Appendix to Intergenerational Risk Sharing and Fiscal Policy:

Notation Table, Proofs, and Supplementary materials

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Table of Contents

All references to equations and section numbers refer to the main text.

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Part A: Notation Table

General conventions:

- Subscript t denotes the time period.
- Superscript *1* indicates working-age variables (first period of life).
- Superscript *2* indicates retirement-age variables (second period of life).
- Hats (^) denote percentage deviations from a deterministic steady state.
- Superscript star (*) denotes objects associated with planning solutions.
- Generic symbols: x_t for a variable; s_t for a state variable;

  $\pi_{x,s}$ = Coefficient on $\hat{s}_t$ in the log-linearized dynamics for $\hat{x}_t$ (or equivalently: derivative of $\hat{x}_t$ with respect to $\hat{s}_t$, elasticity of x_t with respect to s_t);

  σ_x = Steady state share of x in output.
- Variable labels without time subscripts denote deterministic steady state values.

Functions: $U(\cdot)$ = utility; $F(\cdot)$ = production; $G(\cdot)$ = capital accumulation.

Operator: $E_t[\cdot]$ = expectation conditional on period-t information.

Model variables (in sequence as defined, but related variables grouped together):

U_t = lifetime utility of the cohort entering working-age in period t.

c_t^1 = consumption of a worker (first period of life) in period t

c_t^2 = consumption of a retiree (second period of life) in period t

l_t = leisure of a worker in period t.

Y_t = aggregate output in period t.

K_t = aggregate capital stock in period t.

L_t = aggregate labor input into period-t production

N_t = size of the working-age cohort in period t.

I_t = aggregate capital investment in period t.

A_t = period-t value of the stochastic trend in total factor productivity.

$a_t = A_t / A_{t-1}$ = permanent productivity shock.

z_t^F = vector of shocks to the production function F.

z_t^G = vector of shocks to the capital accumulation function G.

$z_t = (z_t^F, z_t^G)$ = vector of shocks to F and G. (Note: In Sections 3-4, z_t reduces to a scalar that represents a temporary productivity shock. Then $z_t = z_t^F$, and there is no z_t^G).

h_t = state of nature in period t.

Q_t = value of capital in period t in terms of consumption (Tobin's Q).

R_t = return on capital between periods t-1 and t.

k_t^1 = savings of a worker in period t (all invested in capital).

w_t = wage rate in periods t (equals wage income in versions without leisure).

b_t = government net transfers per retiree in period t, a.k.a., retirees' generational account.

y_t^1 = disposable income of a worker in period t (wage income minus net taxes).

$\kappa_t = k_t^1 / y_t^1$ = savings rate of a worker in period t (savings/disposable income).

ω_t = welfare weight on generation t relative to generation t-1 in the planning problem.

$\tilde{\omega}_t$ = welfare weight on generation t relative to t-1 implied by a market allocation.

S_t = Markov state vector.

$k_t = K_t / (A_{t-1} N_{t-1})$ = capital-labor ratio.

$\chi_t = c_t^1 / A_t$ = working-age consumption deflated by the productivity index.

Constants:

$\gamma_N = N_t / N_{t-1}$ = population growth rate.

$\gamma_A = E[a_t]$ = average growth rate of the productivity trend *A*.

δ = depreciation rate (applicable only when depreciation is constant).

ε = elasticity of intertemporal substitution.

ρ = discount factor for time preference.

α = capital share (constant in case of Cobb-Douglas; steady state value in general.)

$v = (1-\delta)/R$ = steady state share of old capital in the return on capital.

v_0 = cutoff value for *v* defined in equation (22).

$r = R/(\gamma_N \gamma_A)$ = steady state ratio of returns to population plus productivity growth.

$(\tilde{\varepsilon}, \tilde{h})$ = parameters in the habit preference utility function defined in equation (23).

$\bar{h}$ = steady state ratio of habit stock to retirement consumption in Section 4.1.

ε_{KL} = elasticity of substitution between capital and labor in Section 4.3.

Part B: Proofs of Propositions 1-3

The linearizations in Section 3-4 are taken around the deterministic steady state obtained by suppressing the shocks, as follows. Let time-subscripts be omitted to denote steady state values, and let σ_x denote steady state output-shares (specifically: σ_b for $b_t\, {}^{N_{t-1}}\!/_{Y_t}$, σ_{c1} for $c_t^1\, {}^{N_t}\!/_{Y_t}$, σ_{c2} for $c_t^2\, {}^{N_{t-1}}\!/_{Y_t}$, and σ_K for ${}^{K_{t+1}}\!/_{Y_t}$). For every combination of preference and technology parameters $(\varepsilon,\rho,\alpha,\delta,\gamma_A,\gamma_N)$ and for given ω, equations

(a) $\rho(\frac{\sigma_{c1}\gamma_A\gamma_N}{\sigma_{c2}})^{-1/\varepsilon} R = 1$; (b) $R = \frac{\alpha\gamma_A\gamma_N}{\sigma_K} + (1-\delta)$;

(c) $\sigma_{c1} + \sigma_{c1} + \sigma_K = 1 + \frac{(1-\delta)}{\gamma_A\gamma_N}\sigma_K$; (d) $(\sigma_{c1})^{-1/\varepsilon} = \rho(\sigma_{c1}\gamma_N)^{-1/\varepsilon} / \omega$,

determine unique steady state values of the return R and the output shares $(\sigma_{c1}, \sigma_{c2}, \sigma_K)$ in the efficient allocation. Given this solution, an implied level of transfers σ_b is obtained from

(e) $\sigma_{c2} = \alpha + \frac{(1-\delta)}{\gamma_A\gamma_N}\sigma_K + \sigma_b$.

Equations (a-e) are the steady state analogs of: the Euler equation (4), the return on capital (3), the resource constraint (1), the efficiency condition (8), and the retiree budget equation (5).

For market allocations, parameters $(\varepsilon,\rho,\alpha,\delta,\gamma_A,\gamma_N)$ and given transfers σ_b uniquely determine R and $(\sigma_{c1}, \sigma_{c2}, \sigma_K)$ from (a-c) and (e). Given these values, (d) can be used to infer the weight $\omega = \tilde{\omega}$ that yields the same steady state transfers. This dual interpretation of (a-e) illustrates how a given level of transfers determines a welfare weight, and it shows why market allocations and comparable efficient allocations have, by construction, the same steady states.

<u>Proposition 1(a-b)</u>: Under the assumptions of Section 3, one obtains the log-linearizations:

(A1) $\sigma_K \cdot (\hat{k}_t + \hat{a}_t) + \sigma_{c1} \cdot \hat{c}_t^1 + \sigma_{c2} \cdot \hat{c}_t^2 = (\alpha + \sigma_v) \cdot \hat{k}_{t-1} + (1-\alpha) \cdot (\hat{a}_t + \hat{z}_t)$, from (1-2);

(A2) $\hat{R}_t = \pi_{R,a} \cdot (\hat{a}_t + \hat{z}_t - \hat{k}_{t-1})$, from (3);

(A3) ${}^1\!/_\varepsilon \cdot (\hat{c}_t^1 - \hat{a}_t) + E_t[\hat{R}_{t+1} - {}^1\!/_\varepsilon \cdot \hat{c}_{t+1}^2] = 0$, from (4);

(A4) $\hat{c}_t^2 = (1-\frac{\sigma_b}{\sigma_{c2}})\cdot(\hat{R}_t + \hat{k}_{t-1}) + \frac{\sigma_b}{\sigma_{c2}}\cdot\hat{b}_t$, from (5); and combining (A1,A4),

(A5) $\frac{\sigma_K}{1-\alpha-\sigma_b}\cdot(\hat{k}_t + \hat{a}_t) + \frac{\sigma_{c1}}{1-\alpha-\sigma_b}\cdot\hat{c}_t^1 = \frac{(1-\alpha)}{(1-\alpha-\sigma_b)}\cdot\{\alpha\hat{k}_{t-1} + (1-\alpha)(\hat{a}_t + \hat{z}_t)\} - \frac{\sigma_b}{(1-\alpha-\sigma_b)}\cdot\hat{b}_t$.

Equation (A4) proves (18); (A2, A4) imply $\pi_{c2,k} = (1-\frac{\sigma_b}{\sigma_{c2}})\cdot(1-\pi_{R,a}) + \frac{\sigma_b}{\sigma_{c2}}\cdot\pi_{b,k} = 1-\pi_{c2,a}$, and this can be used in (A3) to obtain:

(A6) $\hat{c}_t^1 - \hat{a}_t = (\varepsilon\pi_{R,a} + 1 - \pi_{c2,a})\cdot\hat{k}_t$.

Let $\hat{k}_t$ and $\hat{c}_t^1$ in (A5, A6) be replaced by their log-linearized policy functions with undetermined coefficients, and equate coefficients for each state variable. This implies

$$\pi_{c1,a} - 1 = (\varepsilon\pi_{R,a} + 1 - \pi_{c2,a})\cdot\pi_{k,a},$$

$$\pi_{c1,z} - 1 = (\varepsilon\pi_{R,a} + 1 - \pi_{c2,a})\cdot\pi_{k,z},$$

$$\kappa(\pi_{k,a} + 1) + (1-\kappa)\pi_{c1,a} = (1-\alpha) + \frac{\sigma_b}{(1-\alpha-\sigma_b)}(1-\alpha-\pi_{b,a}), \text{ and}$$

$$\kappa\pi_{k,z} + (1-\kappa)\pi_{c1,z} = (1-\alpha) + \frac{\sigma_b}{(1-\alpha-\sigma_b)}(1-\alpha-\pi_{b,z}).$$

Substituting away the coefficients on capital, one obtains

(A7) $1-\pi_{c1,a} = \Phi\cdot\{\alpha - \frac{\sigma_b}{(1-\alpha-\sigma_b)}(1-\alpha-\pi_{b,a})\}$, where $\Phi \equiv \frac{(\varepsilon\pi_{R,a}+1-\pi_{c2,a})(\sigma_K+\sigma_{c1})}{\sigma_K+\sigma_{c1}(\varepsilon\pi_{R,a}+1-\pi_{c2,a})}$, and

(A8) $\pi_{c1,z} = \Phi\cdot\{(1-\alpha) + \frac{\sigma_b}{(1-\alpha-\sigma_b)}(1-\alpha-\pi_{b,z})\}$

Then consider different cases, (i-iv) below:

(i) For $(\varepsilon,\nu) = (1,0)$ and $\pi_{b,a} = \pi_{b,z} = 1-\alpha$: (18) implies $\varepsilon\pi_{R,a} + 1 - \pi_{c2,a} = 1$, so (A7-8) reduces to $\pi_{c1,a} = \pi_{c2,a} = \pi_{c1,z} = \pi_{c2,z} = 1-\alpha$.

(ii) For $\varepsilon = 1$, $\pi_{b,a} = 1-\alpha$, and any $\nu > 0$: (18) implies $\pi_{c2,a} = \pi_{c2,z} = (1-\nu)(1-\alpha) < 1-\alpha$ and $1-\nu < \Phi \leq 1$, so $\pi_{c1,a} \geq 1-\alpha > \pi_{c2,a}$ and $\pi_{c1,z} = \Phi(1-\alpha) > \pi_{c2,z}$.

(iii) For $\varepsilon = 1$ and any $\nu \geq 0$: Differentiating (18) and (A7-8) with respect to $\pi_{b,s}$, $s \in \{a,z\}$, yields

$\frac{\partial\pi_{c2,s}}{\partial\pi_{b,s}} = \frac{\sigma_b}{\sigma_{c2}} > 0$ for both shocks, $\frac{\partial\pi_{c1,z^T}}{\partial\pi_{b,z}} = -\Phi\cdot\frac{\sigma_b}{(1-\alpha-\sigma_b)} < 0$, and

$\frac{\partial\pi_{c1,a}}{\partial\pi_{b,a}} = \Phi\cdot\{\frac{\sigma_K}{\sigma_K+\sigma_{c1}(\varepsilon\pi_{R,a}+1-\pi_{c2,})}\frac{\partial\pi_{c2,a}}{\partial\pi_{b,a}} - \frac{\sigma_b}{(1-\alpha-\sigma_b)}\} = \Phi\cdot\frac{\sigma_b}{(1-\alpha-\sigma_b)}\{\frac{\Phi\sigma_K}{(\varepsilon\pi_{R,a}+1-\pi_{c2,a})\sigma_{c2}} - 1\} < 0$.

The latter is negative because $(\varepsilon\pi_{R,a} + 1 - \pi_{c2,a})\sigma_{c^r} > 1-\alpha$ whereas $\Phi\sigma_K < 1-\alpha$.

Given step (ii), this implies $\pi_{c1,a} > \pi_{c2,a}$ and $\pi_{c1,z} > \pi_{c2,z}$ for all $\nu > 0$ and all $\pi_{b,s} \leq 1-\alpha$.

(iv) For $\varepsilon < 1$: Because Φ is strictly increasing in ε, (A7) implies that $\pi_{c1,a} > \pi_{c2,a}$ is strict for all $\varepsilon < 1$ whenever $\pi_{c1,a} \geq \pi_{c2,a}$ for $\varepsilon = 1$, i.e., for all $(\varepsilon, v) \in [0,1] \times [0,1]$ and for $\pi_{b,a} \leq 1-\alpha$ excluding the case $(\varepsilon, v) = (1,0)$ with $\pi_{b,a} = 1-\alpha$. This proves Prop.1(a).

Also because Φ is strictly increasing in ε, (A8) implies that $\pi_{c1,z} - \pi_{c2,z}$ is strictly increasing in ε and strictly decreasing in $\pi_{b,z}$. To show that $\pi_{c1,z} > \pi_{c2,z}$ for all ε and all $\pi_{b,z} \leq 1-\alpha$, it is therefore sufficient to show a positive sign for $\varepsilon = 0$ and $\pi_{b,z} = 1-\alpha$. For this case, $\varepsilon = 0$ and $\pi_{b,z} = 1-\alpha$, (18) and (A8) imply $\frac{\pi_{c1,z} - \pi_{c2,z}}{1-\alpha} = \Phi - [1 - v(1 - \frac{\sigma_b}{\sigma_{c2}})] > 0$ if and only if $v > \pi_{c2,a} \{ \frac{\sigma_K}{\sigma_K + \sigma_{c1}} / (1 - \frac{\sigma_b}{\sigma_{c2}}) + v \frac{\sigma_{c1}}{\sigma_K + \sigma_{c1}} \}$. Using (18) for $\pi_{c2,a}$, and using the steady state relationships (b) and (e) to substitute $\frac{\sigma_K}{\sigma_K + \sigma_{c1}} = \frac{1}{1-v} \frac{\alpha}{1-\alpha-\sigma_b} / \frac{R}{\gamma_A \gamma_N}$ and $1 - \frac{\sigma_b}{\sigma_{c2}} = \frac{\alpha}{\alpha + \sigma_b(1-v)}$, this inequality reduces to $v\{f + v\} > f\tilde{\vartheta}(1-v)$, where $f = \frac{(\alpha+\sigma_b)}{1-\alpha-\sigma_b}$ and $\tilde{\vartheta} = \frac{1-\alpha}{\alpha} \cdot f / \frac{R}{\gamma_A \gamma_N}$. The equation $v\{f + v\} = f\tilde{\vartheta}(1-v)$ has a positive root $v_0 = f/2[\sqrt{(1+\tilde{\vartheta})^2 + 4\vartheta / f} - (1+\tilde{\vartheta})]$, which is equivalent to the formula in the text, using $\vartheta = (1+\tilde{\vartheta})/2$. Given the direction of inequality above, $\pi_{c1,z} > \pi_{c2,z}$ holds for $v > v_0$. This proves Prop.1(b).

Proposition 1(c): Inserting the efficient laws of motion into (A1, A6) and evaluating the undetermined coefficients for $\hat{a}_t$ one finds

$$\pi^*_{c1,a} - 1 = (\varepsilon \pi_{R,a} + 1 - \pi^*_{c2,a}) \cdot \pi^*_{k,a} \text{, and}$$

$$\text{(A9)} \qquad \sigma_K \cdot \frac{1-\pi^*_{c1,a}}{\varepsilon \pi_{R,a} + 1 - \pi^*_{c2,a}} + \sigma_{c1} \cdot (1 - \pi^*_{c1,a}) + \sigma_{c2} \cdot (1 - \pi^*_{c2,a}) = (\alpha + \sigma_v).$$

Invoking the efficiency condition $\pi^*_{c1,a} = \pi^*_{c2,a}$, (A9) implies that $(1-\pi^*_{c2,a}) = \phi(1-\pi^*_{c2,a})$ is the fixed point of $\phi(x) \equiv \frac{(\alpha+\sigma_v)(\varepsilon\pi_{R,a}+x)}{\sigma_K + (\sigma_{c1}+\sigma_{c2})(\varepsilon\pi_{R,a}+x)}$. Because $\phi(0) > 0$, $0 < \phi' \leq \frac{\sigma_{c1}}{\sigma_{c1}+\sigma_{c2}} < 1$, the fixed point is unique; and $\phi(x) > x$ holds if and only if x is below the fixed point.

If $\pi_{c2,a} < \pi_{c1,a}$ holds in a market allocation, $\eta \equiv (1-\pi_{c1,a})/(1-\pi_{c2,a}) < 1$. Inserting the market laws of motion (10) into (A1,A6), and proceeding as above, one obtains

$$\sigma_K \cdot \frac{\eta(1-\pi_{c1,a})}{\varepsilon \pi_{R,a} + 1 - \pi_{c2,a}} + \sigma_{c1} \cdot \eta(1 - \pi_{c2,a}) + \sigma_{c2} \cdot (1 - \pi_{c2,a}) = (\alpha + \sigma_v).$$

Because $\eta < 1$, this implies $1 - \pi_{c2,a} = \frac{(\alpha+\sigma_v)(\varepsilon\pi_{R,a}+1-\pi_{c2,a})}{\sigma_K \eta + (\sigma_{c1}\eta + \sigma_{c2})(\varepsilon\pi_{R,a}+1-\pi_{c2,a})} > \phi(1-\pi_{c2,a})$. Thus, $1-\pi_{c2,a}$ is less than the fixed point, hence $\pi_{c2,a} < \pi^*_{c2,a}$. Comparing (A1) for efficient and market solutions, one obtains $\sigma_K \cdot (\pi^*_{k,a} - \pi_{k,a}) + \sigma_{c1} \cdot (\pi^*_{c1,a} - \pi_{c1,a}) + \sigma_{c2} \cdot (\pi^*_{c2,a} - \pi_{c2,a}) = 0$. From $\pi_{c2,a} < \pi^*_{c2,a}$ and (A6), this implies $\pi_{k,a} \geq \pi^*_{k,a} \frac{[\sigma_K + \sigma_{c1} \cdot (\varepsilon \pi_{R,a} + 1 - \pi^*_{c2,a})]}{[\sigma_K + \sigma_{c1} \cdot (\varepsilon \pi_{R,a} + 1 - \pi_{c2,a})]} > \pi^*_{k,a}$, and $\pi_{k1,a} = 1 + \pi_{k,a} > 1 + \pi^*_{k,a} = \pi^*_{k1,a}$. QED.

<u>Proposition 2</u>: From Section 2, efficient policies exist. Prop.1 shows $0 \leq \pi_{b,s} \leq 1 - \alpha$ for $s \in \{a, z\}$ are inefficient under the same assumptions as assumed in Prop.2, and the proof of Prop.1 rules out negative values. Hence $\pi^*_{b,s} > 1 - \alpha$ for $s \in \{a, z\}$. [More constructively, one could derive formulas for the optimal coefficients $(\pi^*_{b,a}, \pi^*_{b,z})$ by treating $(\pi_{b,a}, \pi_{b,z})$ as derived from (A1-A6) as undetermined coefficients and equating retiree and workers' consumption elasticities. Then $\pi^*_{b,s} > 1 - \alpha$ could be verified directly. But such a direct proof would be lengthy.]

<u>Proposition 3</u>: In the habit model, the Euler equation (4) can be written as

$$(c^1_t)^{-1/\tilde{\varepsilon}} = \rho E_t[(R_{t+1} + \tilde{h})(c^2_{t+1} - \tilde{h} c^1_t)^{-1/\tilde{\varepsilon}}].$$

Its log-linearization is $-1/\tilde{\varepsilon} \cdot (\hat{c}^1_t - \hat{a}_t) = E_t[\frac{R}{R+\tilde{h}} \hat{R}_{t+1} - 1/\varepsilon \cdot \{\frac{1}{1-h} \hat{c}^2_{t+1} - \frac{h}{1-h}(\hat{c}^1_t - \hat{a}_t)\}]$ and implies

$$\hat{c}^1_t - \hat{a}_t = E_t[\hat{c}^2_{t+1} - \tilde{\varepsilon} \tfrac{R}{R+\tilde{h}} (1 - \bar{h}) \hat{R}_{t+1}]$$

By comparison to the CRRA, $\tilde{\varepsilon} \frac{R}{R+\tilde{h}} (1 - \bar{h}) = \varepsilon$ is the elasticity of substitution. From (8):

$$(c^2_t - \tilde{h} c^1_{t-1})^{-1/\tilde{\varepsilon}} = (c^1_t)^{-1/\tilde{\varepsilon}} - \rho \tilde{h} E_t[(c^2_{t+1} - \tilde{h} c^1_t)^{-1/\tilde{\varepsilon}}] = \rho E_t[R_{t+1} (c^2_{t+1} - \tilde{h} c^1_t)^{-1/\tilde{\varepsilon}}]$$

Now use $\hat{\chi}_t = \hat{c}^1_t - \hat{a}_t$ and define $\hat{c}^{ha}_t = \frac{1}{1-\bar{h}} \hat{c}^2_t - \frac{\bar{h}}{1-\bar{h}} \cdot \hat{\chi}_{t-1} - \hat{a}_t$. Then (4) and (8) reduce to

$$\hat{\chi}_t = E_t[\hat{c}^{ha}_{t+1} - \tilde{\varepsilon} \tfrac{R}{R+\tilde{h}} \hat{R}_{t+1}] \text{ and } \hat{c}^{ha}_t = E_t[\hat{c}^{ha}_{t+1} - \tilde{\varepsilon} \hat{R}_{t+1}]$$

These equations and (A1-2), which remain valid, define the log-linearized allocation. Because Prop.3 is about $\hat{a}_t$ set $z_t = 0$ in (A1-2). Then write (A1) as

$$\sigma_K \cdot \hat{k}_t + \sigma_{c1} \cdot \hat{\chi}_t + \sigma_{c2} \cdot (1 - \bar{h}) \hat{c}^{ha}_t = (\alpha + \sigma_v) \cdot \hat{k}_{t-1} - (\alpha + \sigma_v - \sigma_{c2} \cdot \bar{h}) \cdot \hat{a}_t - \sigma_{c2} \cdot \bar{h} \hat{\chi}_{t-1}$$

and use (A2) to substitute $E_t \hat{R}_{t+1} = -\pi_{R,a} \cdot \hat{k}_t$ in the optimality conditions. One obtains

$$\hat{\chi}_t = E_t[\hat{c}^{ha}_{t+1}] + \tilde{\varepsilon} \tfrac{R}{R+\tilde{h}} \pi_{R,a} \cdot \hat{k}_t \text{ and } E_t[\hat{c}^{ha}_{t+1}] = \hat{c}^{ha}_t - \tilde{\varepsilon} \pi_{R,a} \cdot \hat{k}_t$$

hence $\hat{\chi}_t = \hat{c}^{ha}_t - \tilde{\varepsilon} \frac{R}{R+\tilde{h}} \pi_{R,a} \cdot \hat{k}_t = \hat{c}^{ha}_t - \eta \cdot \hat{k}_t$, where $\eta = \tilde{\varepsilon} \frac{R}{R+\tilde{h}} \pi_{R,a} \geq 0$.

Inserting (10) with undetermined coefficients, one obtains $\pi^*_{\chi,s} = \pi^*_{c^{ha},s} - \eta \pi^*_{k,s} \forall s$, and

(A10a) $\sigma_K \cdot \pi^*_{k,k} + \sigma_{c1} \cdot \pi^*_{\chi,k} + \sigma_{c2} \cdot (1-\bar{h})\pi^*_{c^{ha},k} = \alpha + \sigma_v$

(A10b) $\sigma_K \cdot \pi^*_{k,\chi} + \sigma_{c1} \cdot \pi^*_{\chi,\chi} + \sigma_{c2} \cdot (1-\bar{h})\pi^*_{c^{ha},\chi} = -\sigma_{c2}\bar{h}$

and using $\pi^*_{c^{ha},\chi}\hat{\chi}_t + \pi^*_{c^{ha},k}\hat{k}_t = \hat{c}^{ha}_t - \tilde{\varepsilon}\pi_{R,a} \cdot \hat{k}_t$, one obtains

(A10c) $\pi_{c^{ha},\chi}\pi^*_{\chi,k} + (\pi^*_{c^{ha},k} + \tilde{\varepsilon}\pi_{R,a})\pi^*_{k,k} = \pi^*_{c^{ha},k}$

(A10d) $\pi_{c^{ha},\chi}\pi^*_{\chi,\chi} + (\pi^*_{c^{ha},k} + \tilde{\varepsilon}\pi_{R,a})\pi^*_{k,\chi} = \pi^*_{c^{ha},\chi}$

Equations (A10a-d) characterize the four coefficients $\{\pi^*_{k,k}, \pi^*_{k,\chi}, \pi^*_{c^{ha},k}, \pi^*_{c^{ha},\chi}\}$. Noting that $\pi^*_{c^{ha},\chi} = -\frac{\sigma_{c2}\cdot\bar{h}}{\alpha+\sigma_v}\pi^*_{c^{ha},k}$ and $\pi^*_{k,\chi} = -\frac{\sigma_{c2}\cdot\bar{h}}{\alpha+\sigma_v}\pi^*_{k,k}$, this reduces to a pair of equations in $\{\pi^*_{c^{ha},k}, \pi^*_{k,k}\}$ that can be further reduced to a quadratic equation in $\pi^*_{k,k}$, which yields a unique, positive stable root. (Details omitted because the following only relies on $\pi^*_{k,k} \geq 0$.) Noting that $\{\hat{k}_t, \hat{\chi}_t, \hat{c}^{ha}_t\}$ are homogenous of degree zero in $(\hat{a}_t, \hat{\chi}_t, \hat{k}_t)$, $\pi^*_{k,k} \geq 0$ implies $\pi^*_{k,a} = -(1 - \frac{\sigma_{c2}\cdot\bar{h}}{\alpha+\sigma_v})\pi^*_{k,k} < 0$, and therefore $\pi^*_{\chi,a} = \pi^*_{c^{ha},a} - \eta\pi^*_{k,a} \geq \pi^*_{c^{ha},a}$. By construction of $\hat{c}^{ha}_t$ and $\hat{\chi}_t$, one has $\pi^*_{c1,a} = 1 + \pi^*_{\chi,a} \geq 1 + \pi^*_{c^{ha},a}$ and $\pi^*_{c2,a} = (1-\bar{h})(1+\pi^*_{c^{ha},a})$, so $\pi^*_{c2,a} \leq (1-\bar{h})\pi^*_{c1,a}$. QED.

Part C: Supplementary materials on efficiency in Section 2

1. The one-for-one correspondence between ex ante efficient allocations and sets of state-contingent transfers $\{b_t^*(h_t)\}_{t\geq 0}$ (asserted on page 7) works as follows: For arbitrary welfare weights $\{\omega_t\}_{t\geq 0}$ let $\{c_t^{1*}(h_t), c_t^{2*}(h_t), l_t^*(h_t), K_{t+1}^*(h_t)\}_{t\geq 0}$ be the solution to the planning problem. Define transfers for all dates and states by

$$b_t^*(h_t) \equiv c_t^{2*}(h_t) - R_t^*(h_t) \cdot K_t^*(h_{t-1}) / N_{t-1},$$

where stars (*) refer to planning-solution values. In a market economic with these transfers, $c_t^1 = c_t^{1*}(h_t)$, $c_{t+1}^2 = c_{t+1}^{2*}(h_{t+1})$, and $l_t = l_t^*(h_t)$ are optimal choices for generation t-workers, and the resulting savings yield $K_{t+1} = K_{t+1}^*(h_t)$, which means that the planning solution can be decentralized.

2. The efficiency standard (Fn.10, p.6) In the literature, “ex ante” comparisons sometimes refers to “timeless” comparisons of steady states, but such comparisons easily miss transition cost. Hence I condition on the initial state of nature. One could define Pareto optimality more generally conditional on a history h_t. But maximizing W_0 conditional on h_t would yield the same first order condition for subsequent periods and provide no additional insights.

There is also a controversy about ex ante versus interim efficiency. Some authors have favored interim over ex ante efficiency because laissez-faire allocations generally violate (8) and are therefore inefficient (Peled 1982; Wright 1987; Demange and Laroque 1999). Under an interim perspective, agents born in different states of nature are treated as distinct. This yields in a less-demanding welfare standard that is generally satisfied by laissez-faire (Peled 1982). However, for this paper, interim efficiency is uninteresting. This is because alternative policies invariably shift resources across state of nature and are therefore not comparable under the interim standard.

Specifically, interim efficient allocations can be interpreted as solutions to planning problems with state-contingent weights $\omega_t(h_t)$ that maximize $\sum_{t\geq 0}\{\sum_{h_t}\omega_t(h_t)N_t E[U_t \mid h_t]\}$. For any given market allocation, one may compute state-contingent weights

$$\tilde{\omega}(h_t) = 1 / E_t[\tfrac{\partial U_t}{\partial c_t^1} / \tfrac{\partial U_{t-1}}{\partial c_t^2}]$$

for which the allocation can be rationalized as interim efficient, provided the planning problem with weights $\tilde{\omega}(h_t)$ has a solution. Thus, interim efficiency imposes no constraints on policy except to rule out policies that would yield dynamic inefficiency.

Ex ante efficiency, in contrast, imposes non-trivial restrictions on risk sharing and it imposes a feasible efficiency standard, a standard that can be attained regardless of policy makers' redistributional preferences.

3. Non-existence of comparable efficient allocations (Fn.11 on p.7) The welfare weights $\tilde{\omega}_t$ defined in (9) are uniquely defined for all market allocations, but the planning problem with weights $\{\tilde{\omega}_t\}_{t\geq 0}$ may not have a solution. Notably, if the market allocation is dynamically inefficient, the sequence $\{\prod_{t=0}^{k}\tilde{\omega}_t\}_{k\geq 0}$ will typically diverge as $k \to \infty$. If the planning problem with weights $\{\tilde{\omega}_t\}_{t\geq 0}$ does have a solution, however, it is easy to check if the market solution satisfies (8) for all states of nature. Because transfers imply allocations, one may also check if the given transfers $\{b_t(h_t)\}_{t\geq 0}$ match the transfers that maximize welfare for weights $\{\tilde{\omega}_t\}_{t\geq 0}$. There may be "borderline" Pareto efficient allocations for which the sum in (2.7) does not converge, but they are inessential for risk sharing issues and hence disregarded. For this paper, Pareto efficiency and "solution to the planning problem subject to initial conditions" are treated as equivalent.

4. An economic interpretation of why distributional judgments are not needed: The main intuition is that constructing the comparable efficient allocation formalizes the distinction between redistribution and risk sharing. A planner who discounts future generations' utility heavily will tend to transfer resources from young to old, or put differently, assign relatively more consumption to the old than to the young. This applies in all states of nature, which means that the general "level" of transfers in a market allocation reveals the planner's welfare weights, i.e., his/her distributional preferences.

Risk sharing is inherently about fluctuations around a given "level" of activity. The basic requirement for optimal risk sharing is—as shown in (8)—that transfers vary across states of nature so that the marginal utilities of the old and the young vary proportionally. This proportionality applies regardless of welfare weights, which means that ω_t is a nuisance for studying risk sharing. The nuisance is removed by conditioning on the welfare weights $\tilde{\omega}_t$ that are embedded in the market allocation itself. This step separates risk sharing from redistribution and thus provides the conceptual foundation for the study of intergenerational risk sharing.

Once the appropriate welfare weights are identified via (9), efficiency can be assessed without making distributional judgments. One must simply examine to what extent the variables of interest—consumption, leisure, capital investment, transfers, etc—vary across states of nature in the same way as in the comparable efficient policy.

Because redistribution is a distraction, this paper is not interested in market allocations that are inefficient because of an inefficient level of transfers—the case of dynamic inefficiency—nor in "borderline" cases of dynamic efficiency where problem (7) has no finite solution (e.g., Golden Rule-type allocations). To avoid existence issues, the paper restricts attention to allocations for which comparable efficient allocations exist. That is, I restrict the analysis to combinations of preferences, technologies, and policies for which the Pareto problem with weights $\{\tilde{\omega}_t\}_{t\geq 0}$ has a finite solution.

5. Details on why balanced growth implies constant welfare weights (Fn.12, p.8): If a market allocation has balanced growth, marginal rates of substitution must converge to a limiting distribution. Hence the welfare weights in (9) must always converge to a constant

$$\omega = 1 / \left(\lim_{t\to\infty} E_0[\tfrac{\partial U_t}{\partial c_t^1} / \tfrac{\partial U_{t-1}}{\partial c_t^2}] \right).$$

Balanced growth in infinite horizon planning problems similarly requires exponential discounting. This suggests that one may restrict attention to planning problems with constant discount factors and simply compare a market allocation to the efficient allocation with fixed discount factor $\omega_t = \omega$. This comparison applies asymptotically for any initial conditions h_0 (as $t \to \infty$). It applies for all t, if initial values are drawn from the limiting distribution of the market allocation to make $E_0[\tfrac{\partial U_t}{\partial c_t^1} / \tfrac{\partial U_{t-1}}{\partial c_t^2}]$ time-invariant. In any case, researchers commonly invoke ergodicity to estimate or calibrate a model, i.e., use time-averages to proxy expectations. This means that initial conditions and initial periods, though conceptually important to define efficiency, are disregarded in most applications. Treating ω as constant is no more restrictive.

Part D: Supplementary materials on the empirical claims in Section 3, Fn.18.

This section examines empirical co-movements of wages and returns on capital. Though the paper is primarily theoretical, the inequality between wage and return responses to productivity shocks is sufficiently important to deserve empirical analysis.

To start—as a simple extension that gives returns a higher variance than wages without changing any results about productivity—suppose capital accumulation takes the form $G(I_t,K_t,z_t^G)=I_t+(1-\delta+z_t^G)\cdot K_t$, where z_t^G is a mean-zero shock to the value of old capital, or equivalently, stochastic depreciation.

In the presence of z_t^G, the temporary productivity shock must be relabeled to avoid inconsistent notation. For this part of the appendix (Part D only), denote the temporary productivity shock by z_t^F. The overall resource constraint is then

$$K_{t+1}+N_t c_t^1+N_{t-1}c_t^2=K_t^{\alpha}(N_t A_t z_t^F)^{1-\alpha}+(1-\delta+z_t^G)\cdot K_t\,.$$

The wage equation remains unchanged, and the return on capital is

(A11) $$R_t=\alpha(k_{t-1}/\gamma_N)^{\alpha-1}(a_t z_t^F)^{1-\alpha}+1-\delta+z_t^G\,.$$

The stochastic z_t^G-term increases return volatility, breaks the perfect correlation of returns and wages, and may give returns a one-period-ahead variance that exceeds the comparable variance of wages. The introduction of z_t^G-shocks would add another linear term to the law of motion in (10).[1] However, assuming z_t^G, z_t^F, and a_t are independent, this extension does not change any of the elasticities with respect to productivity shocks and hence leaves Prop.1-2 unchanged. (Correlation between capital values and new shocks is addressed in the Q-model below.)

[1] This additional term raises new questions about the efficient policy response. Because z_t^G is a generation-specific shock, it is obvious that efficient policies should help spread its impact across generations.

Turning to the empirical evidence, I estimated variances and covariances of output, wages, and returns at "generational" frequencies for a range of VAR and error-correction specifications, using 1870-2002 U.S. data. Because long-run covariance matrices must be inferred from annual data that cover at most 4-5 generations, the results should be interpreted cautiously. The objective is to document that the theoretical modeling of incomes and asset returns above is broadly consistent with empirical evidence, not to provide precise parameter estimates.

A period in the model is interpreted as 30 years in the data. Because the Cobb-Douglas model implies $\hat{Y}_t - E_{t-1}\hat{Y}_t = \hat{w}_t - E_{t-1}\hat{w}_t = (1-\alpha)\cdot\hat{a}_t$, one-generation-ahead output or wage movements relative to expectations can be used to identify productivity shocks. (For shorter periods—post 1929, for which wage data are available, output and wages are highly correlated. Hence I use shocks to GDP as proxy for wages, permit use of longer time series.)

At a 30-year horizon, stationary shocks (in the empirical sense) have a negligible impact as compared to unit root shocks. Hence I interpret the shock identified by wage movements as a permanent shock ($\hat{a}_t$).[2] For returns, (A11) and (15) imply the log-linearization

(A12) $$\hat{R}_t - E_{t-1}\hat{R}_t = \pi_{R,a}\hat{a}_t + z_t^G = \tfrac{\pi_{R,a}}{\pi_{w,a}}(Y_t - E_{t-1}\hat{Y}_t) + \hat{z}_t^G$$

where the coefficient on z_t^G is normalized to 1 (by choice of units), so z_t^G can be interpreted as residual volatility in R_t. The coefficients $\pi_{R,a}$ have multiple interpretations: According to (A11), or equivalently (14), $\pi_{R,a} = (1-\alpha)(1-\nu)$ and $\pi_{w,a} = 1-\alpha$. However, the same regression specification also fits the Q-model in Section 4.3, then with interpretation $\pi_{R,a} = (1-\alpha)(1-\nu) + \tfrac{\nu}{\varepsilon_{IK}}\cdot\pi_{I,a}$ (see

[2] Because the sample contains so few non-overlapping generational periods, no attempt is made to distinguish permanent shocks from generation-long temporary ones. I focus on 30 year periods to be specific; 20 year intervals yield very similar results. Implicitly, all shocks to output within a generation are interpreted as productivity shocks. This is a broad interpretation of productivity, but appropriate for describing the risks faced by different generations.

below). Either way, the ratio $\pi_{R,a} / \pi_{w,a}$ in (A12) captures relative magnitude of return to wage responses to productivity shocks.

In the data, I proxy returns on capital by log-linearized, leverage-adjusted returns on the S&P500 index. Equity returns ($\ln R^e$) are computed from Shiller's (2003b) equity price and dividend data, using Campbell et al's (1997, ch.7) log-linearization method. To obtain returns on capital ($\ln R$), I take a weighted average of 26% debt (assumed safe) and 74% S&P500 equities. (The 26% debt/asset ratio is from Hall and Hall, 1993.) Output and its components are taken from National Income Accounts for 1929-2002; pre-1929 output is taken from Romer (1989). Because output is available for a much longer period than wages, I use output movements to identify productivity shocks and use 1875-2002 as estimation period.[3]

Table A1 shows the estimation results. As suggested by Hamilton (1994, ch. 20.4), I first estimate a basic vector autoregression (VAR) in levels, with two lags, and then examine the data for unit roots; see Col.1. Because a Johansen test suggest a single cointegrating vector of $0.13 \cdot \ln Y + 0.43 \cdot \ln(Stockprice) - \ln(Dividends)$, I estimate an error corrections model by regressing differenced data on own lags and the cointegrating vector (labeled ECM#1, Col.2). Separately, Phillips-Perron and Dickey-Fuller tests reject a unit root in dividend yields and in the dividend-output ratio. This suggests an ECM with the dividend yield and in the dividend-output ratio as regressors (labeled ECM#2, Col.3).

For each specification, the estimated coefficients are used to compute 30-year-ahead standard deviations and correlations for output ($\ln Y$), equity returns, and returns on capital, as shown. For the

[3] To examine how well output proxies for wages, I estimated a VAR for log-GDP and the log-wage share (computed as suggested by Cooley-Prescott 1995) for 1933-2002, finding a 30-year ahead correlation of 0.997. An error corrections specification yields a similarly high correlation, and both are consistent with the high correlations found by Baxter and Jermann (1997).

key elasticity ratio $\pi_{R,a} / \pi_{w,a}$, all three specifications imply a ratio well below one.[4] The ECM estimates also imply a variance ratio of capital returns and output above one, suggesting that z_t^G has sufficient variance to make returns on capital more risky in an absolute sense than wage income.

The variance estimates are somewhat sensitive to the treatment of time trends. A much lower variance for log-GDP is obtained if time is included in the VAR (Col.4). This effectively treats much of GDP-growth as deterministic. But because this specification also reduces the correlation with equities, it does not yield a higher ratio of elasticities. Moreover, if one estimates ECM#2 with time trends (Col.5), the data again favor a higher variance of output. This specification also yields a high correlation of GDP and equity returns and a high ratio of elasticities, 0.73, the highest value across specifications.

Finally, Col.6 displays annual data for comparison. As is well known, annual equity returns are much more volatile that output growth, as reflected in the high variance ratio. The 0.59 response of $\ln R$ to output shocks is, however, within the range of the generational estimates.

I show this range of specifications to demonstrate robustness. Without arguing about the relative merits of the different specification—the VAR is simple, ECM#1 has a better statistical grounding, and ECM#2 is more intuitive economically—the main point is that all specifications yield a ratio of elasticities below one. The 0.29-0.73 range is low enough to consistent with Cobb-Douglas type models of production. The highest estimate, 0.73, is remarkably consistent with the lower bound estimate $v = 0.27$ in Section 3, which implies $1 - v = 0.73$. The other (lower) estimates for the ratio of elasticities would imply higher v-values and hence greater inefficiencies in the allocation of productivity risk.

[4] A display of the regression estimates would be lengthy and is hence omitted. Details are available from the author. I make no attempt to identify a definitive model (a heroic task, given the long-standing controversies about unit roots in GDP), but focus on robustness. Standard errors are not provided because they would fail to capture specification errors, which are arguably at least as important as sampling error, given that the sample covers only 4-5 generational periods. Specification error is crudely captured by the range of point estimates.

Because capital returns and output are imperfectly correlated, z_t^G-shocks are needed to fit the data. But because the estimated elasticity ratios are within the range of calibrated values for $1-v$, the data do not require a Q-theoretic term nor a deviation from Cobb-Douglas; that is, they are consistent with $\varepsilon_{KL} \approx 1$ and with a linear *G*-function (arbitrarily high ε_{IK}-values). As claimed in Fn.18, the data appear consistent with (17), if supplemented with an independent z_t^G-shock.

Part E: Supplementary materials on the model extensions in Section 4

1. More on the Habit Model: Table A2 (below) demonstrates the quantitative impact of habit formation. Column 1 shows the laissez-faire allocation of risk for the Example Parameters. Col. 2-4 show the efficient allocation of risk for CRRA (labeled $\bar{h}=0$) and for the habit model with habit stocks of $\bar{h}=25\%$ and $\bar{h}=50\%$ of retirement consumption, respectively. Comparing the elasticities of worker and retiree consumption across columns, it is evident that economies with higher habit parameter have higher efficient ratios of worker to retiree exposure to productivity shocks. The ratios in the market allocation are too high when compared to the efficient CRRA allocation, about correct when compared to $\bar{h}=25\%$, and actually too low in case of $\bar{h}=50\%$. Thus, the 25% habit model makes laissez-faire look roughly efficient, and the 50% habit model might provide a rationale for policies to protecting the old.

Note that laissez-faire allocations are always inefficient in the habit model because they omit χ_{t-1} as state variable. The resulting inefficiency in the propagation mechanism is illustrated in Table A2 by the consumption responses to k_{t-1} and χ_{t-1}.

Table A3 presents three policies with different responses to permanent productivity risk and it displays the habit parameters for which each policy would be efficient. Col. 1 assumes $\pi_{b,a}=0.11$, the calibrated elasticity of U.S. retirees' generational account, Col. 2 assumes $\pi_{b,a}=0$, the polar case of safe transfers, and Col. 3 assumes $\pi_{b,a}=0.67$, the limiting case of wage-indexed transfers, all for $\sigma_b=10\%$. The habit parameters needed to justify these policies range from about 0.26 for wage-indexed transfers to 0.49 for safe transfers, and an intermediate 0.45 for the calibrated policy.

To be efficient for the respective habit parameters, these policy would have to response efficiently to other state variables, too, as shown. This means that even though the habit model adds a degree of freedom to the benchmark model, it has testable restrictions. The efficient policy response to temporary productivity shocks turns out to be slightly less than the responses to permanent shocks, which is intuitive because consumption smoothing reduces the need to intervene; the required responses to lagged consumption are large; and a negative response to lagged capital is implied by homotheticity. An intuition for these large lag coefficients is that, because laissez-faire ignores χ_{t-1}, efficient responses to χ_{t-1} must be implemented entirely through the transfer system. To conclude, the fact that efficiency restricts policy in other dimensions and may even add new testable restrictions—here, requiring efficient responses to lagged consumption—is noteworthy for future research, but the ramifications are beyond the scope of the paper.

2. Labor-Leisure Choices:

To see why variable labor supply does not help to rationalize safe transfers, consider an arbitrary time-separable utility function $U_t = u(c_t^1, l_t) + \rho \cdot u(c_t^2, 1)$, which generalizes the utility function in section 3.1. By log-linearizing the efficiency condition (8) one obtains

(A13) $$\left(-\frac{u_{cc}(c^2,1)c^2}{u_c(c^2,1)}\right)\cdot \hat{c}_t^{2*} = \left(-\frac{u_{cc}(c^1,l)c_t^1}{u_c(c^1,l)}\right)\cdot \hat{c}_t^{1*} - \left(\frac{u_{cl}(c^1,1)l}{u_c(c^1,1)}\right)\cdot \hat{l}_t^*$$

which is like (12) but with a term for leisure. Because the sign of the leisure term depends on u_{cl}, the role of leisure in the efficient allocation depends on how leisure and consumption interact in the utility function. This interaction is restricted by balanced growth: leisure is non-stationary unless consumption and leisure have a unit elasticity of substitution.

An instructive special case is log-utility. If utility is logarithmic in consumption, balanced growth requires separability, $u_{cl} = 0$, so $\hat{l}_t^*$ drops out of (A13). Then age-independent risk aversion implies $\hat{c}_t^{2*} = \hat{c}_t^{1*}$, the same efficiency condition (13) as in Section 3, and this has similar implications for the inefficiency (all $v > 0$) or the efficiency (in case $v = 0$) of market allocations.

For elasticities of substitution $\varepsilon < 1$, the empirically relevant range, balanced growth requires $u_{cl} < 0$. This combined with age-independent risk-aversion implies that (A13) can be reduced to $\hat{c}_t^{2*} = \hat{c}_t^{1*} + \phi \cdot \hat{l}_t^*$, where $\phi \equiv {}^{u_{cl} l}\!/_{u_{cc} c} > 0$. The conditions for approximate efficiency are then $\pi^*_{c2,s} = \pi^*_{c1,s} + \phi \pi^*_{l,s}\ \forall s \in S_t$, which differs from (13). For permanent productivity shocks, it is straightforward to show $\pi^*_{l,a} > 0$, i.e., that leisure responds positively. Given $\phi > 0$, this implies $\pi^*_{c2,a} > \pi^*_{c1,a}$, indicating that retirees should be *more* exposed to productivity shocks than workers. Thus, variable labor shifts the efficiency standard in the opposite direction of what one would need to rationalize safe transfers.[5]

3. <u>More on general production</u> (Section 4.3): The main point to note is that for general F, the only relevant new parameter is the elasticity of substitution between capital and labor in steady states. This parameter would be constant if production is CES, but for the log-linearization, a CES assumption is not required. Straightforward algebra yields $\pi_{w,a} = 1 - \alpha / \varepsilon_{KL}$ and $\pi_{R,a} = (1-\alpha)(1-\nu) / \varepsilon_{KL}$. The bound $\varepsilon_{KL} = 0.78$ is based on numerical calculations.

Regarding empirically relevant values for ε_{KL}, note that the $\pi_{R,a} / \pi_{w,a}$ ratios in Table A1 imply estimates for ε_{KL} greater or equal one if one uses (A2) and $\nu = 0.27$ to interpret the empirical data. This suggests the low ε_{KL} is not a plausible rationalization for safe transfers.

4. <u>Formal results for the Tobin's-Q model</u> (Section 4.3): This extension is conceptually important and hence deserves analysis, because it produces an endogenous positive correlation between productivity shocks and the value of old capital. For Tobin's-Q to be a strictly increasing function of

[5] The same line of argument implies that a variable work effort would be more promising if $u_{cl} > 0$, but this would either contradict the empirical evidence on intertemporal substitution or require abandoning balanced growth.

investment, the function G in (2) must be strictly concave. Then constant returns to scale imply that the value of old capital, $G_K(I_t, K_t, z_t^G)$, is also strictly increasing in new investment. Assuming Cobb-Douglas production with temporary productivity shock z_t^F, the total return on capital expressed in terms of stationary variables is

(A14) $$R_t = \alpha(k_{t-1}/\gamma_N)^{\alpha-1}(a_t z_t^F)^{1-\alpha} + \frac{G_K(I_t/A_{t-1}/N_{t-1}, k_{t-1}, z_t^G)}{G_I(I_t/A_{t-1}/N_{t-1}, k_{t-1}, z_t^G)}.$$

The response of investment to shocks is endogenous and best derived by log-linearization. For log-linearizations, the only relevant characteristic of G is the elasticity of substitution between investment and old capital in steady state, denoted ε_{IK}. For $\varepsilon_{IK} \to \infty$ on recovers the linear specification; and if (I_t, K_t) are reasonably close substitutes, ε_{IK} should be well above one. An intuition for how Q-theory affects the allocation of productivity shocks is obtained by log-linearizing (6.2), which implies

(A15) $$\pi_{R,a} = (1-\alpha)(1-v) + \frac{v}{\varepsilon_{IK}} \cdot \pi_{I,a},$$

where $v = QG_K/R^k$ is again the value-share of old capital. If new investment is an increasing function of a_t, so $\pi_{I,a} > 0$, then the investment term in (A15) increases the response of retiree incomes to a productivity shock. However, the same concavity in G that makes the return on capital variable and gives investment a role in (A14) also acts as an adjustment cost that discourages variations in capital investment. Economies with low ε_{IK}, for which v/ε_{IK} in (A15) is large, also have elasticities $\pi_{I,a}$ near zero, which limits the overall effect. More specifically, one can prove:

<u>Proposition 4</u>: *Consider a laissez-faire economy with power utility, Cobb-Douglas production, and concave accumulation function G. If* $\varepsilon_{IK} \geq \sigma_v/\sigma_I$, *then* $\pi_{c2,a} < \pi_{c1,a}$ *applies for all* $(\varepsilon, v) \in [0,1]\times[0,1]$ *except* $(\varepsilon, v) = (1,0)$.

The proof is below. The bound σ_v / σ_I is ratio of old to new capital in production—the lower bound asserted in Section 4.3. Because most capital depreciates over a generation, $\varepsilon_{IK} \geq \sigma_v / \sigma_I$ is fairly weak restriction. For simplicity, the proposition is only proved laissez-faire (which suffices, given the direction of inefficiency).

Proof of Proposition 4: By log-linearizing (A14), one obtains

$\pi_{R,a} = (1-\alpha)(1-v) + \frac{v}{\varepsilon_{KI}} \cdot \pi_{I,a}$ and $\pi_{c2,a} = \pi_{R,a} = (1-\alpha)(1-v) + \frac{v}{\varepsilon_{KI}} \cdot \pi_{I,a}$

where $\frac{1}{\varepsilon_{KI}} = \frac{G_{KI}}{G_K} - \frac{G_{II}}{G_I}$ is used to simplify derivatives.

For workers, the log-linearizations

$$\hat{k}_t^1 - \hat{Q}_t = \hat{k}_t + \hat{a}_t = \frac{G_I I}{G} \cdot \hat{I}_t + \frac{G_K K}{G} \cdot \hat{k}_{t-1} = \frac{\sigma_I}{\sigma_K} \cdot \hat{I}_t + (1 - \frac{\sigma_I}{\sigma_K}) \cdot \hat{k}_{t-1}$$

and $\hat{Q}_t = -\frac{G_{II}}{G_I} \cdot (\hat{I}_t - \hat{k}_{t-1}) = \frac{\sigma_I}{\sigma_K} \frac{1}{\varepsilon_{KI}} \cdot (\hat{I}_t - \hat{k}_{t-1}) = \frac{1}{\varepsilon_{KI}} (\hat{k}_t + \hat{a}_t - \hat{k}_{t-1})$

imply $\pi_{I,a} = \frac{\sigma_K}{\sigma_I} \cdot (1 + \pi_{k,a})$, $\pi_{Q,a} = \frac{1}{\varepsilon_{KI}} \cdot (1 + \pi_{k,a})$, and $\pi_{k1,a} = (1 + \pi_{k,a})(1 + \frac{1}{\varepsilon_{KI}})$.

The first order condition $1/\varepsilon \cdot (\hat{c}_t^1 - \hat{a}_t) + E_t[\hat{R}_{t+1} - \hat{Q}_t - 1/\varepsilon \cdot \hat{c}_{t+1}^2] = 0$

implies $\hat{c}_t^1 - \hat{a}_t = [1 + (1-\varepsilon)\pi_{R,k}] \cdot \hat{k}_t + \varepsilon \cdot \hat{Q}_t$, hence

$$\pi_{c1,a} = 1 + [1 + (1-\varepsilon)\pi_{R,k}] \cdot \pi_{k,a} + \varepsilon \cdot \frac{1}{\varepsilon_{KI}} \cdot (1 + \pi_{k,a})$$

The critical case is again $\varepsilon = 1$ (as in the proof of Prop.1) because $\pi_{c1,a}$ is declining in ε. For $\varepsilon = 1$:

$\pi_{c1,a} = 1 + \pi_{k,a} + \frac{1}{\varepsilon_{KI}} \cdot (1 + \pi_{k,a}) = \pi_{k1,a}$ combined with $\frac{\sigma_K}{\sigma_K + \sigma_{c1}} \pi_{k1,a} + \frac{\sigma_{c1}}{\sigma_K + \sigma_{c1}} \pi_{c1,a} = 1 - \alpha$ imply

$\pi_{c1,a} = \pi_{k1,a} = 1 - \alpha$, hence $(1 + \pi_{k,a}) = (1-\alpha)\frac{\varepsilon_{KI}}{1+\varepsilon_{KI}}$. For retirees, one finds

$\pi_{c2,a} = (1-\alpha)(1-v) + \frac{v}{\varepsilon_{KI}} \cdot \frac{\sigma_K}{\sigma_I} \cdot (1-\alpha)\frac{\varepsilon_{KI}}{1+\varepsilon_{KI}}$. By comparison, $\pi_{c1,a} - \pi_{c2,a} = (1-\alpha) \cdot v \cdot (1 - \frac{\sigma_K}{\sigma_I} \frac{1}{1+\varepsilon_{KI}}) > 0$

for all $v > 0$ provided $\varepsilon_{KI} > \frac{\sigma_K}{\sigma_I} - 1 = \frac{\sigma_v}{\sigma_I}$. QED.

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Table A1: Empirical Evidence about Output and Equity Returns at Generational Frequencies (30-year-ahead projections)

	Values for Generational Periods obtained from					Annual
Estimation Method:	Basic VAR	ECM #1	ECM #2	VAR w/ trends	ECM #2 w/ trends	Raw Data
Column:	(1)	(2)	(3)	(4)	(5)	(6)
Standard Deviations of						
Output (lnY)	30.1%	35.9%	35.1%	10.7%	34.6%	4.9%
Return on equity ($\ln R^e$)	38.3%	64.2%	63.9%	35.8%	49.6%	17.7%
Return on capital (lnR)	28.3%	47.5%	47.3%	26.5%	36.8%	13.1%
Correlation (lnR, lnY)	0.315	0.303	0.414	0.157	0.686	0.166
Variance Ratio (lnR, lnY)	0.89	1.75	1.82	6.17	1.13	7.02
Ratio of elasticities $\pi_{R,a} / \pi_{w,a}$	**0.297**	**0.400**	**0.558**	**0.390**	**0.730**	**0.595**

Notes: Output refers log real GDP, equity returns are log returns on the S&P500 computed from price and dividend data as suggested by Campbell et. al, (1997, ch.7). The return on capital refers to a weighted average of equity and debt returns, assuming 26% debt/asset ratio and treating debt as riskless.

The ratio of elasticities is the covariance of lnR and lnY divided by the variance of lnY, as suggested by the log-linearization in the text. It measures the ratio of the return-on-capital response to productivity shocks over the wage response.

Col.1-5 use 30-year-ahead projections computed from vector autoregression (VAR) and error corrections models (ECM) that are estimated for 1875-2002, using the variables log GDP, log S&P500 prices, and log S&P500 dividends. Col.1 is based on a VAR in levels with two lags and a constant. ECM #1 in Col.2 is estimated in differences with two lags and includes an estimated cointegrating vector (see text). ECM #2 in Col.3 instead includes the log dividend-yield and log dividends-output ratios. Col.4 and Col.5 and analogous to Col.1&3 but include time trends. Col.6 is computed from annual growth rates.

Table A2: The Allocation of Risk without and with Habit Formation (Benchmark Parameters)

Setting:	Laissez-Faire Economy (1)	Efficient with $\bar{h} = 0$ (CRRA) (2)	Efficient with habit $\bar{h} = 0.25$ (3)	Efficient with habit $\bar{h} = 0.50$ (4)
Consumption responses to permanent shocks (a_t):				
Workers: $\pi_{c1,a} =$	0.49	0.63	0.51	0.37
Retirees: $\pi_{c2,a} =$	0.69	0.63	0.69	0.77
Ratio: $\pi_{c1,a}/\pi_{c2,a} =$	143%	100%	135%	205%
Consumption responses to temporary shocks (z_t):				
Workers: $\pi_{c1,z} =$	0.49	0.54	0.46	0.37
Retirees: $\pi_{c2,z} =$	0.61	0.54	0.60	0.69
Ratio: $\pi_{c1,z}/\pi_{c2,z} =$	126%	100%	130%	186%
Implied responses to the lagged capital labor ratio (k_{t-1})				
Workers: $\pi_{c1,k} =$	0.51	0.37	0.32	0.25
Retirees: $\pi_{c2,k} =$	0.31	0.37	0.41	0.47
Implied responses to lagged consumption (χ_{t-1})				
Workers: $\pi_{c1,\chi} =$	0	0	0.17	0.37
Retirees: $\pi_{c2,\chi} =$	0	0	-0.10	-0.24

Notes: The responses are the elasticities of worker/retiree consumption with respect to the state variables. The case $\bar{h} = 0$ in Col.2 is equivalent to the power utility model of Section 3.

Table A3: How the Habit Model Rationalizes Relatively-safe Transfers

Policy:	Safe Transfers (1)	Calibrated U.S. Policy (2)	Wage-indexed Transfers (3)
Assumed $\pi^*_{b,a}$ =	0.0	0.11	0.67
Implied habit $\bar{h}$ =	0.491	0.455	0.258
Implied other policy parameters:			
$\pi^*_{b,z}$ =	0.00	0.075	0.42
$\pi^*_{b,k}$ =	-0.82	-0.77	-0.53
$\pi^*_{b,\chi}$ =	1.82	1.66	0.87

Note: The parameter $\bar{h}$ in each column is set to rationalize the given policy response to permanent shocks ($\pi_{b,a} = \pi^*_{b,a}$). The bottom three rows show the efficient policy responses to the other state variables if the habit stock $\bar{h}$ takes the value required to rationalize the assumed response to permanent shocks.