# Voting over Non-Linear Taxes in a Stylized Representative Democracy

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**Abstract**

Meltzer-Richard (1981) show that voting over linear (one-dimensional) taxes leads to a positive tax rate and government transfer under plausible distributional assumptions. Their result provides a theory of welfare to low-productivity individuals. We study voting over general non-linear (multi-dimensional) tax schedules. We assume the tax schedule is set by an elected politician; this assumption eliminates cycling. Voting outcomes are consistent with Director's law: net taxes favor the middle class.

Preliminary–Comments Very Welcome

## INTRODUCTION

Meltzer and Richard (1981) show that labor is taxed to provide a positive per-individual lump-sum transfer in direct democracy with majority rule if taxes are constrained to be linear and the median voter's income is less than average income. Their result provides a non-altruistic theory of welfare to the poor; roughly, the equilibrium linear tax under majority rule with purely selfish voters transfers from a smaller number of higher-productivity voters to a larger number of lower-productivity voters.

Although a linear tax has two parameters (a tax rate and a per-capita transfer amount), policy is one-dimensional in Meltzer-Richard because the government budget balances so the choice of the tax rate fully determines the size of the per-capita transfer. The linear-tax (one-dimensional policy) restriction matters; without it, voting equilibrium fails generically to exist.[1] This can be seen by considering an electorate of three voters: any tax policy can be upset by a coalition of two voters who support a new policy that would effectively take resources from the third voter.

The linear-tax restriction imposed in Meltzer-Richard is not the outcome of a political process. We assume that policy is determined in a simple form of representative democracy and extend the ideas in Meltzer-Richard to general nonlinear (multi-dimensional) tax policy.[2] Voters elect a single agent who, once elected, implements his/her own most preferred policy.[3] The elected agent maximizes own utility

[1]For instance, Hindriks and de Donder (2000), study a case in which rich and poor voters together extract resources from middle-income voters.

[2]Roemer (1999) assumes a quadratic tax function and shows that progressive taxation results from a political process in which party positions in a general election reflect compromise between internal factions in each party.

[3]Osborne and Slivinski (1996) consider such a setting, which they term one of "citizen-

by choosing a net tax function that transfers resources from individuals with productivities different from his/her own.[4] The maximization is technically similar to that in Mirrlees (1971) in which an integral of utilities is maximized by choosing a tax function that transfers resources to those with greater effective weight in the welfare function. Voters know the policy that would be implemented by each candidate. As in Mirrlees (1971), policy must satisfy incentive-compatibility and budget constraints but is not restricted to any functional form. We end up with a theory of Director's Law instead of a theory of welfare to the poor.[5] Although taxes are still progressive at higher incomes, equilibrium policy tends to transfer net resources from lower-productivity to middle-productivity individuals.[6]

To understand the main forces, consider an election between two candidates who are members of a population of voters. Individuals differ only in productivity, so voter heterogeneity is one-dimensional. The candidate who receives the majority of the votes is elected. This elected agent chooses to impose a negative (low) net tax on himself/herself and generally higher net taxes on others. Suppose: candidates differ in productivity and in the tax functions they would impose; one of the candidates has median productivity; and no voters abstain. Along the single dimension of productivity, voter choices are monotone: any voter with productivity greater than both candidates' productivities prefers the tax function set by the candidate with the greater productivity, and any voter with productivity lower than both candidates' productivities prefers the tax function set by the candidate with the lower

candidates." See also Beasley and Coate (1997).

[4]The *net tax* is the level of payments to government net of transfers from government.

[5]See Stigler (1970), who argued that the middle classes sometimes have relatively great political clout and also disproportionately benefit from many government spending programs. Recent analyses of such spending programs are in Gouveia (1997) and Dixit and Longregan (1998).

[6]We use *equilibrium* to mean that a policy that would win an election against any alternative policy.

productivity.

Monotonicity implies that the candidate that receives the vote of median-productivity voters wins under majority rule. A corollary is that a candidate with median productivity beats any other candidate who would choose a different policy. Specifically, if the other candidate has less-than-median productivity, the median-productivity candidate obtains all votes of those with greater-than-median productivity and some votes of those with less-than-median productivity, and if the other candidate has greater-than-median productivity, then the median-productivity candidate obtains all votes of those with less-than-median productivity and some votes of those with greater-than-median productivity. In this way, the assumption of representative democracy helps rule out cycling, and gives a median-voter outcome.

Resulting positive marginal (net) tax rates at higher incomes are technically similar to progressive taxation in Meltzer-Richard and Mirrlees. Different from Meltzer-Richard and Mirrlees is that individuals with below-median productivity are taxed more than on those with median productivity. This implies that marginal rates at incomes just below that earned by median-productivity individuals may be negative.

Such negative net marginal taxes at "lower-middle" incomes need not occur if government also finances non-redistributional spending or if agents are altruistic. (Meltzer and Richard focus on the implications of self-interested behavior and thus abstract from altruism; they also abstract from non-redistributional spending.) Specifically, positive non-redistributional spending effectively gives government a revenue requirement that can make predicted marginal rates positive at all incomes. Similarly, a strong altruistic preference to provide a utility floor for all individuals effectively gives government a positive revenue requirement that can cause uniformly positive marginal rates.

Predicted negative or at least low marginal net are consistent with spending programs that transfers resources more to middle-income individuals than to lower-income individuals. Such programs make the net tax fall with income more than it otherwise would have, which tends to push net marginal rates down and possibly make them negative.

Section I describes the model. Because Mirrlees-type models are technically complex, section II uses a numerical example to provide an intuition for the results.[7] Formal analysis and a sufficient condition for a median voter to be the elected agent are in sections III-IV. The analysis is by backward induction. First, we characterize the tax functions that would be set by agents with different productivities if they were elected. Second, we characterize the profiles of individual utilities that would be induced if agents with different productivities were elected. An individual votes in a two-candidate election for the candidate who would give the individual greater utility. We show that a median-productivity agent wins an election against a candidate that would implement any policy other that the median agent's policy. Section V shows that inclusion of non-redistributional spending or altruism can cause the elected agent's tax function to have non-negative marginal rates throughout. Section VI concludes.

## I. MODEL

We follow Meltzer-Richard and Mirrlees in assuming that tax-paying units (henceforth, "individuals") differ only in productivity so heterogeneity among individuals is one-dimensional. Specifically, we assume that all individuals have the same utility function and that there is no non-labor wealth. There is a unit mass of in-

[7]Numerical techniques are often used to gain insight into the shape of the welfare-maximizing tax function—see Mirrlees, (1971), Tuomala (1984), Kanbur and Tuomala (1990), Diamond (1998). A survey is in Tuomala (1994). Numerical simulation has also been used to evaluate dynamic voting models—see Krusell and Ríos-Rull (1999).

dividuals. Productivity $x$ has distribution function $F(x)$ with continuous density $f$ and finite mean over support $[x_-, x_+]$, with $x_+ > x_- \geq 0$; $x_+$ may be infinite. Below, "all $x$" means "all $x \in [x_-, x_+]$" and "some $x$" means "some $x \in [x_-, x_+]$."

Each individual has strictly increasing, differentiable, weakly concave utility $u$ defined over consumption $c \geq 0$ and leisure $l$, where $0 \leq l \leq 1$. Each maximizes utility by choosing how much to labor to supply, $n \equiv 1 - l$, which determines income, $y \equiv nx$, and hence consumption, $c \equiv y - T$, where $T$ is the tax the individual pays. To ensure that labor supply is always strictly less than one, we assume that $\lim_{l->0} u_l(c,l)/u_c(c,l) = \infty$ for any $c > 0$, where subscripts denote partial derivatives here and below.[8]

Prior to making labor-leisure choices, individuals elect one of their members (the "elected agent," denoted $e$) to form a government that finances transfers and possibly other government spending out of tax revenue. The government can observe individuals' incomes but not their productivities or work efforts, and hence it imposes a tax function $T^*$ that specifies the positive or negative net taxes paid by each individual as a function of the individual's income.

Elections are between two exogenously given candidates and are settled by majority rule. Each individual is a potential voter. If the two candidates' tax functions give a voter different utilities, then the voter votes for the candidate that would provide the higher utility. We characterize the electoral winner according to the Condorcet criterion, asking whether there is a tax function such that a candidate who would impose that function would beat a candidate who would impose any alternative tax function.

The elected agent maximizes own utility by choosing a net-income-tax function $T^*$ subject to a government budget constraint $\int_{x_-}^{x_+} T^*(Y(x))dF(x) \geq G$, where $G \geq 0$

[8]Meltzer and Richard make similar assumptions. Our analysis also applies to preferences with finite $u_l(c,0)/u_c(c,0)$ provided $l \geq 0$ is not a binding constraint.

is an exogenous level of non-redistributional spending and $Y$ is the profile of incomes chosen by all individuals when the tax function is $T^*$. Meltzer and Richard (1981) assume $G = 0$; we allow $G > 0$ to show how government spending influences the shape of the tax function.

We assume that tax functions $T^*$ exist that strictly satisfy the government budget. Let $\hat{G}$ denotes the supremum of levels of $G$ that can be financed. Specifically, we assume that $\hat{G} > 0$ and $G \in [0, \hat{G})$.

A key constraint on government's power to tax is the "threat" of citizens not to work.[9] We express this as a minimum utility constraint of the form

$$U(x) \geq u_{\min} > -\infty, \text{ for all } x, \tag{1}$$

where $U(x)$ is the utility obtained by an individual with productivity $x$ under the tax function chosen by $e$. (We suppress dependence of $U$, $Y$ , $T^*$ and other functions on $e$ here and below when no ambiguity arises.) The natural interpretation is that $u_{\min}$ is the utility of a non-worker with zero taxes, $u_{\min} = u(0, 1)$.[10]

Following Mirrlees (1971) and Seade (1982), we make the *agent monotonicity assumption* that $nu_l(c, 1-n)/u_c(c, 1-n)$ is strictly increasing in $n$ for all $c > 0$. This ensures that the marginal rate of substitution between $c$ and $l$ declines with $x$ so income is a signal for productivity. Sufficient conditions for agent monotonicity are that consumption is normal or that utility is separable. Meltzer and Richard take consumption to be normal, which is slightly stronger.

[9]Intuitively, taxation without the right to refuse work could lead to tax policies that force low productivity individuals to work essentially all the time ($n \approx 1$) and to pay almost their entire income in taxes. Such slavery-like solutions seem uninteresting.

[10]If $u$ is unbounded, then $u_{\min} = u(0, 1) = -\infty$ could occur, violating (1). In such cases, we redefine $c$ to be consumption financed *from taxed income,* and assume that individuals also obtain consumption from home production, underground labor, or altruistic transfers. These assumptions make $u(0, 1)$ is finite, and the analysis below goes through we the redefined variable $c$.

Agent $e$'s utility maximization induces profiles of indirect utilities $\{U(x)\}_x$ and incomes $\{Y(x)\}_x$ for each individual under the policy implemented by $e$. To characterize the policy choices of a given elected agent $e$, we follow Mirrlees (1971, 1986) by transforming the controls into the choice of a utility and income profiles and then invoking the revelation principle to formulate the problem as a standard control problem. Specifically, the elected agent's problem of choosing the net-income-tax function $T^*$ is equivalent to choosing a tax profile $\{T(x)\}_x$ and an income profile $\{Y(x)\}_x$ subject to incentive-compatibility constraints requiring that any individual with productivity $x$ be induced to work $n(x) = Y(x)/x$ in order to earn the assigned income $Y(x)$. Because utility, income, and taxes are linked by the identity

$$U(x) \equiv u(Y(x) - T(x), 1 - Y(x)/x), \text{ for all } x, \tag{2}$$

the choice of tax and income profiles is equivalent to the choice of utility and income profiles.

To express the incentive-compatibility constraints as functions of the controls $\{U(x)\}_x$ and $\{Y(x)\}_x$, denote the consumption level that provides utility $U$ at work effort $n$ by $c^*(U, n)$; this function is defined by the identity $U = u(c^*, 1-n)$. Incentive-compatibility then requires

$$U(x) \geq u(c^*(U(z), Y(z)/z), 1 - Y(z)/x), \text{ for all } x \text{ and } z \text{ satisfying } x \geq Y(z), \tag{3}$$

which says that income and utility profiles must be such that an individual with true productivity $x$ does not prefer the income-and-tax package of any other individuals ($z$) to his own. The constraints apply only for $Y(z) \leq x$ because an individual with productivity $x$ cannot earn an income $Y(z) > x$ within the time constraint $n \leq 1$.

From Mirrlees (1971, 1986), the incentive-compatibility constraints (3) are equivalent to the combination of a differential equation

$$\begin{aligned} U(x) - U(x_-) &= \int_{x_-}^{x} \omega(U(z), Y(z), z) dz, \\ \text{where } \omega(U, y, z) &\equiv u_l(c^*(U(z), y(z)/z), 1 - y(z)/z) \cdot y(z)/z^2, \end{aligned} \tag{4}$$

plus requirements that $Y(x)$ be monotone non-decreasing on $[x_-, x_+]$ and that $Y(x) < x$.[11] We therefore replace (3) by (4 below, and impose monotonicity by using the income derivative

$$\psi(x) \equiv dY/dx \geq 0$$

as a piecewise continuous control constrained to be non-negative; this treats $U$ and $Y$ as state variables.[12] Our assumption that $\lim_{l->0} u_l(c, l)/u_c(c, l) = \infty$ for any $c > 0$ ensures that $Y(x) < x$ always holds.

The government budget constraint can also be expressed in terms of the profiles $\{U(x)\}_x$ and $\{Y(x)\}_x$,

$$\int_{x_-}^{x_+} [Y(x) - c^*(U(x), Y(x)/x)] dF(x) \geq G. \tag{5}$$

Thus the elected agent solves the problem:

$$\text{maximize } U(x_e),$$
$$\text{by choice of } \{\psi(x)\}_x \geq 0, \text{ subject to (1), (4), and (5),}$$

where $x_e$ denotes $e$'s productivity. We refer to this as **e's problem**. The tax profile $\{T(x)\}_x$ that solves $e$'s problem is obtained from (2).

[11]Specifically, theorem 1 in Mirrlees (1971) shows that (3) implies a non-decreasing income profile and $Y(x) < x$. Lemma 6.1 in Mirrlees (1986) proves that (3) implies (4) for a lower bound of 0 instead of $x_-$; taking the difference $U(x) - U(x_-)$ yields (4) for any $x_- \geq 0$. That (4) implies (3) is a special case of Mirrlees (1986) lemma 6.3, as discussed in Mirrlees (1986, p. 1237).

[12]That is, we follow Brito and Oakland (1977) and Ebert (1992) in using a "second-order" approach; see Miles (1995) for discussion. Of course, $dY(x)/dx$ is undefined at points where $\psi$ is discontinuous.

## II. AN EXAMPLE

This section presents an example that highlights the model's salient features without dwelling on technical complications.

Diamond (1998) reduces Mirrlees' analysis to a single differential equation by assuming preferences are affine: $u(c, l) = c + v(l)$. We follow Diamond in assuming affine preferences with a constant elasticity of labor supply, $u(c, l) = c + v_0\{1 - (1-l)^{1+v_1}\}/(1+v_1)$, where $v_0$ is a constant and $v_1$ is the wage elasticity of labor supply. In the example, we take $v_0 = 5$, $v_1 = 1$, and assume that $F(x)$ is uniform on $[x_-, x_+] = [.2, 1.8]$.[13] We also assume that $G = 0$. We also assume that $G = 0$.

Consider an election between a median-productivity individual $e_{med}$ and an alternative individual $e_{alt}$ whose productivity, $x = .84$, is at the 40th percentile. The solid line in figure 1 is the tax function $e_{med}$ would set if elected and the dashed line is the function $e_{alt}$ would set if elected. Because $G = 0$ and the government budget binds, some individuals gain ($T < 0$) and others lose ($T > 0$) from either imposed tax function.

If elected, either individual would impose a low tax rate on himself/herself and would transfer resources from others by imposing a net tax function that is V-shaped with a local minimum at the individual's own income $y_e = y(x_e)$, as in the figure. The reason for this shape is that the incentive-compatibility constraints imply that voters with similar productivities cannot be taxed too differently. The steepness of the V balances two forces. A steeper V tends to extract greater resources from others by raising taxes paid by others, increasing the net transfer to the elected agent. Because the agent's own income level is tax-favored and taxes are based on income and not productivity, however, individuals with productivities $x \neq x_e$ have an incentive to

[13] Although affine utility violatesour assumption that $\lim_{l->0} u_l(c,l)/u_c(c,l) = \infty$, the parameter values we choose ensure that leisure is bounded away from zero so $Y(x) < x$.

change their labor supplies to earn the tax-favored income level $y_e$. This means that a steeper V also distorts individual incentives toward earning the elected agent's income, reducing the net transfer to the elected agent.

An individual's incentive to distort labor choice to earn the tax-favored income level decreases with the difference between the individual's productivity and $e$'s productivity. Thus there is an interval of productivities around $x_e$, denoted $[x_1, x_2]$ below, such that $e$'s optimal tax function induces all individuals with productivities in $[x_1, x_2]$ to earn $y_e$. This gives a mass-point in the income distribution at $y_e$. The imitation-intervals are illustrated in figure 2, where interval endpoints are slightly higher if $e = e_{med}$ than if $e = e_{alt}$. Note that, if the elected agent were to impose a (suboptimal) tax function with a steeper V, more agents would imitate $e$'s income and the interval $[x_1, x_2]$ would expand.

The minimum-utility constraints (1) limit government ability to impose positive taxes on individuals with low market incomes.[14] This explains why the two tax functions rise from the origin, peak, and then fall at incomes below $e$'s. Without constraints (1), the model would have very-low-productivity individuals paying *more* taxes than median-productivity individuals, a strong form of Director's law. No corresponding effect is present at higher incomes, so marginal rates are positive at all incomes above $y_e$. Thus marginal rates in the example are positive at very low incomes and then negative up to $y_e$, before again becoming positive at higher incomes.

The tax functions in figure 1 yield different utilities for different voters. The tax function set by $e_{med}$ is better for all voters with productivities greater than 1.0 and the function set by $e_{alt}$ is better for all voters with productivities less than .84. For those in between, there is a cut-off productivity at $x = 0.926$ such that all individuals with productivities above the cut-off receive greater utility if $e = e_{med}$ and all agents

[14]Meltzer-Richard and Mirrlees only consider cases in which the optimal or equilibrium policy is to transfer to low-productivity agents, so minimum-utility constraints would not bind in their models.

below the cut-off receive greater utility if $e = e_{alt}$. This is illustrated in figure 3, where the utility profile induced by the median-productivity agent is the solid line and the profile induced by the agent with productivity .84 is the dashed line. (The profiles coincide for low-productivity individuals with utility $u_{\min}$.)

It is clear from figure 2 that a majority of individuals would receive higher utility if the median-productivity candidate were elected. Thus $e_{med}$ would win an election against $e_{alt}$. This would also be true for any alternative candidate, not just one with $x = .84$. Thus a median-productivity candidate would win an election against any opponent.

In the sections below, we show that a median-productivity candidate would win under more general assumptions about $F$, $u$, and $G$. A key to showing this is to establish the single-crossing property of utility profiles in figure 3. With single-crossing, a median-productivity candidate running against any alternative candidate with less-than-median productivity would receive the votes of all individuals with greater-than-median productivities and some individuals with productivities between those of the two candidates, gaining a majority. Similar electoral arithmetic applies if the alternative candidate has greater-than-median productivity.

The specification of simple representative democracy here rules out the cycling that would occur with an infinite-dimensional tax function under Meltzer-Richard's specification of direct democracy. With direct democracy, cycling would occur because it is always possible to find a majority coalition that would prefer an alternative tax function. For instance, majority coalition of "rich" and "poor" voters might design a tax function they prefer over the median voter's function. Were such a coalition to form in our model, however, either a rich or a poor candidate would have to be elected to implement the coalition policy, but once elected, this agent would impose his or her own most preferred policy and not the coalition's.

Figure 4 compares the equilibrium tax function in the example above to the

Meltzer-Richard outcome, which arises if the tax function in our model is constrained to be linear. In both cases, a median-productivity candidate is elected. With non-linear taxation, the elected, median-productivity agent's tax function gives a large net transfer to the elected agent and a negative net transfer to low-productivity individuals. This does not occur under a linear-tax constraint because the median-productivity agent is then constrained to give a large net transfer to low-productivity individuals in order to obtain a small net transfer for him/herself.

Figure 5 illustrates the impact of government spending $G$ on the equilibrium tax functions. For $G = 0$, individuals with incomes near $y_e$ receive transfers while individuals with income near $Y(x_-)$ pay taxes, giving the tax function a mostly negative slope at the lower end. For higher $G$ values, the minimum utility constraint rules out increased taxes on low productivity individuals, forcing the elected agent to impose higher taxes on middle and upper income individuals to collect revenues. As $G$ approaches $\hat{G}$ (about 0.0772 in this example), the negatively sloped segment of the tax function vanishes. Thus, non-linear taxation is consistent with monotonely increasing taxes; this is important to avoid misinterpretations of Figures 1-2. Note that all the equilibrium tax functions display a downward-pointing "kink" at $y_e$. This reflects the elected agent's ability to design a tax function favoring the agent's own equilibrium income level and turns out to be a generic feature of the non-linear tax model.

## III. AN ELECTED AGENT'S POLICY

To solve $e$'s problem, we first remove the complication that the objective $U(x_e)$ is also a point on the utility profile and hence enters a subset of the incentive-compatibility constraints. The approach is to consider a modified version of $e$'s problem that will turn out to have the same solution as $e$'s problem. In the modified problem, $e$ chooses a separate income level $y_e$ and tax level $T_e$ for him/herself, and maximizes $u_e = u(y_e - T_e, 1 - y_e/x_e)$ subject to additional incentive compatibility constraints that no individual capable of earning $y_e$ would prefer earning income $y_e$ and paying taxes $T_e$ over his/her assigned point on the income profile. These additional constraints are

$$U(x) \geq U_e(x) \text{ for all } x \geq y_e, \tag{6}$$

where $U_e(x) \equiv u(\ y_e - T_e, 1 - y_e/x)$ is the utility an individual with productivity $x$ would receive from imitating $e$'s income and paying $e$'s level of taxes. Formally, **e's modified problem** is to

$$\text{maximize } u_e = u(y_e - T_e, 1 - y_e/x_e),$$
$$\text{by choice of } y_e, T_e, \text{ and } \{\psi(x)\}_x \geq 0, \text{subject to (1), (4), (5), and (6).}$$

A given pair $(y_e, T_e)$ is feasible if and only if a profile $\{\psi(x)\}_x$ exists that satisfies (1), (4), and (6), and yields sufficient revenue to fund $G$. We evaluate feasibility by solving the **conditional revenue problem (CRM):**

$$\text{maximize} \int_{x_-}^{x_+} [Y(x) - c^*(U(x), Y(x)/x)] dF(x) \tag{7}$$
$$\text{by choice of } \{\psi(x)\}_x \geq 0, \text{ subject to (1), (4), and (6).}$$

For any pair $(y_e, T_e)$, the solution $\{\psi(x)\}_x$ (if one exists) implies a revenue value $R(y_e, T_e) \equiv \int_{x_-}^{x_+} [Y(x) - c^*(U(x), y/x)] dF(x)$. The pair $(y_e, T_e)$ is feasible if $R(y_e, T_e) \geq G$.

We solve CRM by breaking it into three problems.[15] The first is the **uncon-**

[15] The "simple" approach of applying the maximum principle directly to CRM is problematic because the inequality constraints (6) cause CRM to violate commonly-assumed regularity conditions—see Seierstad and Sydsaeter (1987, ch.5).

**ditional revenue problem (URM):**

$$\text{maximize} \int_{x_-}^{x_+} [Y(x) - c^*(U(x), Y(x)/x)] dF(x)$$

$$\text{by choice of } \{\psi(x)\}_x \geq 0, \text{ subject to (1), and (4),}$$

with optimal profiles denoted $\{\hat{U}(x), \hat{Y}(x)\}_x$; this problem is CRM without (6). Because $y_e$ and $T_e$ enter CRM only through (6), URM is independent of $(y_e, T_e)$.

For $(y_e, T_e)$ pairs with sufficiently low $c_e = y_e - T_e$, the utility from imitating $e$'s income and paying $e$'s level of taxes may be low enough so that $\hat{U}(x) \geq U_e(x)$ for all $x$, in which case $\{\hat{U}(x), \hat{Y}(x)\}_x$ satisfies (6) and therefore solves CRM. For each such pair $(y_e, T_e)$ for which $\hat{U}(x) \geq U_e(x)$ for all $x$, the solution $\{\hat{U}(x), \hat{Y}(x)\}_x$ provides maximum revenue to fund $G$, so $R(y_e, T_e) = \hat{G}$.

For all other $(y_e, T_e)$ pairs, $\hat{U}(x) < U_e(x)$ for at least one value of $x$, which means that $\{\hat{U}(x), \hat{Y}(x)\}_x$ cannot be a solution to CRM and that (6) must bind at some (possibly different) value of $x$ in the solution to CRM. Given a $(y_e, T_e)$ pair with $\hat{U}(x) < U_e(x)$ for at least one $x$, let $x_c$ denote a value of $x$ at which (6) binds in the solution to CRM, or $U(x_c) = U_e(x_c)$. For given $(x_c, y_e, T_e)$, **problem CRM1** then focuses on revenue from individuals with $x \leq x_c$:

$$\text{maximize } R_1(x_c, y_e, T_e) = \int_{x_-}^{x_c} [Y(x) - c^*(U(x), Y(x)/x)] dF(x)$$

$$\text{by choice of } \{\psi(x)\}_x \geq 0, \text{ subject to (1), (4)}, U(x_c) = U_e(x_c), \text{ and } Y(x_c) \leq y_e.$$

Similarly, **problem CRM2** focuses on revenue from individuals with $x \geq x_c$:

$$\text{maximize } R_2(x_c, y_e, T_e) = \int_{x_c}^{x_+} [Y(x) - c^*(U(x), Y(x)/x)] dF(x)$$

$$\text{by choice of } \{\psi(x)\}_x \geq 0, \text{ subject to (1), (4), } U(x_c) = U_e(x_c) \text{ and } Y(x_c) \geq y_e.$$

Problems URM, CRM1, and CRM2 have the same objective function and differential equations. They differ only in domains and boundary conditions. Hence their Hamiltonians and Euler equations have the same form. The Hamiltonian is

$$H(U, Y, \psi, \xi, \mu, x) = [Y - c^*(U, Y/x)] \cdot f(x) + \omega(U, Y, x) \cdot \xi(x) + \psi(x) \cdot \mu(x),$$

where $\xi$ and $\mu$ as the costate variables associated with $U$ and $Y$, and the Euler equations are

$$H_Y = (1 - \frac{u_l}{u_c x}) \cdot f(x) + \omega_Y(U, Y, x) \cdot \xi(x) = -\mu_x(x), \tag{8}$$

$$H_U = (-\frac{1}{u_c}) \cdot f(x) + \omega_U(U, Y, x) \cdot \xi(x) = -\xi_x(x), \tag{9}$$

using $dc^*/dn = u_l/u_c$ and $dc^*/dU = 1/u_c$.

We sidestep the technical complications of existence with:[16]

**ASSUMPTION (CON)**: URM, CRM1, and CRM2 each have at least one solution with a Hamiltonian strictly concave in $(U, Y)$. For CRM1 and CRM2, this is assumed for all $(x_c, y_e, T_e)$ such that $\hat{U}(x_c) \leq U_e(x_c)$.

We then have:

**Lemma 1 (Revenue Maximization)** *Assume CON. Then:*

1. *The solution to URM defines a unique and continuous profile* $\{\hat{U}(x), \hat{Y}(x)\}_x$ *for utility and income.*

2. *For* $(y_e, T_e)$ *such that* $\hat{U}(x) \geq U_e(x)$ *for all* $x$, $\{\hat{U}(x), \hat{Y}(x)\}_x$ *is the unique solution to CRM and* $R(y_e, T_e) = \hat{G}$.

3. *For* $(y_e, T_e)$ *such that* $\hat{U}(x) < U_e(x)$ *for some* $x$, *CRM has a unique solution* $\{U(x), Y(x)\}$ *and:*

[16] The assumption that productivity has a continuous distribution makes existence a technically complicated issue; it would not be had we assumed that productivity had a discrete distribution.

(a) *There is an interval* $[x_1, x_2]$ *such that* $U(x) = U_e(x)$ *and* $Y(x) = y_e$ *on* $[x_1, x_2]$, *and* $U(x) > U_e(x)$ *for all* $x \notin [x_1, x_2]$.

(b) *If* $U(x) = U_e(x)$ *for any* $x \in (x_-, x_+)$ *in the interior of* $[x_-, x_+]$, *then the interval* $[x_1, x_2]$ *is non-degenerate* ($x_1 < x_2$). *The degenerate cases* $x_1 = x_2 = x_-$ *and* $x_1 = x_2 = x_+ < \infty$ *are possible.*

(c) *For any* $x_c \in [x_1, x_2]$ *at which* $U(x_c) = U_e(x_c)$, *the solution to CRM1 on* $[x_-, x_c]$ *together with the solution to CRM2 on* $[x_c, x_+]$ *also solves CRM. The revenue value* $R(y_e, T_e) = R_1(x_c, y_e, T_e) + R_2(x_c, y_e, T_e)$ *does not depend on the choice of* $x_c$.

4. *Revenue* $R$ *is continuous and differentiable in* $(y_e, T_e)$, *and is strictly increasing in* $T_e$ *if* $\hat{U}(x) < U_e(x)$ *for some* $x$.

5. *The function* $T_e^*(y_e; G) = \arg\min\{T_e | R(y_e, T_e) \geq G\}$ *is continuous and differentiable in* $(y_e, G)$, *and is strictly increasing in* $G$ *for* $0 \leq G < \hat{G}$.

Partial Proof (remainder in Appendix): Part 1 follows from CON and the Mangasarian sufficiency theorem. In Part 2, $u(y_e - T_e, 1 - y_e/x) \leq \hat{U}(x)$ ensures that (6) is non-binding. In Part 3, $u(y_e - T_e, 1 - y_e/x_c) > \hat{U}(x_c)$ ensures that $\{\hat{U}(x), \hat{Y}(x)\}_x$ does not solve (CRM) and that there is a point $x_c$ where (6) binds. Given $x_c$, CON implies that CRM1 yields a unique profile $\{U(x), Y(x)\}$ on $[x_-, x_c]$, that CRM2 yields a unique profile $\{U(x), Y(x)\}_x$ on $[x_c, x_+]$. One can show (see Appendix) that the values match at $x_c$ and that the profiles do not depend on the choice of $x_c$, and one can prove by contradiction (exploiting continuity) that (6) must bind on an interval around $x_c$ (denoted $[x_1, x_2]$) except if $x_c = x_-$ or if $x_c = x_+$, and that (6) cannot bind outside this interval; this proves Part 3. Given any $x_c \in [x_1, x_2]$, CON implies that CRM1 yields a unique profile $\{U(x), Y(x)\}$ on $[x_-, x_c]$, that CRM2 yields a unique profile $\{U(x), Y(x)\}_x$ on $[x_c, x_+]$ with matching values at $x_c$, and that these profiles do not depend on the choice of $x_c$; this proves Part 3c and the uniqueness of $\{U(x), Y(x)\}_x$ on $[x_-, x_+]$. In Part 4, continuity is standard (e.g., Seierstad/Sydsaeter 1987, p.213, theorem 9, applies). The strictly increase with respect to $T_e$ follows from a parametric variation of the boundary condition $U(x_c) = U_e(x_c)$ in CRM1 and in CRM2 (which is lengthy, though straightforward; see Appendix). Part 5 follows from the implicit function theorem applied to Part 4. ■

Lemma 1 shows that for any choice of $e$'s own income $y_e$, the revenue requirement $R(y_e, T_e) = G$ defines a unique value $T_e^*(y_e; G)$ of the elected agent's own tax payment. For given $G$, the set

$$C_e = \{(y_e, c_e) \geq 0 | c_e \leq y_e - T_e^*(y_e, G)\} \tag{10}$$

describes $e$'s feasible choices in the income-consumption space. Since income and leisure are negatively related by the identity $l = 1 - y/x_e$, $e$'s indifference curves are positively sloped, convex, and continuous lines in the same space–lines of the form $\{(y, c) \geq 0 | u_e = u(\ c, 1 - y/x_e)\}$ for given utility values $u_e$. The shape of the $C_e$ and of the indifference curves are illustrated in figure 6, using values from Example 1. Thus, $e$'s modified problem reduces to the problem of maximizing $u_e = u(y_e - T_e^*(y_e, G), 1 - y_e/x_e)$ by choice of $y_e$. Finding the solution means finding the highest indifference curve that touches the set $C_e$.

The set $C_e$ is not necessarily convex (as demonstrated by figure 6) and the same set $C_e$ applies to all potentially elected agents, regardless of their own productivity. The former suggests that $e$'s problem may have multiple solutions. The latter will be critical for the voting analysis, because it allows us to compare solutions chosen by different elected agents. The following Proposition characterizes the $e$'s utility maximization and shows that the solution solves $e$'s problem as originally stated in section I.

**Proposition 1 (Utility Maximization)** *For any $x_e \in [x_-, x_+]$, $e$'s modified problem has at least one optimal solution $(y_e, T_e, \{U(x), Y(x)\}_x)$. All solutions have the* *following properties:*

1. *Optimal values of* $y_e$ *maximize* $u_e = u(y_e - T_e^*(y_e, G), 1 - y_e/x_e)$ *for* $0 \leq y_e \leq x_e$.

2. *The solutions to e's problem and e's modified problem coincide, and are given by the income values* $y_e \in \{y_e = \arg\max u(y_e - T_e^*(y_e, G), 1 - y_e/x_e)\}$, *the associated tax values* $T_e = T_e^*(y_e, G)$, *and the profiles* $\{U(x), Y(x)\}_x$ *that solve (CRM1) and (CRM2) for* $(x_c, y_e, T_e) = (x_e, y_e, T_e^*(y_e, G))$.

3. *For* $y_e > 0$, *any optimal value* $y_e$ *must satisfy the first-order condition*

$$\partial T_e^* / \partial y_e = (1 - \frac{u_l}{u_c x_e}),$$

   *with* $u_l$ *and* $u_c$ *evaluated at* $(y_e - T_e^*(y_e, G), 1 - y_e/x_e)$.

4. *If* $x_e \in (x_-, x_+)$ *and* $G < \hat{G}$, *then the income profile* $Y$ *displays bunching on a non-degenerate interval* $[x_1, x_2]$ *that includes* $x_e$, *i.e.,* $Y(x) = y_e$ *for all* $x \in [x_1, x_2]$.

Proof: Part 1 is a restatement of $e$'s modified problem, part 3 is the associated first-order condition, and part 2 referring to the modified problem follows from lemma 1 (uniqueness). From the continuity of $R$, $R(y_e, T_e) = G$ holds at the optimum, that is, $\{U(x)\}_x$ and $\{Y(x)\}_x$ maximize revenue conditional on $(y_e, T_e)$. Solutions to $e$'s modified problem satisfy $U(x_e) \geq u_e$ because (6) applies everywhere, including at $x_e$. If $U(x_e) > u_e$ held, the program cannot be optimal because $e$ could raise own utility by choosing $(Y(x_e), T(x_e))$ instead of $(y_e, T_e)$. Therefore (6) binds at $x_e$, proving $x_e \in [x_1, x_2]$ and $Y(x_e) = y_e$, and showing that any solution to $e$'s modified problem solves $e$'s original problem. Conversely, any feasible profile in $e$'s original problem is feasible in the modified problem, showing that $e$'s original problem cannot have additional solutions. Since $y_e$ maximizes a continuous function on the compact set $[0, x_e]$, at least one solution exists.

In part 4, $G < \hat{G}$ implies $u_e > \hat{U}(x_e)$, so that lemma 1, Part 3 applies; that $[x_1, x_2]$ is non-degenerate follows from Lemma 1, Part 3b.■

The existence of at least one solution is intuitively obvious from figure 6. The first-order condition simply requires that the slope of $e$'s indifference curve, $\frac{u_l}{u_c x_e}$, equal the slope of the constraint set, $dc_e/dy_e = 1 - \partial T_e^*/\partial y_e$. To emphasize the unique mapping from $y_e$ to the income and utility profiles, we will sometimes write the latter as $\{U(x; y_e)\}_x$ and $\{Y(x; y_e)\}_x$. Importantly, proposition 1 implies that these utility and income profiles are solutions to problems CRM1 and CRM2 conditional on $(x_e, y_e, T_e^*(y_e, G))$. Hence, they satisfy the associated Euler equations and boundary conditions. This allows us to characterize tax functions.

Specifically, the tax profile $\{T(x)\}_x$ implied by $e$'s problem is directly determined by the utility and income profiles via identity (2). The tax function $T^*$ is defined by $T^*(y) = T(x)$ for all $x$ and $y = Y(x) \in [Y(x_-), Y(x_+)]$. To determine tax rates, note that the term $1 - u_l/u_c/x$ in (8) can be interpreted as a marginal tax rate. To see this, consider the first-order condition for income for an individual who maximizes utility $u(y - T^*(y), 1 - y/x))$ subject to a given tax function $T^*$. If $T^*$ is differentiable and $y > 0$, the first-order condition $u_c(1 - dT^*(y)/dy) - u_l/x = 0$ implies

$$\frac{dT^*(y)}{dy} = 1 - \frac{u_l(c, 1 - y/x)/x}{u_c(c, 1 - y/x)}. \tag{11}$$

The marginal rate of substitution on the right-hand side is well-defined even at points where $T^*$ is not differentiable, and it is identical to the term $1 - u_l/u_c/x$ in (8). We therefore define the marginal tax rate for all $x$ by

$$\tau(x) \equiv 1 - \frac{u_l(c^*(U(x), Y(x)/x), 1 - Y(x)/x)/x}{u_c(c^*(U(x), Y(x)/x), 1 - Y(x)/x)}. \tag{12}$$

At points where $T^*$ and $Y$ are differentiable, we have $dT^*(Y(x))/dy = \tau(x)$ and $T'(x) = \tau(x) \cdot \psi(x)$. At other points, $\tau(x)$ can be interpreted as an effective or shadow marginal tax rate. Since $\omega_Y > 0$ (as one can show), $sign(\tau(x)) = sign(-\xi(x))$ applies whenever $\mu_x(x) = 0$, that is, whenever $Y$ is increasing. Thus, the sign of $\xi$ determines the slope of the tax function.

Given the monotonicity of $Y$ and bunching at $y_e$, we can partition $[x_-, x_+]$ into distinct subintervals that correspond to different income ranges. Individuals with $x \in (x_2, x_+]$ have income $Y(x) > y_e$ and face tax rates implied by CRM2. Individuals in $[x_1, x_2]$ "mimic" $e$'s income choice and pay taxes $T_e$. For $x < x_1$, there are two possibilities depending on the value of $x_0 = \inf\{x \geq x_- : Y(x) > 0\}$. If $x_0 = x_-$, everyone works (or almost everyone, in case $Y(x_-) = 0$). If $x_0 > x_-$, there is an interval $[x_-, x_0]$ of non-workers with income bunched at $Y = 0$, and $0 < Y(x) < y_e$ hold for $x \in (x_0, x_1)$. In both cases, tax rates on $[x_-, x_1)$ are determined by CRM1.

The following proposition presents the implications of CRM1 and CRM2, shows that the main subintervals are connected in a "smooth" manner, covers special cases, and accounts for the possibility of a binding or non-binding minimum utility constraint (1). For brevity, we use "the tax functions" to mean the profile $T$ and the income-tax function $T^*$ whenever a result applies to both:

**Proposition 2** *The shape of e's tax functions for arbitrary e:*

1. *For a range of productivities $[x_1, x_2]$ that characterize individuals who earn $Y = y_e$ and that includes $x_e$, the tax profile $T$ is flat, $T(x) = T_e$, and the tax function $T^*$ collapses to the point $T^*(y_e) = T_e$. Marginal tax rates $\tau(x)$ are strictly increasing on $[x_1, x_2]$ and may have either sign.*

2. *For $x \in [x_2, x_+]$, which is non-degenerate unless $x_e = x_+$, marginal tax rates are strictly positive, $\tau(x) > 0$, except that $\tau(x_+) = 0$ if $x_+$ is finite. The tax profile $T$ is strictly increasing, except at points (if any) where $Y$ is constant, in which case $T$ is flat. The tax function $T^*$ is strictly increasing for all $y > Y(x_2) = y_e$.*

3. *For $x \in [x_-, x_1]$, which is non-degenerate unless $y_e = 0$ or $x_e = x_-$, marginal tax rates may be all positive, all negative, or they may switch sign once from positive to negative at some point $x_\tau \in (x_-, x_1)$. The tax functions may be increasing, decreasing, or inverted-U shaped on $[x_-, x_1]$. The different shapes are associated with different boundary configurations of CRM1 at $x_-$:*

    (a) *If the minimum utility constraint (1) does not bind, then marginal tax rates are strictly negative, $\tau(x) < 0$, except that $\tau(x_-) = 0$. The tax profile $T$ is strictly decreasing, except at points (if any) where $Y$ is constant, in which case $T$ is flat. The tax function $T^*$ is strictly decreasing for all $y < Y(x_1) = y_e$.*

    (b) *If the minimum utility constraint (1) binds (in the sense that $\xi(x_-) < 0$), then marginal tax rates have a strictly positive segment $[x_-, x_\tau)$, which may or may not extend to $x_1$. That is, the tax functions are increasing or inverted-U shaped.*

4. *If $x_0 > x_-$, the tax profile $T$ is flat on $[x_-, x_0]$, and $T(x) = T(x_0) = T^*(0)$ for all $x \in [x_-, x_0]$.*

5. *At the transition points $\{x_0, x_1, x_2\}$ between intervals, the properties of both adjacent intervals apply. If $y_e = 0$ or $x_e = x_-$, Part 1 applies with $x_1 = x_-$. If $x_e = x_+$, Part 1 applies with $x_2 = x_+$.*

Proof: Lemma 1 shows that $e$'s problem yields bunching on $[x_1, x_2]$, proving Part 1. Since $x_e \in [x_1, x_2]$, the profile $\{U(x), Y(x)\}_x$ solves CRM2 on $[x_e, x_+]$ and CRM1 on $[x_-, x_e]$. Integrating (9), one finds

$$\begin{aligned}\xi(x) &= \int_{x_-}^{x} \frac{\eta(x,z)}{u_c(c^*(U(z), y(z)/z), 1 - y(z)/z)} dF(z) + \xi(x_-) \\ &= -\int_{x}^{x_+} \frac{\eta(x,z)}{u_c(c^*(U(z), y(z)/z), 1 - y(z)/z)} dF(z) + \xi(x_+), \qquad (13)\end{aligned}$$

with $\eta(x,z) \equiv exp\{\int_x^z \omega_U(U(\tilde{z}), y(\tilde{z}), \tilde{z})d\tilde{z}\} > 0$. In CRM2, the transversality condition $\xi(x_+) = 0$ implies $\xi(x) < 0$ for all $x < x_+$. In CRM1, note that (1) is satisfied for all $x$ if and only if $U(x_-) \geq u_{\min}$. This constraint implies the transversality conditions $(U(x_-) - u_{\min}) \cdot \xi(x_-) = 0$ and $\xi(x_-) \leq 0$. If (1) does not bind, $\xi(x_-) = 0$ implies $\xi(x) > 0$ for all $x > x_-$. In (8), $\mu_x(x) = 0$ holds whereever $Y$ is strictly increasing, hence $\tau(x) = 1 - \frac{u_l}{u_c x} = -\omega_Y(U, y, x)/f(x) \cdot \xi(x)$. Since $\omega_Y > 0$ and $f > 0$, $\tau$ and $\xi$ have opposite signs. Part 2 therefore follows from $\xi(x) < 0$ on $x \in [x_2, x_+)$ (with shapes of $T$ and $T^*$ following trivially from the sign of $\tau$), and Part 3a follows from $\xi(x) > 0$ on $x \in (x_-, x_1)$ with (1) non-binding. If (1) binds and $\xi(x_-) < 0$, (13) implies that $\xi(x)$ is continuous and strictly increasing. Hence, one of two cases must apply: either $\xi(x_1) \leq 0$, which implies $\xi(x) < 0$ and $\tau(x) > 0$ for all $x \in [x_-, x_1)$; or $\xi(x_1) > 0$, in which case $\xi(x_\tau) = 0$ at some $x_\tau \in (x_-, x_1)$ and $\xi(x) > 0$ and $\tau(x) < 0$ for all $x > x_\tau$. The former establishes Part 3b, and the latter establishes Part 3c. Parts 4 and 5 are obvious special cases, stated for completeness. ■

Proposition 2 shows that the tax functions always slope upward above $e$'s income. Below $e$'s income, the tax function has negative sloped unless the minimum-utility constraint (1) binds. This reflects the elected agent's desire to extract revenues from the poor. We interpret Part 3a as a counterfactual indication that if taxes are determined by self-interested politicians, the minimum utility constraint (1) will likely bind under plausible conditions. (That is, unless $x_-$ is so high that even the lowest productivity individual has the ability to pay substantial taxes.[17]) If (1) binds, proposition 2 is still consistent with negative marginal tax rates at incomes near $y_e$, but strictly positive marginal tax rates at incomes near $Y(x_-)$. This offers a rationalization for tax-transfer systems with a poverty trap. In Mirrlees-type welfare analyses, in contrast, poverty traps are difficult to rationalize because the well-being

[17] A solution with non-binding (1) is obtained, for example, if one assumes $[x_-, x_+] = [0.7, 1.3]$ instead of $[0.2, 1.8]$ in example 1 (case of $G = 0$, $x_e = 1$).

of the poor tends to receive substantial weight. The intuition here is that burdensome taxes on the poor reduce tax-avoidance opportunities at higher incomes. A selfish elected agent $e$ will assign very low incomes and utilities to the poor, perforce without regard for their well-being, if that helps to extract more taxes from individuals with productivity above $x_-$ (but below $x_e$).

The example in Section II suggests that with a binding minimum-utility constraint (1), the question of a monotonely increasing versus inverted-U shaped taxes depends on the level of required revenue $G$, as shown in figure 5. Generally, one might conjecture that $T_e$ is negative if $G$ is near zero (as taxes on others finance transfers to $e$), while $T(x_-)$ is near zero (near $T_0$). Then the tax function must include a negatively sloped segment, that is, it must have an inverted-U shape (as illustrated in Figure 5 for $G = 0$ and for $G = 0.03$). As $G$ approaches $\hat{G}$, however, $T_e$ rises while $T(x_-)$ is essentially fixed, suggesting a uniformly positive slope for high $G$-value (as illustrated in Figure 5 for $G = 0.06$ and above). General results about the role of $G$ and $u_{\min}$ are derived in Section V.

One robust, universal property of marginal tax rates is $\tau(x_1) < \tau(x_2)$, because $\tau(x)$ increases on $[x_1, x_2]$. Hence, all tax functions $T^*$ display a downward kink at the $e$'s income level, that is, a distinct non-linearity benefitting the "middle classes." (again see figure 5 for illustration). We interpret this as theoretical support for Directors law, the notion that government favors the middle class.

## IV. THE VOTING PROBLEM

The previous section derived the utility each voter would receive from an arbitrary elected agent. In this section, we characterize the voting outcomes in elections between a pair of candidates $(e_L, e_H)$ with $x_{e_L} < x_{e_H}$. The main steps are to establish that candidates' most–favored-income choices $(y_e)$ are monotone in productivity and to show that as $y_e$ varies, resulting shifts in utility profiles satisfy a "single-crossing" property. Specifically, we show that as $x_e$ rises, agents vary $y_e$ such that utilities of individuals with $x > x_e$ rise and utilities of individuals with $x < x_e$ fall.

Our first result is:

**Lemma 2** *Candidates $e_L$ and $e_H$ with $x_{e_L} < x_{e_H}$ choose income $y_{e_L} < y_{e_H}$, except that $y_{e_L} = y_{e_H} = 0$ is possible.*

Proof: The lemma follows from Agent Monotonicity and the common budget set $C_e = \{(c_e, y_e)|c_e = y_e - T_e^*(y_e, G)\}$. For any incomes $y_1 < y_2$ with associated feasible consumption levels $c_e(y_1)$ and $c_e(y_2)$, if agent $e_L$ prefers $(y_2, c_e(y_2))$ over $(y_1, c_e(y1))$, then by Agent Monotonicity any agent $e_H$ with $x_{e_H} > x_{e_L}$ must prefer $(y_2, c_e(y_2))$, too, which establishes weak monotonicity in the sense that $y_{e_L} \leq y_{e_H}$. By Agent Monotonicity, the first-order condition in proposition 2 (Part 3) cannot be satisfied at the same $y_e > 0$ for different $x_e$. Hence $y_{e_L} = y_{e_H}$cannot hold except for the corner case $y_{e_L} = y_{e_H} = 0$, where the first-order condition does not apply.■

The intuition in terms of income-consumption choices is illustrated in figure 7, again using values from example 1 for concreteness. Indifference curves for candidates with different productivities "rotate clockwise" as productivity increases. Since all elected agents face the same set of feasible income-consumption choices $C_e$,[18] candidates' income choices are monotone in productivity. The exception $y_{e_L} = y_{e_H} = 0$ occurs if both $x_e$-values are so low that the indifference curves at $y_e = 0$ are steeper than the boundary of $C_e$. Note that lemma 3 applies for all $(y_{e_L}, y_{e_H})$ pairs, that is, it applies even if there are multiple optimal $y_e$-values for given $x_e$.

Voters are assumed to understand the mapping from the candidates' productivity $x_e$ to their income choices $y_e$ (lemma 2), and to the resulting utility profiles (proposition 1 and lemma 1). Individual $x$ therefore prefers candidate $e_L$, or $e_H$, or is

[18]To be exact, consumption choices are also subject to the additional constraint $y \leq x_e$ that varies with $x_e$, but it is never binding and therefore irrelevant.

indifferent if and only if the utility profile at point $x$ implemented by $e_L$, $U(x; y_{e_L})$, is strictly above, strictly below, or equal to the utility value $U(x; y_{e_H})$ implemented by $e_H$. Because utility profiles are solutions to problems CRM1 and CRM2, the comparisons can be interpreted as parametric variations of CRM1 and CRM2 with respect to $y_e$. The challenge in proving median voter results is to show that as $y_e$ rises from $y_{e_L}$ to $y_{e_H}$, the utility profile on $[x_-, x_{e_L}]$ shifts down while utility profile on $[x_{e_H}, x_+]$ shifts up, with a single crossing between $U(x; y_{e_L})$ and $U(x; y_{e_H})$ on $(x_{e_L}, x_{e_H})$. We call this the **Single Crossing Property** of utility profiles.

The intuition for the Single Crossing Property is simple and illustrated in figure 8. As proved in proposition 2, the tax function implemented by a given elected agent has a downward "dip" or "corner" at $(y_e, T_e)$, and as proved in lemma 2, the "dip" shifts to the right as $x_e$ increases. Recall figure1 for illustration. A dip in the tax function implies that the set $C(y_e) = \{(y, c) \geq 0 | c = y - T^*(y)\}$ of income and consumption pairs available to individuals for a given tax function shows an upward-pointing corner at $(y_e, y_e - T_e)$. Figure 8 shows $C(y_e)$-sets for the $y_e$-values chosen by the median productivity agent ($x_{e_H} = 1$) the 40th-percentile agent ($x_{e_H} = 0.84$) in example 1. As $y_e$ increases from $y_{e_L}$to $y_{e_H}$, the relevant "corner" points $(y_e, y_e - T_e)$ in figure 8 move to the right along the boundary of the agents' feasible set $C_e$. The boundary of $C(y_e)$ shifts "outward" for high-income individuals (those with $y > y_{e_H}$), "inward" for low-income individuals (those with $y < y_{e_L}$), and has a single crossing in between. By Agent Monotonicity, voters with productivity above/below $x_{e_L}$ and/or $x_{e_H}$ have flatter/steeper indifference curves than $e_L$ and/or $e_H$ and are therefore better/worse off.[19]

Our next proposition shows that the intuition illustrated in figure 8 is general, except that we need one additional regularity condition and we must address some special cases where some voters are indifferent. As regularity condition, to do para-

[19]Note that $C(y_{e_L})$ and $C(y_{e_H})$ are close together at low $y$-values but don't touch.

metric variations on problems CRM1 and CRM2, we must assume that incomes are not "bunched" at values other than $y = y_e$ and $y = 0$. Otherwise, the potential for discontinuities in the control $\psi$ (in graphical terms: arbitrary kinks in the $C(y_e)$ sets) would make the analysis intractable. Formally, we define:

**ASSUMPTION (NB)**: For all $x_e \in [x_-, x_+]$, the solutions to CRM1 and CRM2 conditional on $(x_e, y_e(x_e), T_e^*(y_e(x_e), G))$ satisfy $\mu(x) = 0$ for all $x \in (x_0, x_1)$ and $x \in (x_2, x_+)$, that is, they do not display bunching at income levels other than $y = y_e$ or $y = 0$.

We allow bunching at zero income even though it causes complications. Namely, if a positive measure of voters has zero income *and* is assigned utility $U(x; y_e) = u_{\min}$ by both candidates, these individuals are indifferent in their voting decisions. Since utility profiles are strictly increasing for $Y > 0$, an indifference issue arises only at the bottom end of the productivity distribution, and it arises if and only if the solutions to CRM1 for $x_{e_L}$ and for $x_{e_H}$ display both bunching at zero income and a binding minimum utility constraint (as we will show). There are two natural assumptions that might be made about how voters with $U(x; y_{e_H}) = U(x; y_{e_L}) = u_{\min}$ vote. One is that such individuals vote lexicographically for $e_L$, who is closer than $e_H$ to the indifferent individuals' own productivity and income. This assumption may be taken as representing the idea that voter indifference is broken by "ideological" closeness. A motivation is that, in an extended model with randomness in future productivity, a voter who is indifferent based on today's productivity may be more likely prefer $e_L$ than $e_H$ in the future and hence may vote for $e_L$ today. The second assumption is that indifferent individuals abstain.[20] Formally, we let $X_{\min}(y_e) = \{x : U(x; y_e) = u_{\min}\}$ denote the set of voters obtaining $u_{\min}$ under the utility profile associated with $e$ choosing $y_e$, and we examine both assumptions about their behavior.

[20]Indifferent voters might also be assumed to randomize. This is formally similar to abstention under simple assumptions about how randomization occurs.

Our main result is then:

**Proposition 3 (Single-Crossing Property )** *Consider an election between candidates* $(e_L, e_H)$ *with* $x_{e_L} < x_{e_H}$ *and assume CON, NB, and* $y_{e_H} \neq 0$*. Then the induced utility profiles* $\{U(x; y_{e_H})\}_x$ *and* $\{U(x; y_{e_L})\}_x$ *cross at a unique point* $x \in (x_{e_L}, x_{e_H})$*, and:*

1. *all agents with productivity* $x \in [x_{e_H}, x_+]$ *strictly prefer* $e_H$ *over* $e_L$*:* $U(x; y_{e_H}) > U(x; y_{e_L})$*;*

2. *all agents with productivity* $x \in (x_-, x_{e_L}] \backslash X_{\min}(y_{e_L})$ *strictly prefer* $e_L$ *over* $e_H$*:* $U(x; y_{e_H}) < U(x; y_{e_L})$*;*

3. *all agents with productivity* $x \in X_{\min}(y_{e_L})$ *are indifferent:* $U(x; y_{e_H}) = U(x; y_{e_L}) = u_{\min}$.

Overview of proof: The proof is lengthy, straightforward, and in the appendix. The main steps are: (a) We show that any marginal increase in $y_e$ induces a variation in $U_e(x) = u(y_e - T_e^*(y_e)), 1 - y_e/x)$ on $[x_1, x_2]$ that raises $U(x; y_e)$ on $(x_e, x_2]$ and lowers $U(x; y_e)$ on $[x_1, x_e)$, using proposition 1, Part 3 to locate $x_e$ on $[x_1, x_2]$. (b) We show that the differential equations characterizing $\{U(x)\}_x$ on $[x_-, x_1]$ and on $[x_2, x_+]$ have the property that a shift of the boundary points $U(x_1; y_e)$ and $U(x_2; y_e)$ up or down triggers a "shift" of the entire profile $\{U(x)\}_x$ in the same direction. On $[x_2, x_+]$, this exploits the transversality condition $\xi(x_+) = 0$ of (CRM2). On $[x_-, x_1]$, this requires multiple case distinctions to cover all possibilities of $U(x) \geq u_{\min}$ and $Y(x) \geq 0$ being binding or non-binding (four major cases plus several non-generic ones). (c) On $[x_2, x_+]$ unconditionally, and on $[x_-, x_1]$ for all cases that satisfy either $Y(x) > 0$ for $x > x_-$ (allowing $Y(x_-) = 0$) or $U(x_-) > u_{\min}$, we show that the variations in $\{U(x)\}_x$ are non-zero. (d) On $[x_-, x_1]$, in the case that $Y(x) = 0$ and $U(x) = u_{\min}$ on an interval $[x_-, x_0]$, $x_- < x_0 < x_1$, we show that the variations in $\{U(x)\}_x$ are non-zero on $(x_0, x_1]$, identically zero on $X_{\min}(y_{e_L}) = [x_-, x_0]$, and that the cutoff point $x_0$ is non-decreasing in $y_e$. (e) For any pair $(e_L, e_H)$ with $x_{e_L} < x_{e_H}$ and any intermediate point $x_m \in (x_{e_L}, x_{e_H})$, lemma 2 implies $y_{e_L} < y_e(x_m) < y_{e_H}$, and steps (a)-(d) prove the strict inequalities in Parts 1-2, and the uniqueness of a crossing point on $(x_{e_L}, x_{e_H})$. (f) Indifference applies on $X_{\min}(y_{e_L})$, because $x_0$ is non-decreasing in $y_e$, hence $X_{\min}(y_{e_L}) \subset X_{\min}(y_{e_H})$, and individuals in $X_{\min}(y_{e_H}) \backslash X_{\min}(y_{e_L})$ strictly prefer $e_L$. ■

The assumption $y_{e_H} \neq 0$ rules out the trivial case of $y_{e_L} = y_{e_H} = 0$ when all voters are indifferent (recall lemma 2). The issues of bunching and concavity (assumptions like NB and CON) have been discussed extensively in the welfare literature following Mirrlees (1971). Here, concavity is needed because, as usual in comparative statics, one must assume that the relevant first-order conditions uniquely characterize the problem and that the second-order conditions hold strictly.[21] Bunching at $y = y_e$ is characteristic of the model and it is technically tractable–namely by spliting CRM into CRM1 and CRM2. Bunching at $y = 0$, although it makes the analysis more cumbersome, raises a substantively interesting voting issue (Part 3, indifference) that should not be assumed away. By assuming CON and NB, we implicitly restrict our analysis to "well behaved" problems. For less well behaved problems, the results may hold–the assumptions are sufficient and not necessary–but there are no general results.

To verify concavity in examples, it is useful to note that if NB holds, CON is equivalent to the following condition: Let $y(U, \omega, x)$ be defined by the identity $\omega = \omega(U, y(U, \omega, x), x)$ (which is well defined because $\omega_Y < 0$) and consider

$$T^{**}(U, \omega, x) = y(U, \omega, x) - c^*(U, y(U, \omega, x)/x)$$

[21] Mirrlees (1986, p.1235) discusses a concavity condition similar to (CON) arising in his welfare problem and notes that conditions like CON is somewhat "obscure" (the second derivatives of $\omega$ depend on various mixed third partial derivatives of $u$) but unavoidable in variational problems of this type.

as function of $(U, \omega)$. Then $T^{**}$ being strictly concave in $(U, \omega)$ is equivalent to CON, and it easier to verify because it involves neither $\xi$ nor $f$. Separately, if $u$ is separable in consumption and leisure, $H$ being strictly concave in $Y$ and weakly concave in $(U, Y)$ is sufficient for propositions1-3; and if NB holds in addition, $T^{**}$ being strictly concave in $\omega$ and weakly concave in $(U, \omega)$ is sufficient. It is straightforward to verify that our examples in Section II satisfy the latter condition..

Our voting results follow directly from proposition 3. Since individuals' preferences over candidates are single-peaked, median voter arguments become applicable despite the infinite-dimensional policy space. Proposition 3 shows that the one-dimensional nature of candidates' heterogeneity over productivities suffices to eliminate "cycling" over policy choices. We provide several statements to allow for different voting mechanisms and to be careful about technical complications.

**Corollary 1 (Election Outcomes)** *In any election between arbitrary candidates* $(e_L, e_H)$ *with* $x_{e_L} < x_{e_H}$ *that satisfies the Single-Crossing Property:*

1. *If* $X_{\min}(y_{e_L})$ *has measure zero, the candidate who's policy is preferred by median-productivity individuals wins.*

2. *If* $X_{\min}(y_{e_L})$ *has positive measure and indifferent individuals vote based on closeness in productivity, the candidate who's policy is preferred by median-productivity individuals wins.*

3. *If* $X_{\min}(y_{e_L})$ *has positive measure and indifferent agents abstain, the candidate who's policy is preferred by the median of* $(x_0(y_{e_L}), x_+]$ *wins.*

Proof: Follows directly from Proposition 3. ■

If there are more than two potential candidates, candidate selection becomes an issue. It is beyond the scope of this paper to examine explicit voting mechanisms. Hence we limit our discussion to the question under what conditions there is a Condorcet winner, that is, a candidate who would win against any opponent. Winning means obtaining strictly more votes in favor than against not counting abstentions. Let $e_{med}$ be a candidate with median productivity $x_{med}$, where $F(x_{med}) = 0.5$. Also, let $e^*$ be an agent with productivity $x_{e^*} = \inf\{x_e \geq x_{med} \,|F(x_e) = (F(x_0(y_e(x_e))) + 1)/2\}$, that is, the lowest $x_e$-value that is the median productivity on the set $[x_0(y_e(x_e)), x_+]$, and let $e_0$ be the highest productivity candidate with $X_{\min}(y_{e_0})$ having measure zero. If $X_{\min}(y_{e_{med}})$ has measure zero, $x_{e^*} = x_{med}$; if $X_{\min}(y_{e_{med}})$ has positive measure, $x_{e^*}$ exists and lies above $x_{med}$ but below the median of $[x_0(y_e(x_+), x_+]$.

Then we have:

**Corollary 2 (Median Voter Theorem)** *If the Single-Crossing Property is satisfied:*

1. *If $X_{\min}(y_{e_{med}})$ has measure zero, $e_{med}$ is the Condorcet winner.*

2. *If $X_{\min}(y_{e_{med}})$ has positive measure and indifferent individuals vote based on closeness in productivity, $e_{med}$ is the Condorcet winner.*

3. *If $X_{\min}(y_{e_{med}})$ has positive measure and indifferent agents abstain, a Condorcet winner may not exist. However:*

   (a) *Agents with productivity $x < x_{med}$ lose against $e_{med}$ and agents with productivity $x > x_{e^*}$ lose against $e^*$.*

   (b) *Agent $e^*$ is the Condorcet winner, if and only if $e^*$ wins against all candidates $e_L$ with $x_{e_L} \in [x_{e_0}, x_{e^*})$. Otherwise there is no Condorcet winner.*

Proof: Parts 1 and 2 follow directly from the previous corollary. In Part 3, voting cycles cannot be ruled out because the set of voters depends on the candidate pair. Part 3a follows from the definition of $e_{med}$ and $e^*$. In Part 3b, a loss would mean that $e^*$

is not a Condorcet winner. Winning against $e_0$ implies that $e^*$ wins against all lower productivity candidates, too. Given Part 3a, winning against candidates in $[x_{e_0}, x_{e^*})$ is therefore sufficient for $e^*$ to be the Condorcet winner. Candidates other than $e^*$ cannot be Condorcet winners, because higher productivity candidates lose against $e^*$ and lower productivity candidates are below-median on $[x_0(y_e(x_e)), x_+]$ and lose against the median candidate on this set. ■

Under the conditions of Parts 1 and 2 of the corollary, the median-productivity agent is elected in any voting mechanism that selects a Condorcet winner. Under the conditions of Part 3, the elected agent $e^*$ may be close to the median in many empirically plausible cases. Because $X_{\min}(e^*)$ is the set of non-workers under $e^*$'s tax policy, a measure of this set might be roughly the unemployment rate. Given an observed unemployment rate of, say, six percent, the only potential Condorcet winner is the agent $e^*$ with productivity equal to the 53rd percentile of the productivity distribution.

Note that $y_{e_{med}} > 0$ is necessary to apply the Single Crossing Property in corollary 2. Otherwise, if $y_{e_{med}} = 0$, then $y_e = y_{e_{med}} = 0$ would hold for all votes on $e_{med}$ against agents $e$ with below-median productivity by lemma 3, violating the assumption $y_{e_H} \neq 0$. The voting outcomes for $y_{e_{med}} = 0$ are obvious, however: Voters in $[x_-, x_{med}]$ prefer the utility profile associated with $y_e = 0$ over any other. Any agent $e_H$ setting $y_{e_H} > 0$ would therefore lose against any agent–it does not matter which one–who sets $y_e = 0$.

More broadly, one may ask how and why our model avoids voting cycles despite the multi-dimensional policy setting. Two features of our model are crucial for obtaining median voter results. First, we need one-dimensional heterogeneity in the set of candidates. Otherwise, voting over candidates would tend to produce voting cycles, as usual in multi-dimensional voting situations. Second, we must allow the elected agent's ability to impose a fairly general tax function. Additional "ad hoc" constraints that may appear to simplify the problem can easily destroy the median-voter result. For example, if the elected agent were required to impose a quadratic or a piecewise linear function (distinct tax brackets), one would generally lose the single-crossing property and obtain voting cycles–unless of course the problem is drastically simplified to become one-dimensional, as in Meltzer-Richard.

The intuition is that the single-crossing property relies on incentive constraints that tie together the utility levels of "neighboring" (on the $x$ axis) individuals. These constraints ensure that candidates treat individuals with productivity close to their own better than individuals further away on the productivity distribution. Additional constraints that might break the monotone link between utility and distance are likely to destroy the median voter results. (An example is discrete tax brackets that would force $e$ to give entire sets of agents the same marginal rate of substitution.) We conjecture, however, that median voter theorems similar to ours could be obtained in other multi-dimensional voting situations where suitable "local" incentive constraints apply (e.g., in voting over public services). Our analysis suggests that one should not necessarily interpret voting cycles as a indication of excessive complexity of a model and that reducing the voting choice to a single issue is not the only remedy. Instead, voting cycles may indicate that a model imposes inappropriate constraints, a problem that might be avoided by allowing more general solutions, such as general non-linear taxes rather than a finite number of tax brackets. Though the general setup is technically more demanding, our analysis of labor income taxes suggests that the general approach can yield economically insightful results.

The notion that median-voter results rely on the elected agent's ability to exploit individual-specific incentive constraints also provides an intuitive explanation of why the no-bunching assumption (NB) is required in proposition 3. Bunching forces elected agents to offer the same "package" of income& taxes to a set of heterogenous individuals. Though we have not been able to generate examples of voting cycles, we conjecture that if different elected agents are placing entire sets of voters into different intervals with common, bunched income, voting cycles are possible. Individuals would have an incentive to vote for candidates that place them into a "favorable" interval (as in the case of tax brackets), and the choice of intervals may not be monotone in the productivity of the elected agent. Thus, the median voter result is not a universal property of the non-linear tax problem, but one that holds under reasonable regularity conditions.

## V. TAX FUNCTIONS, REVENUE NEEDS, AND A FORM OF ALTRUISM

We now reinterpret minimum utility values $u_{\min} > u(0,1)$ as reflecting a form of altruism, and we examine how altruism alters the tax function of the elected agent. For this reinterpretation, we treat $u_{\min}$ as a parameter that may differ from $u(0,1)$ and use the relationship $u_{\min} = u(TR, 1)$ to define the minimum transfer $TR \geq 0$ due to a non-worker.[22]

One main result is that the differential equations characterizing solutions to CRM1 permit four different boundary configurations at $x_-$, and that three of the four imply a zero measure for the set $X_{\min}(y_e)$ of voters that are assigned utility $u_{\min}$. The other main result is that negative marginal rates necessarily occur with zero government spending and selfish agents, but the tax function is monotonely increasing if non-redistributional spending $G$ is sufficiently high or if agents are sufficiently altruistic.

The utility profile and the shape of the tax function below the elected agent's income $y_e$ are determined by the Euler equations characterizing problem CRM1 and

[22] In principle, we could consider cases with taxes $T_0 = -TR > 0$, e.g., with the interpretation the government can seize some of the non-market income $\epsilon$ from non-workers. We ignore this possibility and assume $TR \geq 0$ to avoid more case distinctions in the analysis below.

by the boundary conditions $U(x_-) \geq u_{\min}$ and $Y(x_-) \geq 0$. The boundary conditions may each bind or not bind, resulting in four basic boundary configurations. Whenever $U(x_-) = u_{\min}$ and/or $Y(x_-) = 0$ hold as equality, the associated costate variables are generally strictly negative ($\xi(x_-) < 0$ and/or $\mu(x_-) < 0$), but the non-generic special cases of $\xi(x_-) = 0$ and/or $\mu(x_-) = 0$ must also be considered. (The latter will be included in the appropriate generic cases.) Our next proposition shows that the boundary configurations depend importantly on the relationship between the lowest productivity ($x_-$) and the marginal rate of substitution of non-workers, $u_l(-T_0, 1)/u_c(-T_0, 1)$, and on the government's revenue needs. The intuition is that if $x_- > u_l(-T_0, 1)/u_c(-T_0, 1)$, all individuals are willing to work (meaning, at a wage equal to their marginal product). Then cases with $Y(x_-) = 0$ will occur only if work incentives are severely distorted, which could occur because of high revenue needs; and $Y(x_-) > 0$ is sufficient for $X_{\min}(y_e)$ to have measure zero. If $x_- \leq u_l(-T_0, 1)/u_c(-T_0, 1)$, however, low-productivity individuals would have to be subsidized to work, and bunching at $Y = 0$ is generic. Specifically, we have:

**Proposition 4 (Boundary Conditions )** *In problem CRM1:*

1. *For given $(x_e, y_e, T_e)$, the following four boundary configurations for $(U, \xi, Y, \mu)$ at $x_-$ and at $x_0 = \inf\{x : Y(x) > 0\}$ are possible:*

   *(a) $x_0 = x_-$ and $\xi(x_-) = 0$. Then $U(x_-) \geq u_{\min}$, $U(x) > u_{\min}$ for all $x > x_-$, $Y(x_-) \geq 0$, $Y(x) > 0$ for all $x > x_-$, $\mu(x_-) = 0$, and $X_{\min}(y_e)$ has measure zero.*

   *(b) $x_0 = x_-$ and $\xi(x_-) < 0$. Then $U(x_-) = u_{\min}$, $U(x) > u_{\min}$ for all $x > x_-$, $Y(x_-) \geq 0$, $Y(x) > 0$ for all $x > x_-$, $\mu(x_-) = 0$, and $X_{\min}(y_e) = \{x_-\}$ has measure zero.*

   *(c) $x_0 > x_-$ and $U(x_-) > u_{\min}$. Then $U(x_-) = U(x) > u_{\min}$ and $Y(x) = 0$ for all $x \leq x_0$, $\xi(x_-) = 0$, $\mu(x_-) < 0$, and $X_{\min}(y_e) = \emptyset$ has measure zero.*

(d) $x_0 > x_-$ *and* $U(x_-) = u_{\min}$. *Then* $U(x_-) = U(x) = u_{\min}$ *and* $Y(x) = 0$ *for all* $x \leq x_0$, $\xi(x_-) \leq 0$, $\mu(x_-) \leq 0$, *and* $X_{\min}(y_e) = \{x \in [x_-, x_0]\}$ *is a set with positive measure* $F(x_0) > 0$

2. (a) *f* $x_- > u_l(-T_0, 1)/u_c(-T_0, 1)$, *then there is an interval of* $T_e$*-values where boundary configuration (a) applies and an interval of higher* $T_e$*-values where boundary configuration (b) applies. Configurations (c) and (d) may or may not occur. With increasing* $T_e$, *the boundary configurations apply (if they occur) in the sequence (c), then (a), then (b), then (d).*

3. (a) *If* $x_- < u_l(-T_0, 1)/u_c(-T_0, 1)$, *then there is an interval of* $T_e$*-values where boundary configuration (c) applies and an interval of higher* $T_e$*-values where boundary configuration (d) applies. Configuration (b) cannot arise, and (a) can only occur if consumption is an inferior good. With increasing* $T_e$, *the boundary configurations apply (if they occur) in the sequence (a), then (c), then (d).*

Proof: The list of possibilities in Part 1 follows from the transversality conditions $(U(x_-) - u_{\min}) \cdot \xi(x_-) = 0$ with $\xi(x_-) \leq 0$ and $U(x_-) - u_{\min} \geq 0$ and $Y(x_-) \cdot \mu(x_-) = 0$ with $\mu(x_-) \leq 0$ and $Y(x_-) \geq 0$. The other claims follow from parametric variations on CRM1 and from a detailed analysis of all possible boundary configurations, which is lengthy and therefore deferred to the appendix. ■

Though the proof is lengthy due to the numerous case distinctions, proposition 4 merely combines and verifies a few simple arguments. Part 1 shows that the case of $X_{\min}(y_e)$ having positive measure–the one creating indifference issue in proposition 3 and its corollaries–is one of four possible cases. Since $U$ is strictly monotone whenever $Y > 0$ (see lemma 1), it requires bunching at $Y = 0$. Parts 2-3 combine four observations: First, as $T_e$ varies, the entire tax profile shifts in the same general direction, and the utility profile shifts in the opposite direction. Hence, the configurations (a) and (c) with non-binding minimum-utility constraint tends to apply for low (or negative) $T_e$-values whereas configurations (b) and (d) apply for higher $T_e$-values. Second, if everyone is willing to work (Part 2), configurations without bunching at $Y = 0$–(a) and (b)–must occur for some range of $T_e$-values, whereas (c) and (d) at apply only at extreme values, if at all. Conversely, if low-productivity workers would have to "bribed" to work (Part 3), configurations with bunching at $Y = 0$–(c) and (d)–must occur for some range of $T_e$-values, and the non-bunching configurations are special cases. Third, the intuition about (a) versus (c) involves the income effect of transfers: Under the conditions of Part 2, configuration (a) applies rather than (c), unless a positive income effect from high transfers $-T(x_-) >> -T_0 \geq 0$ induces low-income individuals to stop working; and under the conditions of Part 3, configuration (c) applies rather than (a), unless consumption is inferior and high transfers $-T(x_-) >> -T_0 \geq 0$ reduce $u_l(-T(x_-), 1)/u_c(-T(x_-), 1)$ to a value at or below $x_-$. Fourth, the intuition for configuration (d) in Part 2 is that pushing low-productivity individuals out of the workforce may enable the elected agent to extract more revenues from higher productivity agents. Algebraically, this is optimal whenever $|\xi(x_-)|$, the shadow value of revenue from reducing $x_-$'s utility, exceeds $\xi_0 \equiv [x_- - u_l(-T_0, 1)/u_c(-T_0, 1)] \cdot f(x_-)/x_-/\omega_Y(u_{\min}, 0, x_-)$, a critical value that depends on $x_-$'s motivation to work (the gap of $x_-$ and $u_l/u_c$) and on the curvature of the utility function (influencing $u_l/u_c$ and $\omega_Y$).

In example 1, only configurations (a) and (b) occur. This is because $u_l(-T_0, 1) = 0$ rules out Part 3 and because $\xi_0$ is high enough to preclude $|\xi(x_-)| \geq \xi_0$, i.e., there is never a problem with indifferent voters. To exhibit cases (c) and (d), one could consider slightly modified preferences of the form $u(c, l) = c + v_0\{1-(1+v_2-l)^{1+v_1}\}/(1+v_1)$ with additional parameter $v_2 \geq 0$. Then $u_l(-T_0, 1) = v_0 \cdot v_2$ exceeds $x_-$ whenever $v_2 > v_0/x_-$, making Part 3 of proposition 4 applicable.

Next, we consider conditions under which the tax function is monotonely increasing or has a negatively sloped segment. For this analysis, we also consider a simple form of altruism in which $u_{\min}$ is interpreted as a preference parameter of the elected agent such that the elected agent wants to give each voter utility no lower than $u_{\min}$. This minimum utility may be sufficiently great that the elected agent must give positive transfers to non-working agents to bring them to $u_{\min}$.[23] That is, we allow $u_{\min} = u(TR, 1)$ to apply for some $TR > 0$, reflecting altruism, or for $TR = 0$, which captures the selfish case. If $TR > 0$ and therefore $u_{\min} > u(0, 1)$, $u_{\min}$ is finite even if $u(0, 1) = -\infty$, so that restrictions on $u(0, 1)$ (or auxiliary assumptions about home production, etc.) are unnecessary. Formally, the impact of variations in $TR$ and in $G$ on the shape of the tax function can be examined like variations in $y_e$ in the voting analysis. One complication is that since $y_e(x_e)$ may have discontinuities, the continuity of $y_e(x_e; G, TR)$ as function of $(G, TR)$ cannot be taken for granted. Hence, we simply provide some limit results that hold if the relevant limits exist; they should be sufficiently intuitive that we present them as remarks without proof.

**Remark 1**: For any $TR \geq 0$, if $G$ is sufficiently high, $T^*$ tends to monotonely increasing, i.e. has no negatively sloped segment. The argument is that as $G$ converges to $\hat{G}$ from below, $e$'s problem becomes dominated by the need to collect revenues and approaches problem URM, which features a monotonely increasing tax function. Formally, if $y_e(x_e; G) \to \hat{Y}(x_e) > \hat{Y}(x_-)$ as $G \to \hat{G}$ (where the existence of the limit must be assumed), then $T_e(y_e(x_e; G), G) \to \hat{T}(x_e) > \hat{T}(x_-) = T_0$ is necessarily increasing one average. Moreover, one can show that $\hat{\xi}(x) < 0$ hold for all $x < x_+$ in problem URM. Thus, if all limits exist, $\xi(x) < 0$ applies for all $x < x_+$ in problems CRM1 and CRM2 conditional on $(y_e, T_e)$ in a neighborhood of $(\hat{Y}(x_e), \hat{T}(x_e))$. This implies monotone tax function because $\xi(x) < 0$ implies $\tau(x) > 0$ whenever $Y$ is

[23] There are other ways one might model altruism, but this formulation fits easily into the model and suffices to show that altruism can make marginal rates everywhere positive.

strictly increasing.

**Remark 2**: For any $G \geq 0$, if $TR$ is sufficiently high, $T^*$ tends to monotonely increasing, too. To make this argument rigorous, one would have make $u_{\min} = u(TR, 1)$ a function of $TR$ and condition on $TR$ in all propositions above. Intuitively, the argument is analogous to the argument about high $G$: As transfers increase, $e$'s problem becomes dominated by the need to finance transfers and, in the limit, approaches problem URM conditional on $u(TR, 1)$. Then monotonicity of $T^*$ follows from $\hat{\xi}(x) < 0$ for $x < x_+$ as above.

**Remark 3**: If $G = TR = 0$, the tax function $T^*$ necessarily has a negatively sloped segment below $y_e$. This is because the elected agent can generate strictly positive revenues from others and afford to pay himself a transfers, setting $T_e < 0 = T_0$. If $U(x_-) = u_{\min}$, $T(x_-) \geq T_0 > T_e = T(x_1)$ implies a negatively sloped segment on $[x_-, x_1]$; and if $U(x_-) > u_{\min}$, Prop.2 implies that $T$ is negatively sloped on all of $[x_-, x_1]$.

Remarks 1-2 explain how our model can be consistent with the monotone or largely monotone tax functions that we observe empirically. This deserves emphasis because the most simple special case, $G = TR = 0$, which has been the focus of the income tax literature, necessarily has a negatively sloped segment in the tax function.

Our example in section II follows this tradition and assumed $G = TR = 0$. Figure 9 shows two simple modifications of the example that yield strictly monotone tax functions. Line (1) shows the tax function implemented by the median voter when $G = 0.06$ instead of $G = 0$ (maintaining $u_{\min} = u(0, 1)$). Line (2) shows the tax function when $TR = 0.06$, $u_{\min} = u(0, 1) + 0.06$, which implies $T_0 = -0.06$ (maintaining $G = 0$). These $G$ and $T_0$ are high enough in magnitude that both tax functions are upward sloping throughout. With affine utility (no income effects), the tax function with $T_0 = -0.06$ has the same shape as the tax function with $G = 0.06$ but is shifted down by $G = -T_0$.

More broadly, proposition 5 and the examples suggest that the slope of the tax function in our voting model depends significantly on government outlays for non-redistributional spending $G$ and for altruistic transfers $TR$. While the model necessarily implies a negatively sloped segment for $G = TR = 0$, the natural base case of pure and selfish redistribution, the model is consistent with consistently positively-sloped tax functions for positive values of $G$ and/or $TR$.

Careful readers may note linkages between proposition 2, proposition 4, and the cases in Remarks 1-3. If $G$ and/or $TR$ are high, the minimum utility constraint (1) tends to bind, i.e., yield configurations (b) or (d) of proposition 4. proposition 2 shows that binding (1) is necessary for a monotonely increasing tax function, but not sufficient. If $G$ and $TR$ are varied for given $y_e$, one can indeed show that for the lowest $G$ and $TR$ values, the tax function has negative slope on $[x_-, x_e]$ and configurations (a) or (c) apply, that for higher $G$ and $TR$ values, the tax function has an inverted-U slope on $[x_-, x_e]$ and configurations (b) or (d) apply, and for even higher $G$ and $TR$ values, the tax function has a positive slope on $[x_-, x_e]$ and still configurations (b) or (d) apply. There is no tight link between monotonicity versus inverted-U shape and configuration (b) versus (d), however; and $y_e$ will vary as $G$ and $TR$ change, and this may break the tendency of the various cases in propositions 2 and 4 to be associated with higher/lower values of $G$ and $TR$.

## VI. CONCLUSIONS

Meltzer and Richard (1981) model voting on one-dimensional tax policy under direct democracy. Allowing multi-dimensional policy in their model results in cycling. We model voting on multidimensional policy by assuming that voters are heterogenous along a single dimension, and by replacing the assumption of direct democracy with an assumption of representative democracy in which the candidate who is elected implements his/her most preferred policy. This characterization of representative democracy eliminates the cycling that would occur if multi-dimensional tax policy were chosen in direct democracy. The solution to the model is consistent with substantial tax breaks for the middle class (Director's law).

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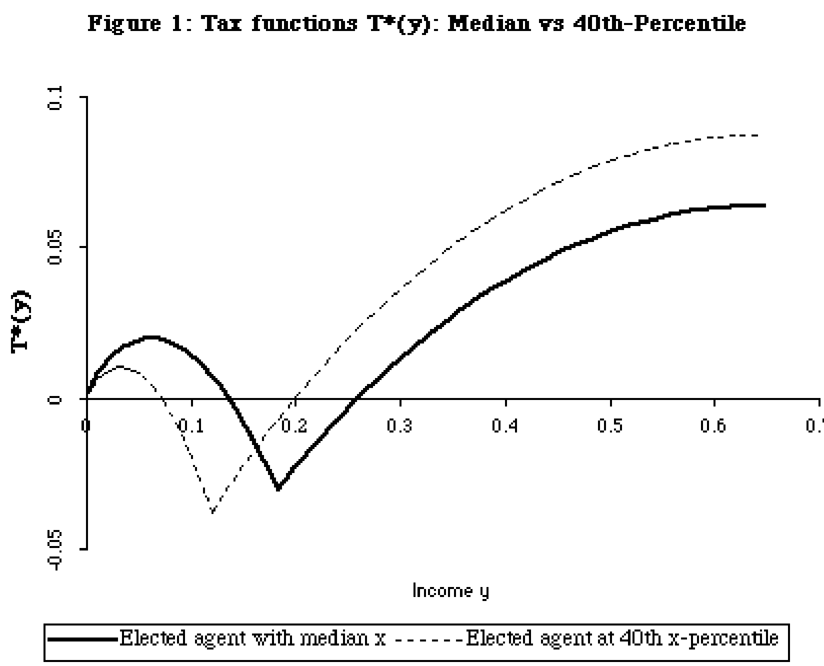
Figure 1: Tax functions T*(y): Median vs 40th-Percentile
T*(y)
0.1
0.05
0
-0.05
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
Income y
Elected agent with median x
Elected agent at 40th x-percentile

Fig. 1.

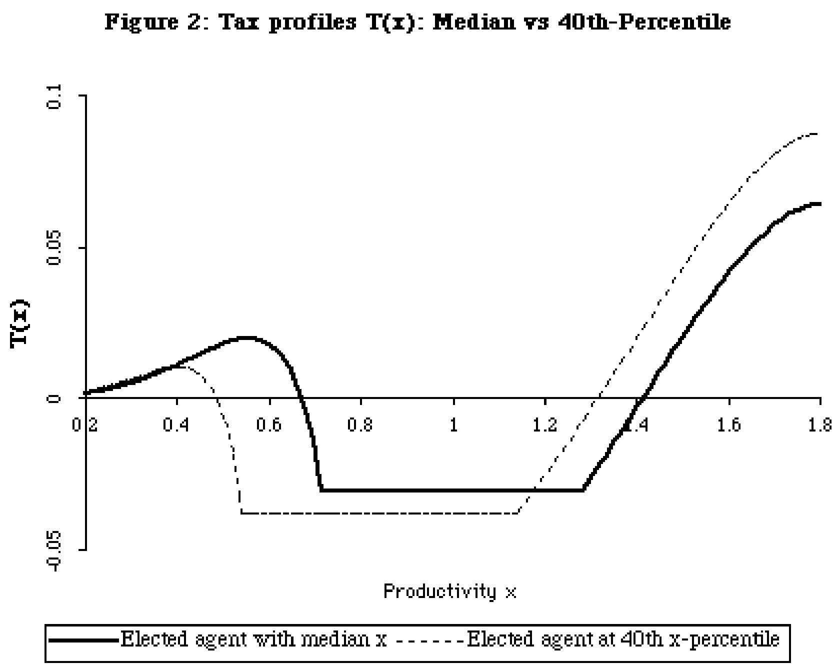

Figure 2: Tax profiles T(x): Median vs 40th-Percentile
T(x)
0.1
0.05
0
-0.05
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
Productivity x
Elected agent with median x
Elected agent at 40th x-percentile


Fig. 2.

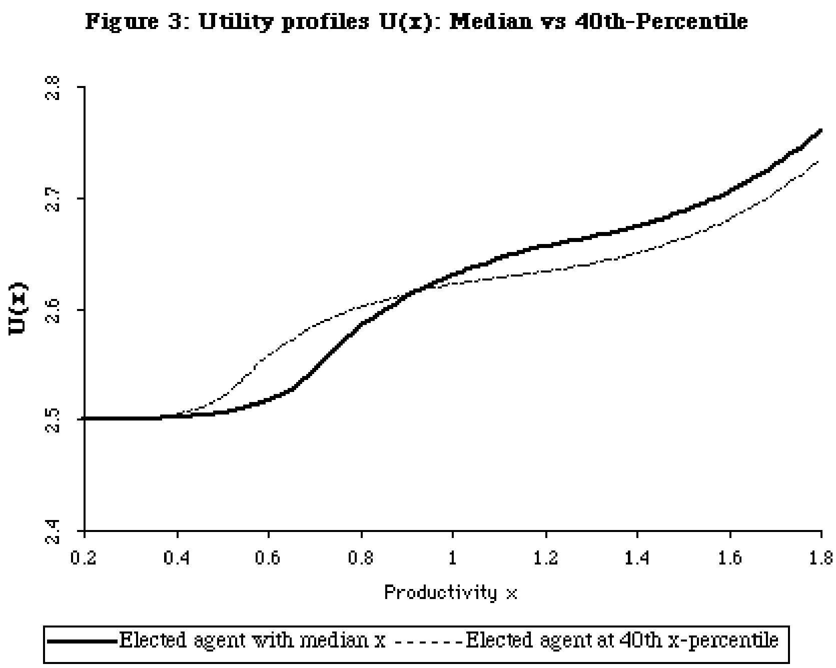
Figure 3: Utility profiles U(x): Median vs 40th-Percentile
U(x)
2.8
2.7
2.6
2.5
2.4
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
Productivity x
Elected agent with median x
Elected agent at 40th x-percentile

Fig. 3.

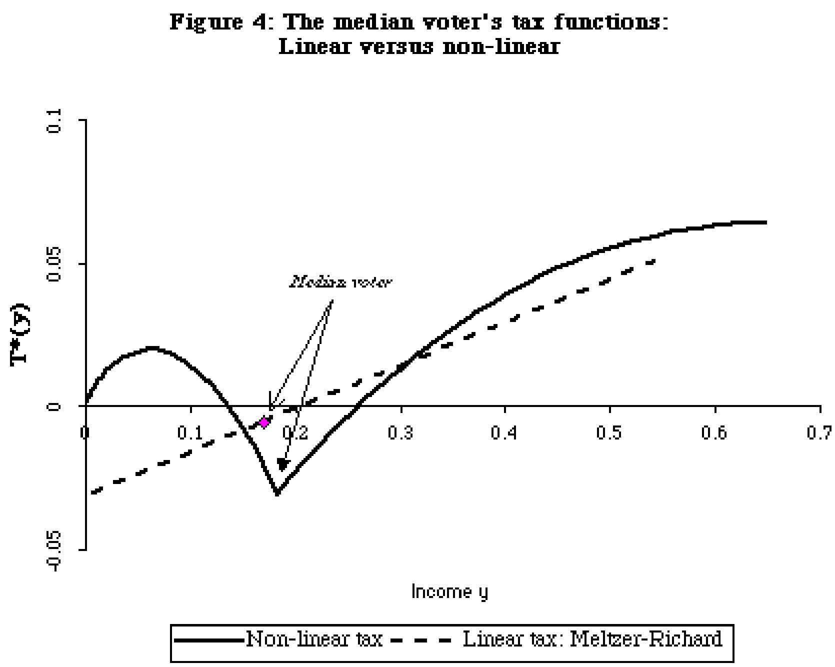
Figure 4: The median voter's tax functions:
Linear versus non-linear
Median voter
T*(y)
0.1
0.05
0
-0.05
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
Income y
Non-linear tax
Linear tax: Meltzer-Richard

FIG. 4.

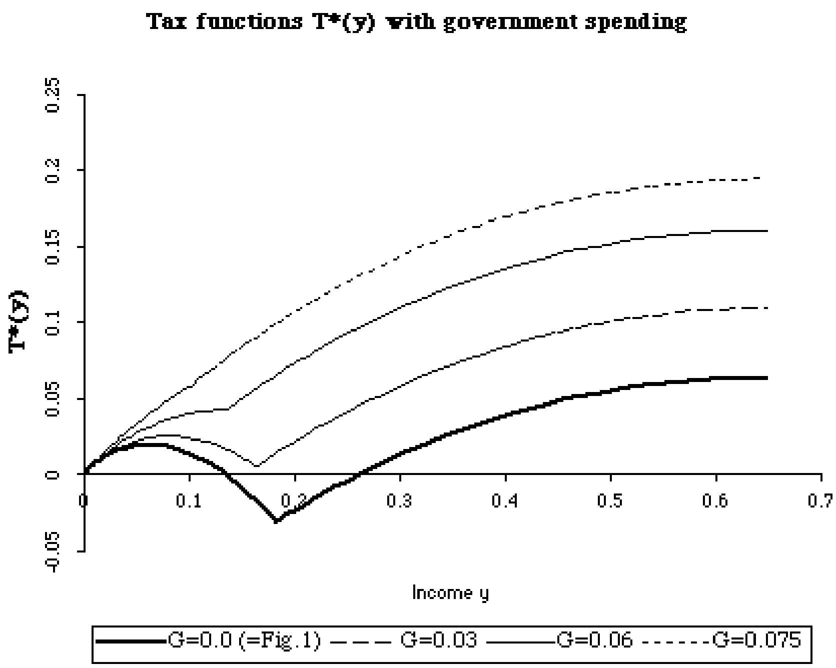
Tax functions T*(y) with government spending
T*(y)
0.25
0.2
0.15
0.1
0.05
0
-0.05
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
Income y
G=0.0 (=Fig.1)
G=0.03
G=0.06
G=0.075

Fig. 5.

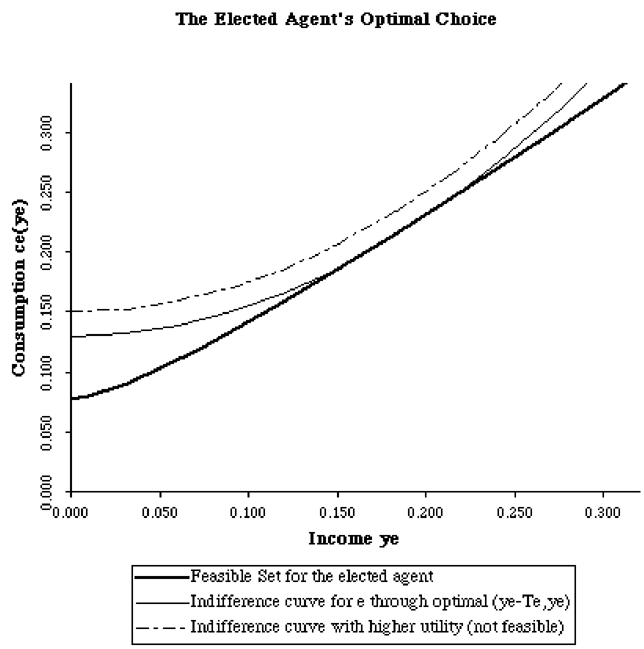
The Elected Agent's Optimal Choice
Consumption ce(ye)
0.000
0.050
0.100
0.150
0.200
0.250
0.300
Income ye
0.000
0.050
0.100
0.150
0.200
0.250
0.300
Feasible Set for the elected agent
Indifference curve for e through optimal (ye-Te,ye)
Indifference curve with higher utility (not feasible)

Fig. 6.

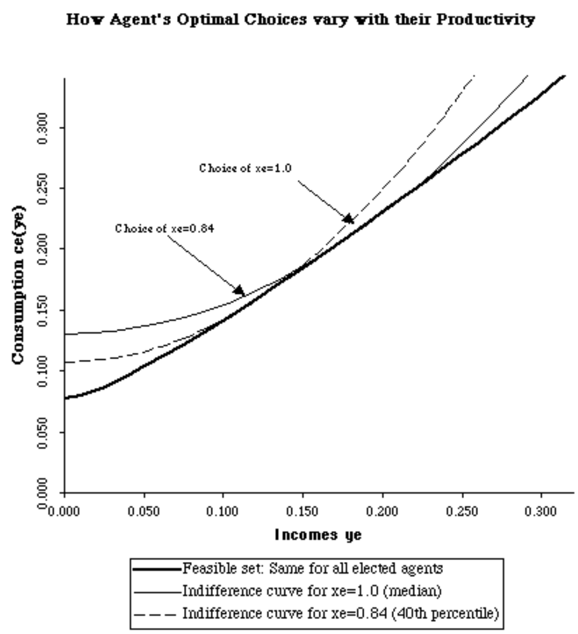
How Agent's Optimal Choices vary with their Productivity
Choice of xe=1.0
Choice of xe=0.84
Consumption ce(ye)
Incomes ye
0.000
0.050
0.100
0.150
0.200
0.250
0.300
Feasible set: Same for all elected agents
Indifference curve for xe=1.0 (median)
Indifference curve for xe=0.84 (40th percentile)

FIG. 7.

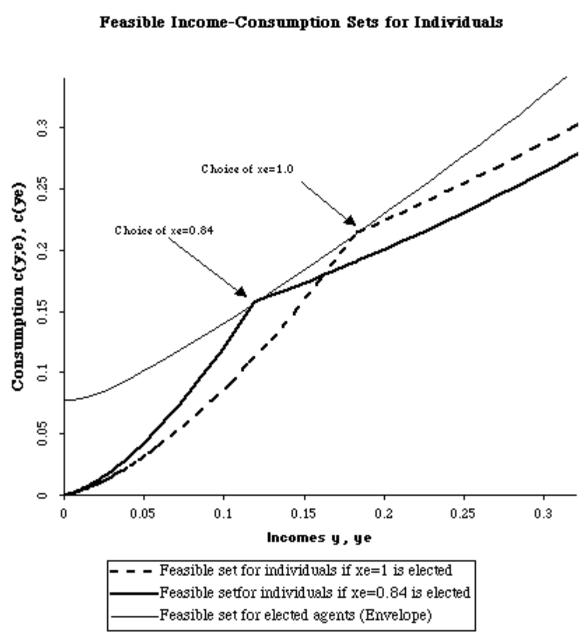
Feasible Income-Consumption Sets for Individuals
Consumption c(y;e), c(ye)
0
0.05
0.1
0.15
0.2
0.25
0.3
Choice of xe=1.0
Choice of xe=0.84
0
0.05
0.1
0.15
0.2
0.25
0.3
Incomes y, ye
Feasible set for individuals if xe=1 is elected
Feasible setfor individuals if xe=0.84 is elected
Feasible set for elected agents (Envelope)

Fig. 8.

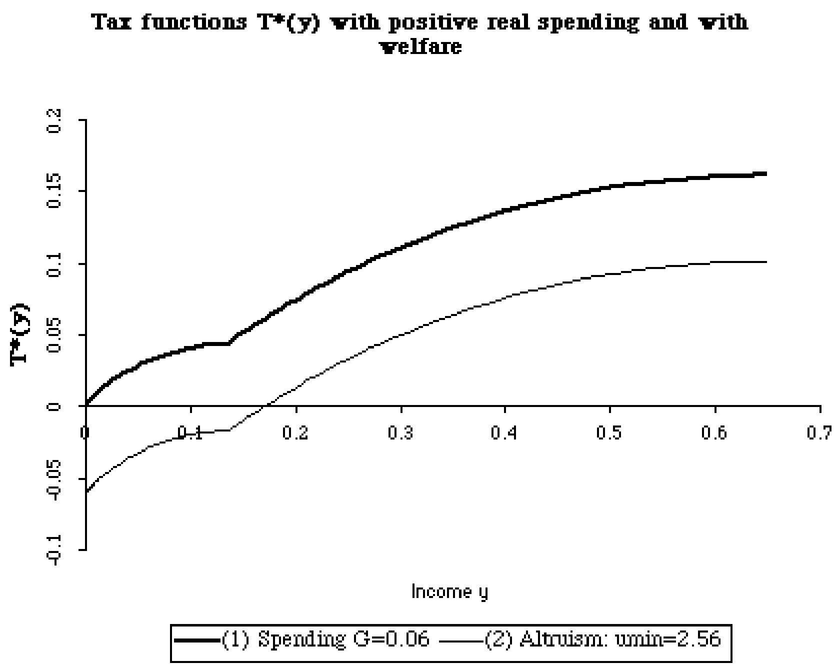
Tax functions T*(y) with positive real spending and with welfare
T*(y)
0.2
0.15
0.1
0.05
0
-0.05
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
Income y
(1) Spending G=0.06
(2) Altruism: umin=2.56

FIG. 9.