Technical Appendix to: Fiscal Policy and the Mehra-Prescott Puzzle

by Henning Bohn

December 1997

University of California at Santa Barbara

A. Proof of Proposition 1:

Any Pareto-optimal allocation can be obtained as solution to a social planning problem with some set of positive welfare weights $\{w_t>0\}$:

$$\max E_0\Big[\sum_{t\geq 0} w_t \frac{(c^T{}_t)^{1-\alpha^T}}{1-\alpha^T}\Big]$$

subject to the budget constraint:

$$E_0\Big[\sum_{t\geq 0} MRS_{0,t}\ Z_t\Big] = D^*{}_0$$

where $MRS_{i,t} = \beta^{t-i}\,(c^I{}_t/c^I{}_i)^{-\alpha} = \beta^{t-i}\,(Y_t/Y_i)^{-\alpha}$ (for $i\leq t$, $\alpha=\alpha^I$) is the pricing kernel, $Z_t = T_t-G_t = (Y_t-c^T{}_t)-g\ Y_t = (1-g)\ Y_t - c^T{}_t$ is the primary surplus, and $D^*{}_0=(1+R_0)\cdot D_{-1}$ is the initial debt. Type-I can be ignored in the welfare function, because type-I is unaffected by debt policy. With λ as the Lagrange multiplier on the budget constraint, the optimal consumption $c^T{}_t(s)$ in state s of period t is

$$c^T{}_t(s) = (w_t\ \ \beta^t/\lambda)^{1/\alpha^T}\ \ (Y_t(s)/Y_0)^{\varphi}, \tag{A1}$$

i.e. a monotone function of the state's income level $Y_t(s)$ with elasticity $\varphi = \alpha/\alpha^T$.

Let $1+R^x{}_t(s) = (1+v)\ (1+x_t(s))$ denote the gross return on income-indexed securities, for some constant v. The existence of a finite type-I consumption plan requires that a finite initial wealth translates into a finite consumption stream. This requires that the discount rate on future income exceeds the income growth rate, i.e. that β is small enough that

$$E_{t-1}\big[MRS_{t-1,t}\ (1+x_t)\big] = \beta\ E_{t-1}\big[(1+x_t)^{1-\alpha}\big] < 1.$$

Since $E_{t-1}\left[MRS_{t-1,t}\ (1+R^x{}_t(s))\right] = (1+v)\ \beta E_{t-1}\left[(1+x_t)^{1-\alpha}\right] = 1$, the above existence condition implies $v>0$, i.e., a return on income indexed claims above the growth rate, $R^x = E[R^x{}_t] > x$. For $v>0$, the present value of future income discounted back to any period t is

$$E_t\left[\sum_{i\geq 1} MRS_{t,t+i}\ Y_{t+i}\right] = Y_t\ \left[\sum_{i\geq 1} E_t[\beta^i \prod_{j=1}^{i}(1+x_{t+j})^{1-\alpha}\right] = Y_t\ [\sum_{i\geq 1} (1+v)^{-i}] = \frac{Y_t}{v}$$

a finite multiple of current income. If welfare weights are such that the expectation

$$PVC_0 \equiv E_0\left[\sum_{t\geq 0} MRS_{0,t}\ (w_t\ \beta^t)^{1/\alpha^T}\ (Y_t(s)/Y_0)^{\varphi}\right] = E_0\left[\sum_{t\geq 0} MRS_{0,t}\ c^T{}_t\ \lambda^{1/\alpha^T}\right]$$

exists and is finite, and if the government budget constraint can be satisfied for non-negative consumption(i.e., $D^*{}_0 \leq (1-g)\ (1+v)/v\ Y_0$), then the multiplier λ is uniquely determined as $\lambda = \left(\frac{PVC_0}{(1-g)\ (1+v)/v\ Y_0 - D^*{}_0}\right) > 0$.

To implement the Pareto-optimal consumption path (A1), the government must run state-contingent primary surpluses of

$$Z_t(s) = (1-g)\ Y_t(s) - (w_t\ \beta^t/\lambda)^{1/\alpha^T}\ (Y_t(s)/Y_0)^{\varphi}, \qquad (A2)$$

which implies a set of state-contingent start-of-period debt levels

$$D^*{}_t = E_t\left[\sum_{i\geq 0} MRS_{t,t+i}\ Z_{t+i}\right]$$

and corresponding state-contingent end-of-period government debt levels

$$D_{t-1} = E_{t-1}\left[\sum_{i\geq 0} MRS_{t-1,t+i}\ Z_{t+i}\right] = E_{t-1}\left[MRS_{t-1,t}\ D^*{}_t\right]$$

To get from D_{t-1} to $D^*{}_t(s)$, government debt must consist of securities with state-contingent returns $1+R_t(s) = D^*{}_t(s)/D_{t-1}$. Using (A2), one can write

$$D^*{}_t = (1-g)\ Y_t\ E_t\left[\sum_{i\geq 0} MRS_{t,t+i}\ (Y_{t+i}/Y_t)\right] - (Y_t/Y_0)^{\varphi}\ PVT_t$$

where $PVC_t = E_t\left[\sum_{i\geq 0} MRS_{t,t+i}\ (w_{t+i}\ \beta^{t+i}/\lambda)^{1/\alpha^T}\ (Y_{t+i}(s)/Y_t)^{\varphi}\right]$ is time-dependent through $w_{t+i}\ \beta^{t+i}/\lambda)^{1/\alpha^T}$, but not state-dependent (because of the i.i.d. growth assumption). Hence, one can write debt as a linear combination of an income-indexed security and a security indexed to $(1+x_t(s))^{\varphi}$, namely

$$D^*{}_t(s) = (1-g)\ (1+v)/v\ Y_{t-1}\ (1+x_t(s)) - PVC_t\ (Y_{t-1}/Y_0)^{\varphi}\ (1+x_t(s))^{\varphi}$$

$$\equiv d_{1t}\ (1+x_t(s)) - d_{\varphi t}\ (1+x_t(s))^{\varphi},$$

where $d_{1t} = (1\text{-}g)\,(1+v)/v\ Y_{t-1}>0$ and $d_{\varphi t} = PVC_t\,(Y_{t-1}/Y_0)^{\varphi}>0$. Thus, $D^*_t(s)$ is a function of the time-t income growth $1+x_t(s)$ that is linear, convex, or concave, if and only if φ is equal, less than, or larger than one, respectively.

If $\varphi=1$ (i.e., $\alpha^I=\alpha^T$), this clearly implies income-indexed debt, $D^*_t = (d_{1t}-d_{\varphi t})\,(1+x_t)$. As shown above, income-indexed claims have an expected return $E[1+R_t(s)] = 1+R^X > 1+x$, proving the $\alpha^I=\alpha^T$ case. For general φ, the value of debt at the end of period t-1 is

$$D_{t-1} = E_{t-1}\left[MRS_{t-1,t}\,\left(d_{1t}\,(1+x_t(s)) - d_{\varphi t}\,(1+x_t(s))^{\varphi}\right)\right]$$

To simplify the notation, define $X=1+x_t(s)$, note that $MRS_{t-1,t}=\beta X^{-\alpha}$, and let H_φ be the relative price of a security with payoff $(1+x_t(s))^{\varphi}$ relative to the price of a security paying $(1+x_t(s))$, $H_\varphi = E\left[X^{\varphi}\,MRS_{t-1,t}\right]/E\left[X\ MRS_{t-1,t}\right] = E\left[X^{\varphi-\alpha}\right]/E\left[X^{1-\alpha}\right]$. Then

$$D_{t-1} = \beta\,E\left[d_{1t}\,X^{1-\alpha}-d_{\varphi t}\,X^{\varphi-\alpha}\right] = \beta\,E\left[X^{1-\alpha}\right](d_{1t}-H_\varphi\,d_{\varphi t}) = \frac{d_{1t}-H_\varphi\,d_{\varphi t}}{1+v}.$$

With the same notation, period-t debt can be written as

$$D^*_t(s) = d_{1t}\,X - d_{\varphi t}\,X^{\varphi} = (d_{1t}-H_\varphi\,d_{\varphi t})\,X + d_{\varphi t}\,[H_\varphi\,X-X^{\varphi}]$$

The present value of the second term is zero, since $E[X^{\varphi-\alpha}-H_\varphi\,X^{1-\alpha}]=0$. Hence, the government's total debt portfolio can be interpreted as consisting of income indexed debt--an amount $(d_{1t}-H_\varphi\,d_{\varphi t})$--plus a "hedge fund." The hedge fund has a zero present value and returns proportional to $H_\varphi\,X-X^{\varphi}$. For $\varphi<1$ (i.e., $\alpha<\alpha^T$), the hedge position obliges the government to make payments when income growth is high ($X^{\varphi}<H_\varphi\,X \Leftrightarrow X^{1-\varphi}>1/H_\varphi$) and it provides payments to the government whenever income growth is low. The more risk tolerant debtholders effectively insure the more risk averse taxpayers. The converse applies for $\varphi>1$.

To show that the expected return on the overall debt portfolio is above R^X whenever $\varphi<1$, it remains to be shown that investors demand a positive expected return for providing an income-insurance fund to the government. This can be shown as follows. For general φ, the return on debt is

$$\begin{aligned}1+R_t(s) &= \frac{D^*_t(s)}{D_{t-1}} = \frac{d_{1t}\,X-d_{\varphi t}\,X^{\varphi}}{(d_{1t}-H_\varphi\,d_{\varphi t})/(1+v)}\\ &= (1+v)\,X + \frac{d_{\varphi t}}{(d_{1t}-H_\varphi\,d_{\varphi t})/(1+v)}\ [H_\varphi\,X-X^{\varphi}],\end{aligned}$$

The proposition assumes positive debt, i.e., $d_{1t} > H_\varphi d_{\varphi t}$, so that the ratio is well-defined and the hedge position positive. (If the government were holding positive assets, $D_{t-1}<0$, a $\varphi<1$ hedge position would—as shown below—reduce R_t, but that would be a reduction in the return on assets rather than a low return on debt. To prevent confusion, the proposition is only stated for positive debt.) The difference between the expected return on government debt and the expected return on income-indexed securities is then

$$E[R_t]-R^x = \frac{d_{\varphi t}}{(d_{1t}-H_\varphi d_{\varphi t})/(1+v)} \; E[H_\varphi X-X^\varphi] \tag{A3}$$

so that

$$E[H_\varphi X-X^\varphi] = H_\varphi E[X]-E[X^\varphi] \geq 0 \tag{A4}$$

is a sufficient condition for the expected return on debt to be above R^x and strictly above the growth rate $x<R^x$. To prove (A4), note first that by definition of H_φ, (A4) is equivalent to

$$E[X^{\varphi-\alpha}] \;\; E[X] \geq E[X^\varphi] \;\; E[X^{1-\alpha}] \tag{A5}$$

Second, note that for any p $(0,1)$ and $q=1-p$ and any random variables (X_1,X_2), the Hölder inequality implies $E[X_1]^p \, E[X_2]^q \geq E[X_1{}^p \, X_2{}^q]$. For $\varphi<1$ and $\alpha>0$, $p = \alpha/(1-\varphi+\alpha)$ and $q = (1-\varphi)/(1-\varphi+\alpha)$ satisfy the Hölder assumptions. For $X_1=X^{\varphi-\alpha}$ and $X_2=X$, $X_1{}^p \, X_2{}^q = X^{p\,(\varphi-\alpha)+q} = X^{1-\alpha}$ and $X_1{}^q \, X_2{}^p = X^{q\,(\varphi-\alpha)+p} = X^\varphi$. Hence $E[X^{\varphi-\alpha}] \;\; E[X] = \left\{E[X^{\varphi-\alpha}]^p \, E[X]^q\right\} \left\{E[X^{\varphi-\alpha}]^q \, E[X]^p\right\} \geq E[X^\varphi] \;\; E[X^{1-\alpha}]$ follows from a double application of Hölder's inequality, proving (A4) and (A5) for $\varphi<1$.

The argument here provides some intuition for the case $\varphi>1$ not covered by Prop.1: Then the hedge fund will have a negative expected payoffs ($E[H_\varphi X-X^\varphi] \leq 0$ by analogous arguments), so that the government earns income from selling income insurance to bondholders. For the overall return on government debt to fall below x, such selling would have be done on a sufficiently large scale that the expected payoffs reduces the return on debt by more than $R^x-x=v\,E[X]$. Thus, $E[R]<x$ is possible for $\varphi>1$ (most likely if v is near zero and $\varphi>>1$), but $\varphi>1$ not a sufficient condition.

B. Proofs of Propositions 2-4:

To start, I will formally state the model and derive some relevant properties. Let $L^i_t \geq 0$ and $W^i_t \geq 0$ denote the amounts of individual borrowing and lending (private plus to the government) of agent i. Then the individual budget equations are

$$c^i_t + W^i_t - L^i_t = (Y_t - T_t) + (1+R_t)\ W^i_{t-1} - (1+R^*_t)\ L^i_{t-1}, \tag{B1}$$

if agent i receives income in period t (i=A for t even, i=B for t odd), and

$$c^i_{t+1} + W^i_{t+1} - L^i_{t+1} = (1+R_{t+1})\ W^i_t - (1+R^*_{t+1})\ L^i_t, \tag{B2}$$

because the agent receiving income in period t does not receive income in period t+1. The Kuhn-Tucker conditions imply that for all t and i,

$$\beta\,(1+R_{t+1})(c^i_{t+1}/c^i_t)^{-\alpha} \geq 1, \tag{B3a}$$

with equality if $W^i_t > 0$,

$$\beta\,(1+R^*_{t+1})(c^i_{t+1}/c^i_t)^{-\alpha} \leq 1 \text{ if } L^i_t > 0, \tag{B3b}$$

with equality if $L^i_t > 0$.

Equilibrium on financial markets requires that total lending equals end of period government debt D^*_t plus private borrowing, i.e.

$$L^A_t + L^B_t + D^*_t = W^A_t + W^B_t. \tag{B4}$$

The discussion in the text focuses on symmetric steady states, i.e., allocations in which the following income-ratios are constant: Agents i receiving income in period t have a consumption-income ratio $c^i_t/Y_t = \gamma$, asset holdings $W^i_t/Y_t = w$, and debt $L^i_t/Y_t = l$. Agents j not receiving income have a consumption-income ratio $c^j_t/Y_t = \gamma^*$, asset holdings $W^j_t/Y_t = w^*$, and debt $L^j_t/Y_t = l^*$. I have to show that such allocations exist and that they have the properties claimed in the text.

Lemma 1: $D_t \geq 0$ and $\kappa > 0$ $\quad$ $w>0$ and $l=0$.

Proof: Since $R^*_t > R_t$ for all t, agents will not simultaneously borrow and lend. Since marginal utility diverges to infinity for $c^i_t \rightarrow 0$, agents either lend a non-zero amount in the period with income ($W^i_t > 0$) or borrow a non-zero amount in the period without income ($L^i_{t+1} > 0$), or both; the same applies to agent j. Suppose (for contradiction) that w=0. Then $l^* > 0$ is necessary to avoid $\gamma^* = 0$, and

$l^*>0$ implies $w^*=0$. But $w=w^*=0$ and $l^*>0$ violates (B4); hence, $w=0$ must be impossible; $w>0$ implies $l=0$. QED.

The above lemma confirms that agents with income are always lenders.

<u>Lemma 2</u>: Interest rates are constant any symmetric steady state; $R=R_t$ satisfies $1+R = [\gamma^*/\gamma\ (1+x)]^{\alpha}/\beta$; if there is private borrowing, $R^*=R^*_t$ satisfies $1+R^* = [\gamma/\gamma^*\ (1+x)]^{\alpha}/\beta$.

Proof: Since $w>0$, (B3a) implies

$$\beta\,(1+R_{t+1})\ (\gamma^*\ Y_{t+1})^{-\alpha}/(\gamma\ Y_t)^{-\alpha} = \beta\,(1+R_{t+1})\ (\gamma^*\ (1+x)/\gamma)^{-\alpha} = 1;$$

and if $l^*>0$, (B3b) implies

$$\beta\,(1+R^*_{t+1})\ (\gamma\ Y_{t+1})^{-\alpha}/(\gamma^*\ Y_t)^{-\alpha} = \beta\,(1+R^*_{t+1})\ (\gamma\ (1+x)/\gamma^*)^{-\alpha} = 1. \text{ QED}$$

The above result motivates why I focus on symmetric steady states. Steady states with asymmetric allocations would have two-period deterministic cycles that would complicate the analysis without adding substantive insights.

<u>Lemma 3</u>: In a symmetric steady state without credit market frictions (if $\kappa=0$), the equilibrium interest rate r satisfies $1+r = (1+x)^{\alpha}/\beta$ and the consumption-income ratios are $\gamma=\gamma^*=(1-g)/2$.

Proof: For $\kappa=0$, agents with and without income face a common interest rate $r=R=R^*$, which implies $\gamma/\gamma^*=\gamma^*/\gamma$, so that $\gamma=\gamma^*$ and $(1+x)^{\alpha}/\beta$. The resource constraint $\gamma + \gamma^* + g = 1$ then implies $\gamma=\gamma^*=(1-g)/2$.

<u>Lemma 4</u>: For $D_t\geq 0$ and $\kappa>0$, all symmetric steady states fall into one of the following three categories:

(a): $(w>0, l^*>0, w^*=0)$; (b): $(w>0, l^*=0, w^*=0)$; (c): $(w>0, l^*=0, w^*>0)$.

Proof: Since l^* and w^* cannot both be positive and since $w>0$ by Lemma 1, the above is an exhaustive list of cases. QED.

I will show that the three cases correspond to the cases in Prop.2.

<u>Lemma 5</u>: In case (a), $1+R^* = (1+r)\sqrt{1+\kappa} = 1+R_{\kappa}$, $1+R = (1+r)/\sqrt{1+\kappa}$, and the individual consumption and wealth positions are the following linear functions of the initial debt-income ratio d:

$$w = w(d) = \bar{w} + w_d\ \ d;\ l^* = l^*(d) = \bar{w} - (\frac{1+x}{1+R} - w_d)\ d;$$

$$\gamma^* = \gamma^*(d) = (1+\frac{1+R}{1+x})\ \bar{w} + \gamma^*_d\ d;\ \text{and}\ \gamma=(1+\kappa)^{1/(2\alpha)}\ \bar{\gamma};$$

where $\bar{w} = (1\text{-}g)/\{(1+\frac{1+R}{1+x})\ (1+\kappa)^{1/(2\alpha)} + 1+\frac{1+R^*}{1+x}\}>0$

$$w_d = \{\frac{x\text{-}R}{1+R} +(1+\kappa)^{1/(2\alpha)}+ \frac{1+R^*}{1+R}\}/\{(1+\frac{1+R}{1+x})\ (1+\kappa)^{1/(2\alpha)}+1+\frac{1+R^*}{1+x}\} \quad (0,\frac{1+x}{1+R}),$$

and $\gamma^*{}_d = \frac{1+R}{1+x}\ \ \frac{R^*\text{-}R}{1+x}\ /\{1+\frac{1+R^*}{1+x} + (\frac{1+R}{1+x} +1)\ (1+\kappa)^{1/(2\alpha)} > 0$

Proof: Since $l^*>0$, R and R^* are given by Lemma 2. The two-period return on savings for any agent is therefore

$$(1+R)\ (1+R^*) = [\gamma^* /\gamma\ \ (1+x)]^{\alpha} /\beta\ [\gamma\ /\gamma^*\ (1+x)]^{\alpha} /\beta = (1+r)^2.$$

Combined with $(1+R^*)=(1+R)\ (1+\kappa)$, this implies the asserted rates. Note that $1+R = (1+x)]^{\alpha} /\beta/\sqrt{1+\kappa}$ implies $\gamma\ \ /\gamma\ = (1+\kappa)^{-1/(2\alpha)}$. In steady state, the government budget equation $D^*{}_t = G_t - T_t + (1+R)\ D^*{}_{t-1}$ implies

$$d^* = g - T_t / Y_t + (1+R)/(1+x)\ \ d^* = g - T_t / Y_t + d$$

where $d^*=D^*{}_t / Y_t$ is the end-of-period debt-income ratio. In terms of the start-of period debt-income ratio d, the disposable income of an agent with income is then

$$(Y_t\text{-}T_t)/ Y_t = (1\text{-}g) + d\text{-}d^* = (1\text{-}g) + (x\text{-}R)/(1+R)\ d$$

and the budget equations (B1, B2) reduce to

$$\gamma\ + w = (1\text{-}g) + (x\text{-}R)/(1+R)\ \ d - (1+R^*)/(1+x)\ l^* \qquad \text{(B5a)}$$

$$\gamma^* = (1+R)/(1+x)\ \ w + l^* \qquad \text{(B5b)}$$

Combined with $\gamma^* = \gamma\ (1+\kappa)^{-1/(2\alpha)}$ and the financial market equilibrium condition $l^*=w\text{-}d^*=w\text{-}(1+x)/(1+R)\ d$, (B5) implies

$$[\frac{1+R}{1+x}\ w + l^*]\ (1+\kappa)^{1/(2\alpha)} + w + \frac{1+R^*}{1+x}\ l^* = (1\text{-}g) + \frac{x\text{-}R}{1+R}\ d$$

$$w\ \{(1+\frac{1+R}{1+x})\ (1+\kappa)^{1/(2\alpha)} + 1+\frac{1+R^*}{1+x}\} = (1\text{-}g) + \{\frac{x\text{-}R}{1+R} +(1+\kappa)^{1/(2\alpha)} + \frac{1+R^*}{1+R}\}\ d$$

which implies the assertions about w(d). Notably, $w_d>0$ and

$$\frac{1+x}{1+R} - w_d = \frac{1+x}{1+R}\ \left(1 - \frac{\{\frac{x\text{-}R}{1+x} +(1+\kappa)^{1/(2\alpha)}+\frac{1+R^*}{1+x}\}}{\{(1+\frac{1+R}{1+x})\ (1+\kappa)^{1/(2\alpha)} + 1+\frac{1+R^*}{1+x}\}}\right)= \frac{1+x}{1+R}$$

$$\frac{\{\frac{1+R}{1+x} +(1+\kappa)^{1/(2\alpha)}\ \frac{1+R}{1+x}\}}{\{(1+\frac{1+R}{1+x})\ (1+\kappa)^{1/(2\alpha)} + 1+\frac{1+R^*}{1+x}\}} = \frac{1+(1+\kappa)^{1/(2\alpha)}}{\{(1+\frac{1+R}{1+x})\ (1+\kappa)^{1/(2\alpha)} + 1+\frac{1+R^*}{1+x}\}} >0.$$

The claims about $l^*(d)$ then follow from $l^*(d) = w(d) - \frac{1+x}{1+R}\, d$. Note that l^* is a decreasing function of d because of $w_d < (1+x)/(1+R)$. For consumption, the results about w and l^* imply

$$\gamma^*(d) = \frac{1+R}{1+x}\, w(d) + l^*(d)$$

$$= (1+\frac{1+R}{1+x})\, \bar{w} + \{(1+\frac{1+R}{1+x})\, w_d - \frac{1+x}{1+R}\}\, d.$$

so that $$\gamma^*_d = (1+\frac{1+R}{1+x})\, w_d - \frac{1+x}{1+R}$$

$$= \frac{R^*-R}{1+x} / \{1+\frac{1+R^*}{1+x} + (\frac{1+R}{1+x}+1)\,(1+\kappa)^{1/(2\alpha)}\} > 0. \text{ QED.}$$

Note that consumption depends positively on debt; this is because government debt reduces private debt that “wastes” intermediation cost (see below).

<u>Lemma 6</u>: In case (b) of Lemma 4, $w=(1+x)/(1+R)\, d$, $\gamma^* = d$, $\gamma = 1-g-d$, and $R = R(d)$ $[R_K,r]$ is a strictly increasing function of d,

$$1+R = [\frac{d}{1-g-d}\,(1+x)]^{\alpha}/\beta \tag{B6}$$

Proof: $\gamma^* = d$ and $\gamma = 1-g-d$ follow from (B5) for $l^*=0$ and $w=d$. Inserted into the equation for R in Lemma 2, we find $\beta(1+R)[d/(1-g-d)\,(1+x)]^{-\alpha} = 1$, which implies (B6). The restriction R $[R_K,r]$ can be proven by contradiction. For any R, (B3a) implies $\beta(1+R)[\gamma^*/\gamma\,(1+x)]^{-\alpha} = 1$. If $R<R_K$, this would imply $[\gamma^*/\gamma\,(1+x)]^{\alpha}/\beta < 1+R_K = (1+x)^{\alpha}/\beta/\sqrt{1+\kappa}$ $[\gamma^*/\gamma]^{\alpha} < 1/\sqrt{1+\kappa}$ $[\gamma/\gamma^*\,(1+g)]^{\alpha}/\beta > \sqrt{1+\kappa}\,(1+x)]^{\alpha}/\beta = 1+R^*$, a violation of (B3b). (Agents without income would want to borrow.) in periods without income. If R>r, then $[\gamma^*/\gamma\,(1+x)]^{\alpha}/\beta = (1+x)^{\alpha}/\beta$ $\gamma^*>\gamma$ $[\gamma/\gamma^*\,(1+x)]^{\alpha}/\beta\,(1+R)<1$, a violation of (B3a). (Agents without income would want to buy bonds.) QED.

Note that while (B6) defines a well-defined function R(d), a substitution of $d=(1+R)/(1+x)\, d^*$ in (B6) would yield a correspondence $R(d^*)$ with potentially multiple solutions. This because for high α the income effect of changes in R may dominate the substitution effect so that the government cannot necessarily sell more debt by offering marginally higher interest rates. But a unique solution for R is obtained, if the government sets the face value of discount bonds maturing next period (i.e., next period’s initial debt-income ratio d) and lets the market determine the discount rate and the current value D_t.

Lemma 7: In case (c) of Lemma 4, consumption and interest rates are the same as in the frictionless markets allocation, $\gamma=\gamma^*=(1-g)/2$, and R=r. Asset holdings are

$$w^* = w^*(d) = \{d - (1-g)/2\}/(1+\frac{1+R}{1+x}) \qquad \text{(B7)}$$

and

$$w = w(d) = \{\frac{1+x}{1+R}\ d + (1-g)/2\}/(1+\frac{1+R}{1+x})$$

Proof: If $w^*>0$, (B3a-b) imply $1+R = [\gamma^*/\gamma\ (1+x)]^\alpha/\beta=[\gamma/\gamma\ \ (1+x)]^\alpha/\beta$ so that $\gamma=\gamma^*$ and R=r; $\gamma=\gamma^*=(1-g)/2$ follows from the resource constraint. The budget equations

$$\gamma + w = (1-g) + (x-R)/(1+R)\ d + (1+R)/(1+x)\ w^*$$

$$\gamma^* = (1+R)/(1+x)\ w - w^*$$

and the market equilibrium condition $w=(1+x)/(1+R)\cdot d-w^*$ imply

$$\frac{1+R}{1+x}\cdot w-w^* + w = (1-g) + \frac{x-R}{1+R}\cdot d + \frac{1+R}{1+x}\cdot w^*$$

$$\Leftrightarrow \qquad (1-g) + 2\cdot(1+\frac{1+R}{1+x})\cdot w^* = \{\frac{1+R}{1+x} + 1 - \frac{x-R}{1+x}\}\cdot\frac{1+x}{1+R}\cdot d = 2\cdot d$$

$$\Leftrightarrow \qquad w^* = w^*(d) = \{d - (1-g)/2\}/(1+\frac{1+R}{1+x}).$$

as claimed; w(d) then follows from $w=(1+x)/(1+R)\cdot d-w^*$. QED.

Proof of Proposition 2:

Define $d_1 = \bar{w}/[(1+x)/(1+R) - w_d] > 0$ to be the unique value of d at which $l^*(d)=0$ in Lemma 5. Define $d_2 = (1-g)/2$ to be the unique value of d at which $w^*(d)=0$ in Lemma 7. At $d=d_1$, Lemma 5 and Lemma 6 specify the same allocation, so that d_1 describes the boundary between cases a and b. At $d=d_2$, Lemma 6 and Lemma 7 specify the same allocation, so that d_2 describes the boundary between cases b and c. Case a applies for $d<d_1$ and case c applies for $d>d_2$. Since

$$d_1 = \bar{w}/(\frac{1+x}{1+R} - w_d) = \frac{1-g}{1+(1+\kappa)^{1/(2\alpha)}} < d_2$$

for $\kappa>0$, case b is well defined and cases a and c are non-overlapping. For $d<d_1$, (B6-B7) would imply $R<R_\kappa$, making case b inapplicable, and for $d>d_2$, (B6-B7) would imply R>r, making case b also inapplicable. Hence, for any value $d\geq 0$, one and only one of the three cases applies. The properties claimed in Prop.2 are proven in Lemmas 5-7. Lemma 4 shows that there are no other symmetric steady states. QED.

Moving on to the proof of Prop.3, a preliminary discussion of initial conditions is needed. The easy part will be to show that more debt increases welfare for $d<d_1$, because for $l^*(d)>0$, more debt reduces the resources "wasted" in monitoring loans and yields an increase in both γ and γ^* (see Lemma 5). It is more difficult to show a Pareto-improvement for d $[d_1,d_2)$, because there is no wasteful monitoring for $d \geq d_1$ and because higher debt raises γ^* at the expense of a lower γ value (see Lemma 6). More debt is nonetheless efficient, because agents without income operate at a corner solution at which the shadow value of borrowing strictly exceeds the interest rate. But a Pareto-improvement is obtained only if the debt-income ratio is increased in (at least) two steps so that each type receives a share of the government receipts from the additional debt issues. This is explained in the following lemmas. The proofs are constructive and show which initial distribution is required to satisfy Prop.3.

Lemma 8: A symmetric steady state requires a set of appropriately chosen initial conditions. In period t, agents i without income must be endowed with initial assets valued at $(1+R)/(1+x) \cdot w \cdot Y_t$. Depending on which case applies, agents who receive income must be endowed either with initial liabilities $(1+R^*)/(1+x) \cdot l^* \cdot Y_t$, if $l^*>0$ in case (a), or with initial assets $(1+R)/(1+x) \cdot w \cdot Y_t$, if $w>0$ in case (c), or with no assets and no liabilities, if case (b) applies.

Proof: The above asset positions are necessary to satisfy the budget equations (B1, B2) in period zero. QED.

Note that in case (b), the assets-income ratio equals the debt-income ratio, $(1+R)/(1+x) \cdot w = d$.

Lemma 9: The government can implement a shift from any symmetric steady state with debt level $d\sim$ $[d_1,d_2]$ within case (b) to another symmetric steady state with a marginally different debt-income ratio d (d_1,d_2) within two periods. For the debt level after the first transition period, d^+, the government can choose an arbitrary value between $d\sim$ and d without affecting the new steady state.

Proof: Without loss of generality, let period t=0 be the first transitional period, so that agent A earns income and agent B does not. (For t>0, the alternative allocations are identical prior to the transition). Since we start with a case (b) symmetric steady state at $d\sim$, $W^A_{-1}=0$ and $(1+R_0) \cdot W^B_{-}$

$_{1}/Y_0 = d\sim$ by Lemma 8. For $t \geq 2$, a symmetric steady state at d is supposed to be implemented, and d^+ is supposed to be arbitrary in between d and d~.

Provided agents without income do not borrow or save during the transition, their consumption is $\gamma^B{}_0 = c^B{}_0 / Y_0 = d\sim$ and $\gamma^A{}_1 = c^A{}_1 / Y_1 = d^+$. For agents with income, the taxes necessary to obtain an debt-income ratio of d^+ and d in periods 1 and 2 must satisfy

$$D_0{}^* / Y_0 = (1+x)/(1+R_1) \cdot d^+ = g - T_0 / Y_0 + d\sim$$

in period 0, taking into account the new debt issue and the interest rate effect of the change, and

$$D_1{}^* / Y_1 = (1+x)/(1+R_2) \cdot d = g - T_0 / Y_0 + d^+$$

in period 1. Therefore agent A in period 0 has income and consumption

$$(Y_0 - T_0)/Y_0 = (1-g) + (1+x)/(1+R_1) \cdot d^+ - d\sim$$

$$\gamma^A{}_0 = c^A{}_0 / Y_0 = (Y_0 - T_0)/Y_0 - (1+x)/(1+R_1) \cdot d^+ = 1-g-d\sim$$

showing that the period-0 consumption is still at the old steady state. Similarly, agent B in period 1 has income

$$(Y_1 - T_1)/Y_1 = (1-g) + (1+x)/(1+R_1) \cdot d - d^+,$$

so that $\gamma^B{}_1 = c^B{}_1 / Y_1 = 1-g-d^+$ is in between the old and the new steady state consumption ratio, if d^+ is in between d and d~. Given these consumption values, the interest rates in periods 0 and 1 are

$$1+R_0 = \left[\frac{d^+}{1-g-d\sim} \cdot (1+x)\right]^{\alpha} / \beta \text{ and } 1+R_1 = \left[\frac{d}{1-g-d^+} (1+x)\right]^{\alpha} / \beta.$$

Since $d^+/(1-g-d\sim) \to d/(1-g-d)$ and $d/(1-g-d^+) \to d/(1-g-d)$ for $|d-d\sim| \to 0$, keeping in mind $|d-d^+| \leq |d-d\sim|$, R_0 and R_1 converge to $R(d)$ (R_K, r) for $|d-d\sim| \to 0$ and therefore satisfy the restrictions of case (b) for a sufficiently small change in debt, i.e., R_0, R_1 (R_K, r). This confirms that agents without income do not borrow or save during the transition. Since the initial allocation at $t=2$ satisfies Lemma 8, the allocation for $t \geq 2$ is the symmetric steady state associated with the debt-income ratio d. QED.

<u>Lemma 10</u>: For any d~ $[d_1, d_2)$ below the upper bound of case (b), a Pareto-improving transition to a symmetric steady state with a higher debt-income ratio d (d_1, d_2) can be implemented over two periods by setting the debt-income ratio in the transition period equal to $d^+ = d\sim + \delta/(1+\delta)\,(d-d\sim)$, where $\delta = (1+x)/(1+r) < 1$.

Proof: Using the notation of Lemma 9, one has to show that the allocation $\{\gamma^A{}_0=\gamma(d\sim),\gamma^B{}_0=\gamma^*(d\sim)\}$ at t=0, $\{\gamma^A{}_1=d^+,\gamma^B{}_1=1-g-d^+\}$ at t=1, and $\{\gamma=\gamma(d),\gamma^*=\gamma^*(d)\}$ for $t\geq 2$ is Pareto-preferred to $\{\gamma(d\sim),\gamma^*(d\sim)\}$ for all t. The utility values are

$$U^A = (c^A{}_0)^{1-\alpha}/(1-\alpha) + \beta\,(c^A{}_1)^{1-\alpha}/(1-\alpha) + \sum_{t\geq 1} \beta^{2t}\,(\gamma\ Y_{2t})^{1-\alpha}/(1-\alpha)$$

$$+ \sum_{t\geq 1} \beta^{2t+1}\,(\gamma^*\ Y_{2t+1})^{1-\alpha}/(1-\alpha)$$

$$= \frac{Y_0{}^{1-\alpha}}{(1-\alpha)}\ \left[(\gamma^A{}_0)^{1-\alpha} + \delta\,(\gamma^A{}_1)^{1-\alpha} + \frac{\delta^2}{1-\delta^2}\,\{\gamma^{1-\alpha} + \delta\,(\gamma\ \ )^{1-\alpha}\}\right]$$

using $\beta(1+x)^{(1-\alpha)} = (1+x)/(1+r) = \delta < 1$, and

$$U^B = (c^B{}_0)^{1-\alpha}/(1-\alpha) + \beta\,(c^B{}_1)^{1-\alpha}/(1-\alpha) + \sum_{t\geq 1} \beta^{2t}\,(\gamma^*\ Y_{2t})^{1-\alpha}/(1-\alpha)$$

$$+ \sum_{t\geq 1} \beta^{2t+1}\,(\gamma\ Y_{2t+1})^{1-\alpha}/(1-\alpha)$$

$$= \frac{Y_0{}^{1-\alpha}}{(1-\alpha)}\ \left[(\gamma^A{}_0)^{1-\alpha} + \delta\,(\gamma^B{}_1)^{1-\alpha} + \frac{\delta^2}{1-\delta^2}\,\{(\gamma\ \ )^{1-\alpha} + \delta\,(\gamma)^{1-\alpha}\}\right].$$

Since $d\gamma^B{}_0/dd = d\gamma^A{}_0/dd = 0$, $d\gamma^B{}_1/dd = -d\gamma^A{}_1/dd = -\delta/(1+\delta)$ and $d\gamma^*/dd=-d\gamma/dd = 1$, the derivatives of U^A and U^B with respect to d are

$$dU^A/dd = \frac{Y_0{}^{1-\alpha}\ \delta}{(1-\alpha)}\left[(\gamma^A{}_1)^{-\alpha}\ \frac{\delta}{1+\delta} + \frac{\delta}{1-\delta^2}\ \{-(\gamma)^{-\alpha} + \delta\,(\gamma\ \ )^{-\alpha}\}\right]$$

$$= \frac{Y_0{}^{1-\alpha}}{(1-\alpha)}\ \frac{\delta^2}{1-\delta^2}\ \left[(1-\delta)\ \{(\gamma^A{}_1)^{-\alpha} - (\gamma)^{-\alpha}\} + \delta\,\{(\gamma\ \ )^{-\alpha} - (\gamma)^{-\alpha}\}\right]$$

$$dU^B/dd = \frac{Y_0{}^{1-\alpha}\cdot\delta}{(1-\alpha)}\cdot\left[-(\gamma^B{}_1)^{-\alpha}\cdot\frac{\delta}{1+\delta} + \frac{\delta}{1-\delta^2}\cdot\{(\gamma\ \ )^{-\alpha} - \delta\cdot(\gamma)^{-\alpha}\}\right]$$

$$= \frac{Y_0{}^{1-\alpha}}{(1-\alpha)}\cdot\frac{\delta^2}{1-\delta^2}\cdot\left[(1-\delta)\cdot\{(\gamma\ \ )^{-\alpha} - (\gamma^B{}_1)^{-\alpha}\} + \delta\cdot\{(\gamma\ \ )^{-\alpha} - (\gamma^A)^{-\alpha}\}\right]$$

Both derivatives are strictly positive, because $\gamma\ <\gamma$ and because $\gamma^B{}_1>\gamma$ and $\gamma^A{}_1<\gamma$ for small changes in d. QED.

<u>Proof of Proposition 3</u>:

For $d\geq d_2$, $\gamma=\gamma^*=(1-g)/2$. The allocation of consumption does not depend on debt. (Ricardian neutrality applies.)

For $d\leq d_1$, Lemma 5 shows that γ and γ^* are strictly increasing functions of d. Adding up (B5a) and (B5b), one finds $\gamma + \gamma^* = (1-g) - (R^*-R)/(1+x))\cdot l^*(d)$ so that $1-(\gamma+\gamma^*+g) = (1+\kappa)(1+R)/(1+x))\cdot l^*(d)$ can be interpreted as the resources "wasted" in monitoring loans. Since $l^*(d)$ is declining in d, more debt reduces waste. Intuitively, debt is efficiency-enhancing because it avoids costly private monitoring.

For d $[d_1,d_2)$, Lemma 10 shows that a marginal increase in d can be implemented in a Pareto-improving way for any d $[d_1,d_2)$. QED.

Proof of Proposition 4:

For Prop.4, the model must be modified slightly. With collection cost k>0, the disposable income Y_t-T_t in (B1) is replaced by $Y_t - T_t \cdot (1+k)$. In case (a), (B5a) is replaced by

$$\gamma + w = (1-g\cdot(1+k)) + (x-R)/(1+R)\cdot d\cdot(1+k) - (1+R^*)/(1+x)\cdot l^*.$$

If $x>R=R_K$, k>0 reinforces the positive effect of d on γ and γ^*, so that higher debt is still improves welfare. For case (b), we have $\gamma^* = d$,

$$\gamma = (1-g\cdot(1+k)) - d - k\cdot(R-x)/(1+R)\cdot d, \tag{B8}$$

and (B6) must be replaced by the implicit function

$$1+R = \left[\frac{d}{(1-g\cdot(1+k))-d-k\cdot(R-x)/(1+R)\cdot d}\cdot(1+x)\right]^{\alpha}/\beta. \tag{B6'}$$

Define $\Omega(d) = d[(R-x)/(1+R)\cdot d]/dd = (R-x)/(1+R) + d\cdot(1+x)/(1+R)^2\cdot R'(d)$, then the derivative of R(d) is implicitly characterized by

$$R'(d) = (1+R)\cdot\alpha\cdot\{1/d - 1/\gamma\cdot d\gamma/dd\} = (1+R)\cdot\alpha\cdot\{1/d + 1/\gamma + 1/\gamma\cdot k\cdot\Omega(d)\}$$

$$= (1+R)\cdot\alpha\cdot\{1/d + 1/\gamma + 1/\gamma\cdot k\cdot(R-x)/(1+R)\} + d/\gamma\cdot k\cdot(1+x)/(1+R)^2\cdot R'(d)$$

$$\Rightarrow \qquad R'(d) = \frac{(1+R)\cdot\alpha/d\{1+d/\gamma+d/\gamma\cdot k\cdot(R-x)/(1+R)\}}{1-1/\gamma\cdot k\cdot d\cdot(1+x)/(1+R)^2} > 0$$

shows that R(d) is still a well-defined, strictly increasing function of d. In case (c), the resource constraint $\gamma + \gamma^* + g = 1 - k\cdot(g+(r-x)/(1+r)\cdot d)$ implies

$$\gamma = \gamma^* = (1-g\cdot(1+k))/2 - k/2\cdot(r-x)/(1+r)\cdot d,$$

so that utility is a strictly <u>decreasing</u> function of debt (since r>x), unlike the case without collection cost. Thus, the Pareto-optimal debt level must be within the range of case (b). The boundary between cases (b) and (c) is now

$$d_2 = \frac{(1-g\cdot(1+k))/2}{1 + k\cdot(r-x)/(1+r)/2}.$$

The utility effects of marginal changes in d within case (b) can be computed as follows. First note that a shift between steady states is still always feasible, because even with k>0, the transitional interest rates R_0 and R_1 in Lemma 9 will remain in the (R_K,r) interval for $|d-d\sim| \to 0$. Second, note that $d\gamma^*/dd=1$ in as in Lemma 10; but different from Lemma 10,

$$d\gamma / dd = -1 - k\cdot\Omega(d)$$

For any transitional debt $d^+ = d\sim + \Delta\cdot(d-d\sim)$, where $0\leq\Delta\leq 1$, we have $d\gamma^A{}_1/dd = \Delta$ and $d\gamma^B{}_1/dd = -\Delta - k\cdot\Omega(d)$. Taking the derivatives of U^A and U^B with respect to d, one finds

$$dU^A/dd = \frac{Y_0{}^{1-\alpha}\cdot\delta}{(1-\alpha)}\cdot\left[(\gamma^A{}_1)^{-\alpha}\cdot\Delta) + \frac{\delta}{1-\delta^2}\cdot\{-(\gamma)^{-\alpha}\cdot(1+k\cdot\Omega(d))+\delta\cdot(\gamma\ \ )^{-\alpha}\}\right]$$

$$= \frac{Y_0{}^{1-\alpha}}{(1-\alpha)}\cdot\frac{\delta^2}{1-\delta^2}\cdot(\gamma)^{-\alpha}\cdot\left[(1-\delta)\cdot\{\Delta\cdot\frac{1+\delta}{\delta}\cdot(\gamma^A{}_1)^{-\alpha}/\gamma^{-\alpha}-1\}+\ \delta\cdot\{(\gamma\ \ )^{-\alpha}/\gamma^{-\alpha}-1\}\right]$$

$$-\ k\cdot\Omega(d)\cdot\frac{Y_0{}^{1-\alpha}\cdot\delta}{(1-\alpha)}\cdot\frac{\delta}{1-\delta^2}\cdot(\gamma)^{-\alpha}$$

$$dU^B/dd = \frac{Y_0{}^{1-\alpha}\cdot\delta}{(1-\alpha)}\cdot\left[-(\gamma^B{}_1)^{-\alpha}\cdot(\Delta+k\cdot\Omega(d)) + \frac{\delta}{1-\delta^2}\cdot\{(\gamma\ \ )^{-\alpha}-\delta\cdot(\gamma)^{-\alpha}\cdot(1+k\cdot\Omega(d))\}\right]$$

$$= \frac{Y_0{}^{1-\alpha}}{(1-\alpha)}\cdot\frac{\delta^2}{1-\delta^2}\cdot(\gamma)^{-\alpha}\cdot\left[(1-\delta)\cdot\{(\gamma\ \ )^{-\alpha}/\gamma^{-\alpha}-\Delta\cdot\frac{1+\delta}{\delta}\cdot(\gamma^B{}_1)^{-\alpha}/\gamma^{-\alpha}\}+\delta\cdot\{(\gamma\ \ )^{-\alpha}/\gamma^{-\alpha}-1\}\right]$$

$$-\ k\cdot\Omega(d)\cdot\frac{Y_0{}^{1-\alpha}\cdot\delta}{(1-\alpha)}\cdot\left[(\gamma^A{}_0)^{-\alpha} + \frac{\delta^2}{1-\delta^2}\cdot(\gamma)^{-\alpha}\right]$$

To eliminate Δ, note that an increase in d cannot be Pareto-improving, if there is some Δ value for which both dU^A/dd and dU^B/dd are negative, because any different Δ value would further reduces either dU^A/dd or dU^B/dd. Hence, I will work with $\Delta=\delta/(1+\delta)$. From Lemma 6, the ratio between any pair of γ and γ^* values is bounded from above by $(1+\kappa)^{1/2\alpha}$ so that $0<(\gamma\ \ )^{-\alpha}/\gamma^{-\alpha}-1\leq\sqrt{1+\kappa}-1$. Hence

$$dU^A/dd \leq \frac{Y_0{}^{1-\alpha}}{(1-\alpha)}\cdot\frac{\delta}{1-\delta^2}\cdot(\gamma)^{-\alpha}\cdot\left[\delta\cdot(\sqrt{1+\kappa}-1) - \delta\cdot k\cdot\Omega(d)\right]$$

and

$$dU^B/dd \leq \frac{Y_0{}^{1-\alpha}}{(1-\alpha)}\cdot\frac{\delta}{1-\delta^2}\cdot(\gamma)^{-\alpha}\cdot\left[\delta\cdot(\sqrt{1+\kappa}-1) - k\cdot\Omega(d)\right]$$

For $d\geq d_K$, we have $R\geq x$, $R'(d)\geq\alpha\cdot(1+R)\cdot(1/d+1/\gamma)$, and $\Omega(d)\geq\alpha\cdot(1+x)/(1+r)\cdot(1+d/\gamma)$. Hence,

$$k\geq (\sqrt{1+\kappa}-1)/\{\alpha\cdot(1+x)/(1+r)\cdot(1+d_K/\gamma)\} \tag{B9}$$

is a sufficient condition for $dU^A/dd\leq 0$ and $dU^B/dd<0$ for all $d\geq d_K$ and $\Delta=\delta/(1+\delta)$. Given (B9), a debt-reduction is Pareto-improving for $d\geq d_K$. Hence, any Pareto-optimal allocation must have a debt-income ratio within the $[d_1,d_K)$ interval. QED.

Note that $dU^A/dd\leq 0$ and $dU^B/dd<0$ may be negative for all $d\geq d_1$ (recall that the formulas apply only for $d\geq d_1$), in which case there is a unique Pareto-optimum at $d=d_1$; this is plausible for high k. But $\Omega(d)$ may turn negative at some d value $d^{-}>d_1$, in which case it is straightforward to show (following Lemma 10) that any Pareto-optimum must lie in the interval (d^{-},d_K).

Sketch of an intermediation cost model with uncertainty

This section outlines a model with intermediation cost and uncertainty as mentioned at the end of Section 4. Consider the model of Section 4, but now assume that income Y_t grows at an i.i.d growth rate x_t, as in Section 3. Since types A and B have equal risk aversion, it is clear that the efficient form of private credit is income-indexed debt. Assume for now that government debt is also income-indexed. In this modified economy, results analogous to Prop.2-4 and Lemmas 1-10 can be derived, as follows.

Income-indexed Debt: Let r_t, R_t, and R^*_t now denote the returns on income-indexed loans that satisfy the Euler equations

$$\beta \cdot E[(1+r_{t+1})(c^i_{t+1}/c^i_t)^{-\alpha}] = 1 \text{ in case of frictionless markets,}$$

$$\beta \cdot E[(1+R_{t+1})(c^i_{t+1}/c^i_t)^{-\alpha}] \geq 1 \text{ for lenders, and}$$

$$\beta \cdot E[(1+R^*_{t+1})(c^i_{t+1}/c^i_t)^{-\alpha}] \leq 1 \text{ for borrowers.}$$

If one divides these returns into income growth times a proportionality factor, $1+r_{t+1} \equiv (1+x_{t+1})\cdot(1+v)$, $1+R_{t+1} \equiv (1+x_{t+1})\cdot(1+V)$, $1+R^*_{t+1} \equiv (1+x_{t+1})\cdot(1+V^*)$, the proportionality factors must be constant in a symmetric steady state, as in Lemma 2. Moreover, one can calculaten them as $1+V = [\gamma^*/\gamma]^{\alpha}/E[\beta\,(1+x)^{1-\alpha}]$, $1+V^* = [\gamma/\gamma^*]^{\alpha}/E[\beta\,(1+x)^{1-\alpha}]$, and $1+v = 1/E[\beta\,(1+x)^{1-\alpha}]>1$. Note that $v>0$ is as in Section 3.

In Lemma 5 and following, the statements about returns can then be replaced by analogous statements about v, V, and V^*, because $(1+V)\cdot(1+V^*)=(1+v)^2 = (1+V)^2\cdot(1+\kappa)$. Hence, $V<0$ is possible for $\kappa>0$ even though $v>0$. With uncertainty, the expected primary surplus in steady state is $E_{t-1}Z_t = E[R_t-x_t]\cdot d\cdot Y_{t-1} = V\cdot d\cdot Y_{t-1}\cdot(1+x)$; $V<0$ implies permanent expected deficits. In this sense, the results in Section 4 do not rely critically on the absence of uncertainty. Both in Sections 3 and 4, expected primary deficits occur if and only if $E[R_t-x_t]<0$. In Section 3, this inequality requires safe debt, i.e. risk taking by the government on behalf of taxpayers. In Section 4, this inequality can be obtained for any type of debt for sufficiently high intermediation cost.

Other Debt: The above assumes income-indexed government debt. To generate non-trivial welfare implications for alternative debt management policies, a model with distinct taxpayers and debtholders is again instructive; and to prevent asset pricing complications, it is best to add a production setting. Specifically, one may further modify the above model as follows: Let there be three types of agents, A, B, and T. Type T is a sequence of one-period lived taxpayers with endowments Y_t, as in Section 3. Types A and B are infinitely lived, have initial endowments and access to production, like type I in Section 3, but only every second period. That is, for each state of nature s in period t+2, there is a production technology that produces one consumption good in period t+2, state s, at a cost of $p^2{}_t(s)$ units of consumption in period t. Productivities are such that $[\beta\cdot\pi^2{}_t(s)/p^2{}_t(s)]^{1/\alpha} = [1+x_{t+1}(s)]\cdot[1+x_{t+2}(s)]$ for all t and s, where $\pi^2{}_t(s)$ is the conditional probability of state s. Agent A has access to production in even periods, agent B in odd periods. These assumptions assure that (1) the consumption growth of type A and B agents matches the growth rate of endowments over any two-period interval for any government tax and debt policy; and (2), if the government issues income-indexed debt, the allocation is the same as the endowment model with income-indexed debt described above (under "income-indexed debt").

With income-indexed debt, the economy suffers from intertemporal consumption distortions induced by intermediation cost, but marginal rates of substitution are proportional across states of nature for all agents. Government debt not indexed to income is then inefficient because it destroys the proportionality between marginal rates of substitution. Hence, the main claim of Prop.1 that income-indexed debt is optimal with equal risk aversion remains valid in this more elaborate model.