TECHNICAL APPENDIX TO:

THE BEHAVIOR OF U.S. PUBLIC DEBT AND DEFICITS*

Henning Bohn

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A1. A General Theoretical Condition for Sustainability

In the text, I claim that ρ>0 in eq. (2) is sufficient for sustainability. To be precise, one can show the following:

PROPOSITION 1: If the primary surplus to income ratio can be written as

(2) $$s_t = \rho \cdot d_t + \mu_t,$$

where $\rho > 0$ and μ_t is a bounded stochastic process, and provided the stream of aggregate income Y_t has a finite present value, then government policy satisfies the intertemporal budget constraint.

COROLLARY: Eq. (2) can be replaced by the non-linear relationship $s_t = f(d_t) + \mu_t$, if there is value d^* such that $f(\cdot)$ satisfies $f'(d) \geq \rho > 0$ for all $d \geq d^*$.

PROOF: Bohn (1995) has shown that a stochastic general equilibrium model implies the transversality condition

$$\lim_{N \to \infty} E_t[u_{t,N} \cdot D_{t+N}] = 0, \tag{A1}$$

where $u_{t,N}$ is the marginal rate of substitution between periods t and t+N and $D_{t+N} = d_{t+N} \cdot Y_{t+N}$ the level of government debt in period N periods ahead. The transversality condition implies the general intertemporal budget constraint

$$D_t = \sum_{i=1}^{\infty} E_t[u_{t,i} \cdot (s_{t+N} Y_{t+N})].$$

The expectation in (A1) can be interpreted as product of debt times state-contingent claims prices summed over all states of nature. Equation (A1) reduces to the standard deterministic or expected value condition

$$\lim_{N \to \infty} \tfrac{1}{(1+r)^N} E_t[D_{t+N}] = 0 \tag{A2}$$

if individuals are risk-neutral or if there is no uncertainty, but not otherwise.

The rate of return R_{t+1} on any security (including government debt secuties) must satisfy the Euler equation $E_t[u_{t,1} \cdot (1 + R_{t+1})] = 1$. But since $u_{t,1} \cdot (1 + R_{t+1})$ may differ from one for any or all states of nature, the marginal rate of substitution in (A1) cannot be measured by the discount rate on government debt. The appropriate discount rate in (A1) depends on how the overall level of government debt is distributed across state of nature and not on the promied return on any particular debt security. (See Bohn, 1995 for more details.)

To show that the policy defined by (2) satisfies (A1), we have to prove that $z_n = E_t[u_{t,n} \cdot D_{t+n}]$ converges to zero as $n \to \infty$. That is, given any $\varepsilon > 0$, there must be a value N^* such that $|z_n| < \varepsilon$ for all $n \geq N^*$. By iterating on (1) and (2), one obtains

$$d_{t+n} = (1-\rho)^n (\prod_{j=1}^{n} x_{t+j}) \cdot d_t - \sum_{i=1}^{n} (1-\rho)^{n-i} (\prod_{j=i}^{n} x_{t+j}) \cdot \mu_{t+i-1} \qquad \text{(A3)}$$

Using the relations $z_n = Y_t \cdot E_t[u_{t.n} \prod_{j=1}^{n}(1+y_{t+j})] \cdot d_{t+n}$ and $E_t[u_{t+i,1} \cdot (1+R_{t+i+1})] = 1$, and taking expectations, (A3) implies

$$z_n / Y_t = (1-\rho)^n \cdot d_t - \sum_{i=1}^{n} (1-\rho)^{n-i} \cdot E_t[a_{i-1}],$$

where $a_k = u_{t.k} \prod_{j=1}^{k}(1+y_{t+j})] \cdot \mu_{t+k}$.

By assumption, the present value of future income,

$$V_t = Y_t \cdot \sum_{k \geq 1} E_t[u_{t,k} \prod_{j=1}^{k}(1+y_{t+j})]$$

is finite. Finiteness of this sum implies that the elements in the sum must converge to zero, i.e., $E_t[u_{t,k} \prod_{j=1}^{k}(1+y_{t+j})] \to 0$ as $k \to \infty$. Combined with a bound on μ_t, this implies $E_t a_k \to 0$ as $k \to \infty$. That is, for any $\delta > 0$ there is an N such that $|E_t a_k| \leq \delta$ for all $k \geq N$.

Denote $\sum_{i=1}^{N} (1-\rho)^{N-i} \cdot E_t[a_{i-1}] = \Omega$. Then for $n > N$ we have

$$|z_n / Y_t - (1-\rho)^n \cdot d_t + (1-\rho)^{n-N} \cdot \Omega| \leq \sum_{i=N}^{n} (1-\rho)^{n-i} \cdot |E_t[a_{i-1}]| \leq \delta / \rho$$

Since $(1-\rho)^n \to 0$ as $n \to \infty$, the absolute value of z_n will be less than ε for high enough n, provided one picks $\delta < \varepsilon \cdot \rho / Y_t$. Q.E.D.

The corollary follows from a dominance argument, since future debt under the policy $s_t = f(d_t) + \mu_t$ is less than under the linear policy $s_t = f(d^*) + \rho \cdot (d_t - d^*) + \mu_t$.

The key element in this proposition is the requirement that the government responds to higher initial debt by increasing the primary surplus ($\rho > 0$), at least linearly for high debt-income ratios. If the debt-income ratio becomes high (for whatever reason—high interest rates, low growth, or high μ_t realizations), the government will then run sufficient primary surpluses to slow down the debt-accumulation process and to satisfy the intertemporal budget constraint. The other assumptions are technical—sufficient, and much stronger than necessary, to make the non-debt determinants of s_t small relative to the $\rho \cdot d_t$ term at high levels of the debt-income ratio.

A2. Tax Smoothing under Uncertainty

Barro (1979) argues that, under certain assumption, tax-smoothing implies random walk behavior for the debt-GDP ratio. This section provides an example demonstrating that tax-smoothing in a stochastic model does not imply a non-stationary path for the debt-income ratio. Moreover, optimal taxes respond positively to unexpected jumps in government debt.

Consider the following simple Lucas (1978) exchange economy with government and with cost of tax collection (as in Barro 1979). Consumers maximize $\sum_{t\geq 0}\beta^t u(c_t)$ subject to $A_{t+1} = (1+R_{t+1}) \cdot [A_t + Y_t(1-\tau_t - h(\tau_t)) - c_t]$, where c=consumption, A=assets = claims on the government, R = return on government debt, Y = exogenous income, τ = tax rate, and $h(\tau) = h/2 \cdot \tau^2$ = cost of tax collection. Government debt satisfies $D_{t+1} = (1+R_{t+1}) \cdot [D_t + G_t - Y_t \tau_t]$. To simplify, assume that utility is CRRA with risk aversion α, that $G_t = g \cdot Y_t$ for all t and that income growth y_t is lognormal i.i.d. Then the individual first order condition for R_{t+1} is

$$E_t[(1+R_{t+1}) \cdot (1+y_{t+1})^{-\alpha} \cdot \beta \cdot \left(\tfrac{1-g-h(\tau_{t+1})}{1-g-h(\tau_t)}\right)^{-\alpha}] = 1. \quad \text{(A1)}$$

If the government were able to borrow on complete markets, it would be straightforward to show that the welfare maximizing policy for “small” values of h would be to issue income-indexed debt and to stabilize the tax rate at a fixed value at all times and for all states of nature. Therefore, to explain why there are any movements in tax rates, one has to impose restrictions on debt management. Specifically, I will impose the same assumption that Barro (1979) apparently imposes implicitly, namely that the government has to use safe debt with return $R_{t+1} = r_t$. Consider a marginal change in (τ_t, τ_{t+1}) that increases D_{t+1} and leaves D_{t+2} unchanged; the first order condition is

$$\tau_t = E_t[\tau_{t+1} \cdot \{1 + r_t + [d_t + g - \tau_t]\tfrac{dr_t}{d\tau_t}\} \cdot (1 + y_{t+1})^{-\alpha} \cdot \beta \cdot \left(\tfrac{1-g-h(\tau_{t+1})}{1-g-h(\tau_t)}\right)^{-\alpha}].$$

It is straightforward to show by differentiating (A1) that $dr_t / d\tau_t$ converge to zero in the limit as h becomes small. In the limit, optimal tax policy is characterized by

$$\tau_t = E_t[\tau_{t+1} \cdot (1+r) \cdot \beta(1+y_{t+1})^{-\alpha}], \tag{A2}$$

where $r_t = r$ is determined by $E_t[(1+r) \cdot \beta(1+y_{t+1})^{-\alpha}] = 1$. To examine what this condition implies for debt service, consider the class of linear polices $\tau_t = g + \rho \cdot d_t$. Since this class of policies implies $d_{t+1} = (1+r)/(1+y_{t+1})(1-\rho) \cdot d_t$, substitution into (A2) yields

$$(1-\rho)E_t[(1+r)^2 \cdot \beta(1+y_{t+1})^{-\alpha-1}] = 1, \tag{A3}$$

For comparison, note that the optimal policy in a hypothetical economy with complete markets will also fall into this linear class (suggesting that the linearity restriction is not unreasonable) with a parameter $\rho_v = v/(1+v) > 0$, where v is defined by $(1+v) \cdot E_t[\beta(1+y_{t+1})^{-\alpha+1}] = 1$. (The value v can be interpreted as ratio of income to the present value of income, which is positive because of dynamic efficiency.) Using the lognormality assumption, one has

$$\log(1+r) = -\log(\beta) + \alpha \cdot x - \alpha^2 \cdot \sigma^2/2$$

$$\log(1+v) = -\log(\beta) + (\alpha+1) \cdot x - (\alpha+1)^2 \cdot \sigma^2/2$$

$$\log(1-\rho) + 2\log(1+r) = -\log(\beta) + (\alpha-1) \cdot x - (\alpha-1)^2 \cdot \sigma^2/2$$

where (x, σ^2) are the mean and variance of $\log(1+y_t)$. After some algebra, this implies

$$\log(1-\rho) + \log(1+v) = -\sigma^2 < 0,$$

hence $\rho > \rho_v > 0$. That is, regardless of the interest rate r, the optimizing government will run a primary budget surplus that is at least as large as the surplus it would have run under the optimal policy with complete markets. In particular, the government will not try to exploit an interest rate on safe debt below the average growth rate to run primary deficits.

A3. Description of the Data

Except for the budget surplus s_t, the data are based on Barro (1986a). For 1916-83, the series YVAR and GVAR were taken directly from Barro (1986a), except that an adjustment was made for YVAR in 1925 and 1930 where the values did not match those in Barro (1986b). The variables were updated for 1984-95 using the methods explained in Barro (1981, equation 14; 1986a, p. 204) and Sahasakul (1986; equations 20, 21).

The debt series d_t is the ratio of privately held public debt (from the WEFA database, Federal Reserve Banking and Monetary Statistics, and recent issues of the Economic Report to the President) relative to either GDP (1959-95) or GNP (prior to 1959). Prior drafts of the paper used GNP as scale variable throughout the sample (yielding very similar results); following a referee request, I have shifted to GDP series for the period over which it is published in the Economic Report. The series differs lightly from Barro's debt-GNP series because of data revisions (mainly in GDP & GNP) and because of the inclusion of minor amounts of non-interest bearing debt. In lines 3 and 4 of Tables I and II, Barro's original debt-GNP series was used for comparison.

The primary budget surplus s_t was constructed by dividing the difference of federal receipts and non-interest outlays by nominal GDP, where all series were taken from the National Income and Product Accounts (WEFA database, updated from recent issues of the Economic Report). The calendar year surplus for 1916-28 was obtained by interpolating a series for fiscal years from Bohn (1991a).

The average real return on government debt is computed by averaging the ratio of interest outlays over debt and subtracting the inflation rate measured by the GDP deflator. Compared to other measures (say, using real T-bill returns), this procedure recognizes that government debt is a portfolio of securities with different interest rates. The measure ignores year-to-year capital gains and losses. But since government bonds are issued and redeemed at par, capital gains and losses should average out to zero for long samples. Even if capital gains and losses do not exactly cancel out over a particular sample, it would still be appropriate to exclude them for estimating an ex ante real rate provided the capital gains/losses are unanticipated. In any case, the values here are not far from estimates of real interest rates that one would obtain from Treasury bill data.

The main data series are attached.

Data Set

ENTRY	s(t)	GVAR(t)	YVAR(t)	d(t)
1916:01	-0.007614879647	-0.012000000000	0.000100000000	0.0200000000000
1917:01	-0.077863727740	0.100000000000	-0.000200000000	0.0146533523546
1918:01	-0.137360425379	0.199000000000	-0.005100000000	0.0890883902169
1919:01	-0.065962237772	0.089000000000	-0.003900000000	0.2365673158331
1920:01	0.014944984665	-0.022000000000	0.000300000000	0.2565480609257
1921:01	0.022525373998	0.012000000000	0.004100000000	0.3244166239106
1922:01	0.022840755078	-0.020000000000	0.001300000000	0.2927053847553
1923:01	0.020893495830	-0.024000000000	-0.001600000000	0.2457790599749
1924:01	0.020021875754	-0.019000000000	0.000100000000	0.2377833885848
1925:01	0.017135177572	-0.022000000000	-0.001100000000	0.2058835405765
1926:01	0.018167760447	-0.024000000000	-0.001800000000	0.1976818469274
1927:01	0.018432965562	-0.020000000000	-0.001900000000	0.1805893142338
1928:01	0.015424837091	-0.019000000000	-0.000400000000	0.1660725353409
1929:01	0.013170324315	-0.019000000000	-0.000900000000	0.1486366234356
1930:01	0.005364582970	-0.012000000000	0.002303225806	0.1598645725003
1931:01	-0.023028398500	-0.002000000000	0.007800000000	0.1893445046937
1932:01	-0.016708169013	0.008000000000	0.010600000000	0.2690012521009
1933:01	-0.015708646765	0.015000000000	0.012600000000	0.3176637456875
1934:01	-0.034269561169	0.012000000000	0.012200000000	0.3188557783299
1935:01	-0.028194997460	0.014000000000	0.009600000000	0.3396827502056
1936:01	-0.037638597656	0.002000000000	0.006500000000	0.3552141543723
1937:01	0.002141121861	-0.002000000000	0.004300000000	0.3586397049831
1938:01	-0.018312450103	0.018000000000	0.007700000000	0.3932026395606
1939:01	-0.017129071242	0.001000000000	0.006700000000	0.3853396280944
1940:01	-0.005841196672	0.014000000000	0.005000000000	0.3704486929483
1941:01	-0.032710701364	0.047000000000	0.001300000000	0.3208764038572
1942:01	-0.197329522205	0.182000000000	-0.003500000000	0.3221205665027
1943:01	-0.228252267352	0.303000000000	-0.006100000000	0.4773008417899
1944:01	-0.242276116610	0.341000000000	-0.006200000000	0.6547003331541
1945:01	-0.179088060794	0.302000000000	-0.004900000000	0.8780353261957
1946:01	0.034053479909	0.037000000000	-0.001800000000	1.0464542339468
1947:01	0.072309639682	-0.024000000000	-0.002500000000	0.8527551100874
1948:01	0.046704244233	-0.029000000000	-0.002800000000	0.7424113196481
1949:01	0.006381012416	-0.024000000000	0.000800000000	0.7173055076606
1950:01	0.046108239406	-0.012000000000	-0.000500000000	0.6709431829330
1951:01	0.032248647969	0.010000000000	-0.003900000000	0.5822346805978
1952:01	0.002223980445	0.017000000000	-0.005000000000	0.5420884573986
1953:01	-0.006838836455	0.007000000000	-0.005600000000	0.5221186593345
1954:01	-0.003935923672	0.006000000000	-0.000200000000	0.5313528838552
1955:01	0.021913244871	-0.017000000000	-0.002400000000	0.4950949055579
1956:01	0.025337313689	-0.020000000000	-0.003800000000	0.4688544775211
1957:01	0.016904500879	-0.019000000000	-0.002700000000	0.4321483942698
1958:01	-0.010912608193	-0.007000000000	0.002300000000	0.4219541834609
1959:01	0.010054194184	-0.017000000000	-0.000200000000	0.4015771315919
1960:01	0.018610832265	-0.018000000000	-0.000200000000	0.3999430280587
1961:01	0.004405284508	-0.001000000000	0.002300000000	0.3808737151248
1962:01	0.004442733991	-0.016000000000	-0.000200000000	0.3627664573455
1963:01	0.012309684159	-0.017000000000	0.000200000000	0.3501781665047
1964:01	0.007088722145	-0.021000000000	-0.000800000000	0.3275894574111
1965:01	0.012376151573	-0.021000000000	-0.002000000000	0.3043977055449
1966:01	0.009393247017	-0.013000000000	-0.003600000000	0.2750698146738
1967:01	-0.004078696017	-0.001000000000	-0.003400000000	0.2585172744722
1968:01	0.005820498037	0.000000000000	-0.004000000000	0.2411662960217
1969:01	0.021481839701	-0.004000000000	-0.004000000000	0.2303952760315
1970:01	0.001641520821	-0.009000000000	-0.001200000000	0.2135908267954
1971:01	-0.007375319338	-0.011000000000	0.000600000000	0.2035765855826
1972:01	-0.001939746621	-0.015000000000	0.000000000000	0.1996322563698
1973:01	0.008968268031	-0.023000000000	-0.001500000000	0.1892707975482
1974:01	0.006079366670	-0.019000000000	0.000000000000	0.1742992951866

1975:01	-0.028394517356	-0.003000000000	0.006300000000	0.1661964920888
1976:01	-0.014733835455	-0.008000000000	0.004500000000	0.1920887874873
1977:01	-0.008338066458	-0.011000000000	0.003200000000	0.2020376446210
1978:01	0.002574829524	-0.016000000000	0.001100000000	0.2013201680214
1979:01	0.010322619746	-0.016000000000	0.000600000000	0.1988661192571
1980:01	-0.002873318907	-0.007000000000	0.003400000000	0.1941259394277
1981:01	0.002760098206	-0.004000000000	0.004400000000	0.1978240315800
1982:01	-0.018907708798	0.008000000000	0.008500000000	0.2142146619063
1983:01	-0.023246387492	0.008000000000	0.008000000000	0.2413981690271
1984:01	-0.013837524104	0.000591724334	0.002789975557	0.2620421917141
1985:01	-0.015978257100	-0.000657689466	0.002258186619	0.2900249361646
1986:01	-0.016123310362	-0.001813603144	0.001846372121	0.3204757839751
1987:01	-0.003367200695	-0.005867188418	0.000407955866	0.3414085767716
1988:01	0.001881317332	-0.010948483464	-0.001008184274	0.3428786438530
1989:01	0.006931835419	-0.013151494506	-0.001398702245	0.3417192606655
1990:01	0.004387320296	-0.011847536863	-0.000816633639	0.3509508036892
1991:01	-0.000566196386	-0.004822348258	0.001796193032	0.3867543848530
1992:01	-0.013624152744	-0.002554864756	0.003329827579	0.4104781465067
1993:01	-0.009659771553	-0.005968603896	0.002101056719	0.4333773338725
1994:01	0.001604019782	-0.010566045628	0.000622337870	0.4394221203339
1995:01	0.008729937171	-0.013572905255	-0.000414530310	0.4367381122243