note that $(1 - \theta) MPC_y = MPC_y - MPC_T = 1/\left(f_T(u_{cc} + U_{KK})\right) \cdot (U_{KK} - U_{DK})$, where $u_{cc} + U_{KK} < 0$. Hence, we have to show that $U_{KK} - U_{DK} < 0$ for $u_{gg} < 0$ and $U_{KK} - U_{DK} = 0$ for $u_{gg} = 0$. As in the proof of Theorem A1, we start with a truncated economy and use induction, which yields $U_{KK}^{t-1} - U_{DK}^{t-1} = \delta R \, E_{t-1}u_{cc}^t(c_K^t - c_D^t) + u_{cg}^t(g_K^t - g_D^t)$. If the game ends in period T, $U_{KK}^{T-1} - U_{DK}^{T-1} = \delta R^2 \, E_{T-1}[h_{TT}U_c^T \, u_{cg}^T - f_T|u^T|]/A < 0$. If $U_{KK}^t - U_{DK}^t < 0$ for some period $t \leq T$, then

$$U_{KK}^{t-1} - U_{DK}^{t-1} = \delta R^2 E_{t-1}[\,|u^t|(h_{TT}u_c^t u_{cg}^t - f_T|u^t|) + h_{TT}u_c|u^t|(U_{KK}^t -$$

$$U_{DK}^t)]/(-|\Omega|) < 0.$$

By induction, $U_{KK}^t - U_{DK}^t < 0$ for all periods, hence $U_{KK} - U_{DK} \leq 0$ in the infinite horizon game. But if $U_{KK} - U_{DK} \leq 0$ in period t, the recursion

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formula implies that $U_{KK} - U_{DK} \leq 0$ in period $t - 1$. Hence $U_{KK} - U_{DK} < 0$ holds for all t.

If $u_{gg} \to 0$ the assumption $u_{gg} < (u_g/u_c)u_{cg} \leq 0$ implies $u_{cg} \to 0$. Hence $|u| = u_{cc}u_{gg} - u_{cg}^2 \to 0$ and $U_{KK}^{t-1} - U_{DK}^{t-1} \to 0$ in the resursion formula above.

QED.

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Table A1

Effect of on	K_{t-1}	D_{t-1}	ϵ_t	γ_t	Y_t
c_t	+	?	+	−	+
g_t	+	−	?	+	+
T_t	+	+	−	?	+
D_t	?	+	+	+	?
K_t	+	?	?	?	+

Legend: + = the marginal effect is positive;
                − = the marginal effect is negative;
                ? = the marginal effect may be positive or negative.