EXERCISES

20.16 Using the arguments in Example 20.13 and in Exercise 20.13, show that we can replace "homogeneous" by "homogeneous of degree one" in the definition of homothetic.

20.17 Which of the following functions are homothetic? Give a reason for each answer.

a) $e^{x^2y}e^{xy^2}$, *b)* $2\log x + 3\log y$, *c)* $x^3y^6 + 3x^2y^4 + 6xy^2 + 9$,
d) $x^2y + xy$, *e)* $x^2y^2/(xy+1)$.

20.18 Use Theorems 20.9 and 20.10 to check the homotheticity of the functions in Exercise 20.17 and to determine whether or not $f(x, y) = x^4 + x^2y^2 + y^4 - 3x - 8y$ is homothetic.

20.19 Write out a complete, careful proof of Theorem 20.9.

20.20 Show that for a strictly monotone function u, the two inequalities in condition (12) can be replaced without loss of generality by equalities.

CHAPTER 21

Concave and Quasiconcave Functions

Concave functions play a role in economic theory similar to the role that homogeneous functions play. Both classes arise naturally in economic models — homogeneous functions as demand functions, concave functions as expenditure functions. Profit functions and cost functions are naturally both homogeneous and concave. Both classes have desirable properties for utility and production functions. Both classes have straightforward calculus-based characterizations — homogeneous functions via Euler's theorem, concave functions via a second derivative test. Finally, both classes are cardinal and need to be modified for full use in utility theory.

On the other hand, concavity is a concept that is very different from homogeneity. As we will see, there are functions which are homogeneous but not concave or convex, and there are functions which are concave or convex but not homogeneous. In a sense, these two properties are complementary; economists often prefer to work with production functions that have both properties.

21.1 CONCAVE AND CONVEX FUNCTIONS

Students first meet concave and convex functions in their study of functions of one variable in Calculus I, as we did in Section 3.2. The definitions of concavity and convexity are the same for functions of n variables as they are for functions of one variable.

Definition A real-valued function f defined on a convex subset U of $\mathbf{R}^n$ is **concave** if for all $\mathbf{x}$, $\mathbf{y}$ in U and for all t between 0 and 1,

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \geq tf(\mathbf{x}) + (1-t)f(\mathbf{y}). \tag{1}$$

A real-valued function g defined on a convex subset U of $\mathbf{R}^n$ is **convex** if for all $\mathbf{x}$, $\mathbf{y}$ in U and for all t between 0 and 1,

$$g(t\mathbf{x} + (1-t)\mathbf{y}) \leq tg(\mathbf{x}) + (1-t)g(\mathbf{y}). \tag{2}$$

Remark Notice that f is concave if and only if $-f$ is convex. To every property of concave functions, there is a naturally corresponding property of convex functions.

Remark Many introductory calculus texts call convex functions "concave up" and concave functions "concave down," as we did in Section 3.2. From now on, we will stick with the more classical terms: "convex" and "concave."

Remark Do not confuse the notion of a convex *function* with that of a convex *set*. A set U is a **convex set** if whenever **x** and **y** are points in U, the line segment joining **x** to **y**,

$$\ell(\mathbf{x}, \mathbf{y}) \equiv \{(t\mathbf{x} + (1-t)\mathbf{y} : 0 \leq t \leq 1\},$$

is also in U. In Figure 21.1, the ball in (a) and the interior of the triangle in (b) are convex sets, while the *annulus* (region between two concentric circles) in (c) and the star in (d) are not convex sets, as the line segments in these last two shapes indicate. The definition of a concave or convex function f requires that whenever f is defined at **x** and at **y**, it is defined on the segment $\ell(\mathbf{x}, \mathbf{y})$. Thus convex and concave functions are required to have convex domains. In this section, all functions will be defined on convex sets, whether the function is concave, convex, or neither. This is not the only connections between convex sets and convex and concave functions. Check that f is concave if and only if $\{(x, y) : y \leq f(x)\}$ is a convex set, and work out the corresponding statement for convex functions. Almost all functions in economics, especially utility and production functions, have convex sets as their natural domains

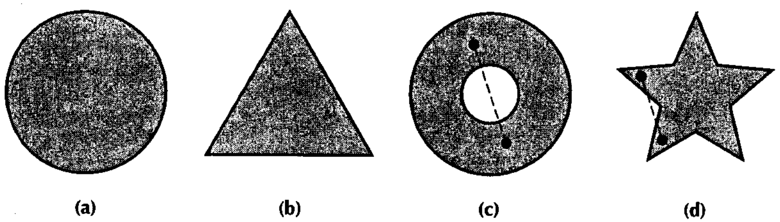
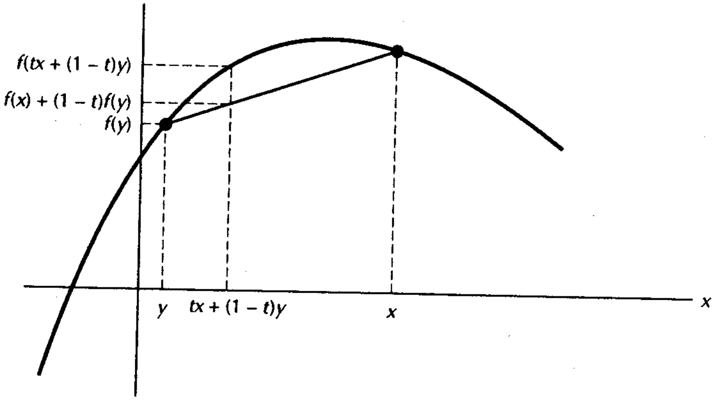


Figure 21.1 *Parts (a) and (b) represent convex sets; (c) and (d) illustrate nonconvex sets.*

Students usually develop a geometric intuition for concave and convex functions of one variable in Calculus I. They can recognize a concave function by its graph because, as Figure 21.2 illustrates, the inequality (1) in the definition of a concave function has the following geometric interpretation:

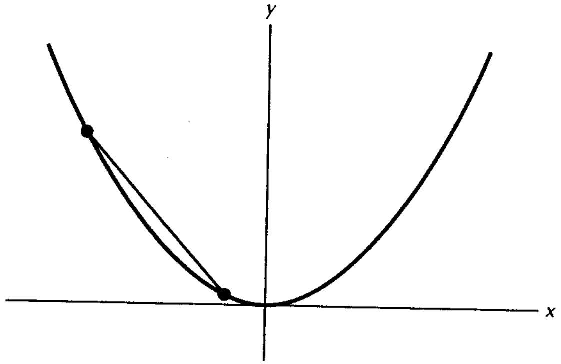
> A function f of n variables is concave if and only if any secant line connecting two points on the graph of f lies *below* the graph. A function is convex if and only if any secant line connecting two points on its graph lies *above* its graph.



Figu 21.2 *The geometric interpretation of the definition of a concave function.*

This property is illustrated in Figures 21.3 and 21.4, which present the graphs of two prototypical convex functions: $y = x^2$ and $z = x_1^2 + x_2^2$.

In developing an intuition for concave functions of several variables and in proving theorems about their properties, it is useful to notice that a function of n variables defined on a convex set U is concave if and only if its restriction to any line segment in U is a concave function of *one* variable. This should be intuitively clear since the definition (1) of a concave function is a statement about its behavior on line segments. Because it is such a useful fact, we provide a careful analytical proof. In the remainder of this section, we will use this result to reduce the proofs of theorems about concave functions on $\mathbf{R}^n$ to statements about concave functions of a single variable.



Figur 21.3 *The graph of the convex function $y = x^2$.*

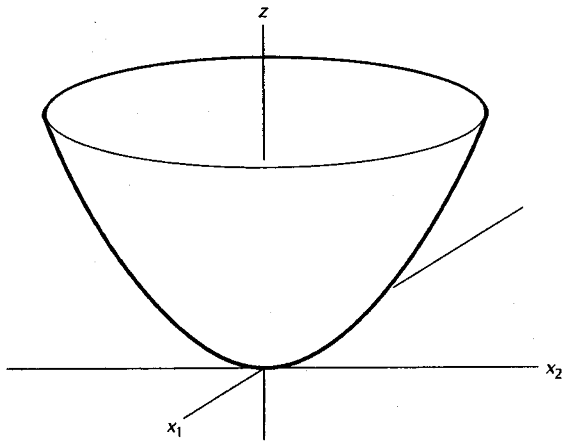


Figure 21.4 *The graph of the convex function $z = x_1^2 + x_2^2$.*

> **Theorem 21.1** Let f be a function defined on a convex subset U of $\mathbf{R^n}$. Then, f is concave (convex) if and only if its restriction to every line segment in U is a concave (convex) function of one variable.

Proof Suppose that the restriction of f to every line segment in U is a concave function. To prove that f is a concave function on U, let $\mathbf{x}$ and $\mathbf{y}$ be arbitrary points in U. Let $g(t) \equiv f(t\mathbf{x} + (1-t)\mathbf{y})$. By hypothesis, g is concave. So, for t between 0 and 1,

$$\begin{aligned} f(t\mathbf{x} + (1-t)\mathbf{y}) &= g(t) && \text{(definition of } g) \\ &= g(t \cdot 1 + (1-t)\cdot 0) \\ &\geq tg(1) + (1-t)g(0) && \text{(since } g \text{ is concave)} \\ &= tf(\mathbf{x}) + (1-t)f(\mathbf{y}). && \text{(definition of } g) \end{aligned}$$

Consequently, f is concave.

Conversely, suppose that f is concave. We want to show that $g(t) \equiv f(t\mathbf{x} + (1-t)\mathbf{y})$, the restriction of f to the line containing $\mathbf{x}$ and $\mathbf{y}$ is concave. To do this, fix s_1 and s_2 and let t be between 0 and 1. Then,

$$\begin{aligned} g(ts_1 + (1-t)s_2) &= f((ts_1 + (1-t)s_2)\mathbf{x} + (1 - (ts_1 + (1-t)s_2))\mathbf{y}) \\ &\qquad \text{(definition of } g) \\ &= f(t(s_1\mathbf{x} + (1-s_1)\mathbf{y}) + (1-t)(s_2\mathbf{x} + (1-s_2)\mathbf{y})) \\ &\qquad \text{(rearranging)} \end{aligned}$$

$$\begin{aligned} &\geq tf(s_1\mathbf{x} + (1-s_1)\mathbf{y}) + (1-t)f(s_2\mathbf{x} + (1-s_2)\mathbf{y}) \\ &\qquad \text{(concavity of } f) \\ &= tg(s_1) + (1-t)g(s_2) \quad \text{(definition of } g). \end{aligned}$$

Therefore, g is concave. The proof for convex functions is nearly identical. ∎

The goal of the remainder of this section is to understand concave and convex functions more deeply by working toward three concrete goals:

(1) to develop simple calculus-based tests for concavity or convexity,
(2) to discover the desirable properties that concave and convex functions have; and
(3) to see how concave and convex functions arise in economic models.

In our discussions, we will usually work with concave functions rather than convex functions, since any statement about one type can easily be translated into a statement about the other. When we summarize our results in the statements of theorems, we will state the results for both types of functions.

Calculus Criteria for Concavity

As the discussion thus far illustrates, one can tell whether or not a function on $\mathbf{R^n}$ is concave by looking at its graph in $\mathbf{R^{n+1}}$. In fact, a more geometric way of phrasing the definition of concavity is to say: a function of n variables is concave if and only if the set **below** its graph in $\mathbf{R^{n+1}}$ is a convex set, as in Figure 21.2; a function is convex if and only if the set **above** its graph in $\mathbf{R^{n+1}}$ is a convex set, as in Figures 21.3 and 21.4. (See Exercise 21.5.)

Of course, it is usually not practical or even possible to draw the graph of a function to test for concavity. We need a more analytic criterion. Students studying functions of one variable in Calculus I learn two simple analytic tests for concavity:

(1) A C^1 function on an interval I is concave if and only if its first derivative $f'(x)$ is a decreasing function of x for x on I.
(2) A C^2 function f is concave on an interval I if and only if its second derivative $f''(x)$ is ≤ 0 for all x in I.

(See Section 3.2.) As one might guess from Theorem 21.1, the generalizations of these criteria work in all dimensions. We must first figure out what these generalizations are.

The natural generalization of the first derivative $f'(x)$ to functions of several variables is the (Jacobian) matrix of the first order partial derivatives of f:

$$Df(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}) \quad \frac{\partial f}{\partial x_2}(\mathbf{x}) \quad \cdots \quad \frac{\partial f}{\partial x_n}(\mathbf{x}) \right).$$

Since this first derivative $Df(\mathbf{x})$ can be thought of as n functions of n variables, that is, as a function from $\mathbf{R^n}$ to $\mathbf{R^n}$, we need to work a little to interpret the statement "$Df(\mathbf{x})$ is a decreasing function." The following theorem provides a closely related first order condition for concavity on $\mathbf{R}^1$ that has an obvious generalization to functions of several variables.

Theorem 21.2 Let f be a C^1 function on an interval I in $\mathbf{R}$. Then, f is concave on I if and only if

$$f(y) - f(x) \leq f'(x)(y - x) \quad \text{for all } x, y \in I. \tag{3}$$

The function f is convex on I if and only if

$$f(y) - f(x) \geq f'(x)(y - x) \quad \text{for all } x, y \in I.$$

Remark First we show that condition (3) means that f' is a decreasing function. Divide both sides of (3) by $(y - x)$; remember to reverse the inequality when $y - x < 0$. The results are

$$\frac{f(y) - f(x)}{y - x} \leq f'(x) \quad \text{for all } y > x \in I \tag{4}$$

and

$$\frac{f(y) - f(x)}{y - x} \geq f'(x) \quad \text{for all } y < x \in I. \tag{5}$$

To see that (4) and (5) imply that f' is decreasing, suppose $z_1 < z_2$ in I. Then,

$$\begin{aligned} f'(z_1) &\geq \frac{f(z_2) - f(z_1)}{z_2 - z_1} && \text{(by (4) with } x = z_1 \text{ and } y = z_2\text{)} \\ &= \frac{f(z_1) - f(z_2)}{z_1 - z_2} && \text{(multiplying top and bottom by } -1\text{)} \\ &\geq f'(z_2) && \text{(by (5) with } x = z_2 \text{ and } y = z_1\text{)} \end{aligned}$$

Proof of Theorem 21.2. Suppose that f is a concave function on I. Let $x, y \in I$ and let $t \in (0, 1]$. Then,

$$tf(y) + (1 - t)f(x) \leq f(ty + (1 - t)x)$$

or

$$\begin{aligned} f(y) - f(x) &\leq \frac{f(x + t(y - x)) - f(x)}{t} \\ &= \frac{f(x + t(y - x)) - f(x)}{t(y - x)}(y - x). \end{aligned}$$

Condition (3) follows by letting $t \to 0$ in the last expression.

On the other hand, suppose (3) holds for all x, y in I. Then,

$$\begin{aligned} f(x) - f((1 - t)x + ty) &\leq f'((1 - t)x + ty)(x - ((1 - t)x + ty)) \\ &= -tf'((1 - t)x + ty)(y - x). \end{aligned}$$

Similarly,

$$f(y) - f((1 - t)x + ty) \leq (1 - t)f'((1 - t)x + ty)(y - x).$$

Multiply the first inequality through by $(1 - t)$ and the second by t; then add the two to obtain

$$(1 - t)f(x) + tf(y) \leq f((1 - t)x + ty). \quad \blacksquare$$

The natural generalization of condition (3) to functions of several variables is now straightforward.

Theorem 21.3 Let f be a C^1 function on a convex subset U of $\mathbf{R^n}$. Then, f is concave on U if and only if for all $\mathbf{x}$, $\mathbf{y}$ in U:

$$f(\mathbf{y}) - f(\mathbf{x}) \leq Df(\mathbf{x})(\mathbf{y} - \mathbf{x});$$

that is,

$$f(\mathbf{y}) - f(\mathbf{x}) \leq \frac{\partial f}{\partial x_1}(\mathbf{x})(y_1 - x_1) + \cdots + \frac{\partial f}{\partial x_n}(\mathbf{x})(y_n - x_n). \tag{6}$$

Similarly, f is convex on U if and only if $f(\mathbf{y}) - f(\mathbf{x}) \geq Df(\mathbf{x})(\mathbf{y} - \mathbf{x})$ for all $\mathbf{x}$, $\mathbf{y}$ in U.

Proof Let $\mathbf{x}$ and $\mathbf{y}$ be arbitrary points in U. Let

$$\begin{aligned} g_{\mathbf{x},\mathbf{y}}(t) &\equiv f(t\mathbf{y} + (1 - t)\mathbf{x}) \\ &= f(x_1 + t(y_1 - x_1), \ldots, x_n + t(y_n - x_n)). \end{aligned}$$

Then, by the Chain Rule,

$$g'_{\mathbf{x},\mathbf{y}}(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))(y_i - x_i) \tag{7}$$

and

$$g'_{\mathbf{x},\mathbf{y}}(0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x})(y_i - x_i) = Df(\mathbf{x})(\mathbf{y} - \mathbf{x}).$$

By Theorems 21.1 and 21.2, f is concave if and only if every such $g_{\mathbf{x},\mathbf{y}}$ is concave if and only if for every $\mathbf{x}, \mathbf{y} \in U$,

$$g_{\mathbf{x},\mathbf{y}}(1) - g_{\mathbf{x},\mathbf{y}}(0) \leq g'_{\mathbf{x},\mathbf{y}}(0)(1-0) = g'_{\mathbf{x},\mathbf{y}}(0)$$

if and only if for every $\mathbf{x}, \mathbf{y} \in U$,

$$f(\mathbf{y}) - f(\mathbf{x}) \leq Df(\mathbf{x})(\mathbf{y} - \mathbf{x}). \quad \blacksquare$$

Corollary 21.4 If f is a C^1 concave function on a convex set U and if $\mathbf{x}_0 \in U$, then

$$Df(\mathbf{x}_0)(\mathbf{y} - \mathbf{x}_0) \leq 0 \quad \text{implies} \quad f(\mathbf{y}) \leq f(\mathbf{x}_0). \tag{8}$$

In particular, if $Df(\mathbf{x}_0)(\mathbf{y} - \mathbf{x}_0) \leq 0$ for all $\mathbf{y} \in U$, then $\mathbf{x}_0$ is a global max of f.

Let's stop to consider the geometry of this situation. To this end, we'll use the more geometric concept of the gradient vector $\nabla f(\mathbf{x}_0)$ instead of the derivative matrix $Df(\mathbf{x}_0)$. Recall from Section 15.2 that $\nabla f(\mathbf{x}_0)$ is a vector perpendicular to the level set of f through $\mathbf{x}_0$. Inequality (8) says that if the vector from $\mathbf{x}_0$ to $\mathbf{y}$ makes an obtuse angle with $\nabla f(\mathbf{x}_0)$ at $\mathbf{x}_0$, that is, if $\nabla f(\mathbf{x}_0) \cdot (\mathbf{y} - \mathbf{x}_0) \leq 0$, then $f(\mathbf{y}) \leq f(\mathbf{x}_0)$. Alternatively, since $\nabla f(\mathbf{x}_0)$ is perpendicular to the tangent hyperplane of the level set of f at $\mathbf{x}_0$, condition (8) says that for a concave function the set $\{\mathbf{z} : f(\mathbf{z}) \geq f(\mathbf{x}_0)\}$, including the level set $\{\mathbf{z} : f(\mathbf{z}) = f(\mathbf{x}_0)\}$, lies above the hyperplane tangent to the level set of f at $\mathbf{x}_0$. In short, if f is concave, then every level set of f lies above any of its tangent planes, where "above" means in the direction of increasing values of f. See Figure 21.5.

Example 21.1 Let us apply the test of Theorem 21.3 to show that $f(x_1, x_2) = x_1^2 + x_2^2$ is convex on $\mathbf{R}^n$. The function f is convex if and only if

$$(y_1^2 + y_2^2) - (x_1^2 + x_2^2) \geq (2x_1 \quad 2x_2)\begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \end{pmatrix}$$
$$= 2x_1y_1 - 2x_1^2 + 2x_2y_2 - 2x_2^2$$

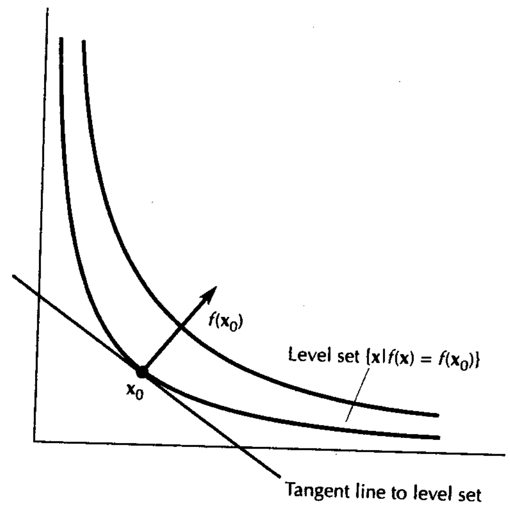
if and only if

$$y_1^2 + y_2^2 + x_1^2 + x_2^2 - 2x_1y_1 - 2x_2y_2 \geq 0$$

if and only if

$$(y_1 - x_1)^2 + (y_2 - x_2)^2 \geq 0,$$

which is true for all (x_1, x_2) and (y_1, y_2) in $\mathbf{R}^2$.



Fi[g] 21.[5] *Relationship between grad $f(\mathbf{x}_0)$ and the level set through $\mathbf{x}_0$ for a concave f.*

Theorem 21.3 is a very useful technique for proving properties about concave and convex functions. However, since it involves checking an *inequality* for all $\mathbf{x}$ and $\mathbf{y}$ in the domain, it is usually not a practical test for checking whether any given function is concave or convex. For the latter purpose, we will find it more practical to use the generalization of the second derivative test: f is concave on an interval I if and only if $f''(x) \leq 0$ for all x in I. (See Section 3.2.) The natural generalization of the second derivative $f''(x)$ to functions of several variables is the Hessian matrix of all the second order partial derivatives of f on $\mathbf{R}^1$:

$$D^2f(\mathbf{x}) = \begin{pmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{pmatrix},$$

where we write $f_{x_ix_j}$ for $\partial^2 f/\partial x_i \partial x_j$ and each entry is evaluated at the point $\mathbf{x}$. The natural generalization of $f''(x) \leq 0$ is the statement that the Hessian matrix $D^2f(\mathbf{x})$ is negative semidefinite at every $\mathbf{x}$ in the domain of f. The following theorem summarizes the second order test for concave and convex functions on $\mathbf{R}^n$.

Theorem 21.5 Let f be a C^2 function on an open convex subset U of $\mathbf{R}^n$. Then, f is a concave function on U if and only if the Hessian $D^2f(\mathbf{x})$ is negative semidefinite for all $\mathbf{x}$ in U. The function f is a convex function on U if and only if $D^2f(\mathbf{x})$ is positive semidefinite for all $\mathbf{x}$ in U.

Remark Recall from Section 16.2 that a matrix H is positive definite if and only if $\mathbf{v}^T H \mathbf{v} > 0$ for all $\mathbf{v} \neq \mathbf{0}$ in $\mathbf{R}^n$; H is negative definite if and only if $\mathbf{v}^T H \mathbf{v} < 0$ for all $\mathbf{v} \neq \mathbf{0}$ in $\mathbf{R}^n$. Replacing the strict inequalities above by weak ones yields the definitions of positive semidefinite and negative semidefinite. Theorem 16.1 from Chapter 16 provides analytic necessary and sufficient conditions for a matrix to be definite or semidefinite:

(1) A matrix H is positive definite if and only if its n leading principal minors are all > 0.
(2) A matrix H is negative definite if and only if its n leading principal minors alternate in sign with the odd order ones being negative and the even order ones being positive.
(3) A matrix H is positive semidefinite if and only if its $2^n - 1$ principal minors are all ≥ 0.
(4) A matrix H is negative semidefinite if and only if its $2^n - 1$ principal minors are alternate in sign so that the odd order ones are ≤ 0 and the even order ones are ≥ 0.

Proof of Theorem 21.5. As in the previous proof, let $\mathbf{x}$ and $\mathbf{y}$ be arbitrary points in U and let $g_{\mathbf{x},\mathbf{y}}(t) \equiv f(t\mathbf{y} + (1-t)\mathbf{x})$. Then, f is concave on U if and only if each $g_{\mathbf{x},\mathbf{y}}(t)$ is concave, which is equivalent to each $g''_{\mathbf{x},\mathbf{y}}(t) \leq 0$. Now, by equation (7) and the Chain Rule,

$$\begin{aligned} g''_{\mathbf{x},\mathbf{y}}(t) &= \frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x} + t(\mathbf{y}-\mathbf{x}))(y_i - x_i)\right) \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x} + t(\mathbf{y}-\mathbf{x}))(y_j - x_j)(y_i - x_i) \\ &= \sum_{i,j=1}^{n} (y_j - x_j)\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x} + t(\mathbf{y}-\mathbf{x}))(y_i - x_i) \\ &= (\mathbf{y}-\mathbf{x})^T \cdot D^2 f(\mathbf{x} + t(\mathbf{y}-\mathbf{x})) \cdot (\mathbf{y}-\mathbf{x}). \end{aligned}$$

If every $D^2 f(\mathbf{z})$ is negative semidefinite, then it follows that:

(*a*) every $g''_{\mathbf{x},\mathbf{y}}(t) \leq 0$,
(*b*) every $g_{\mathbf{x},\mathbf{y}}$ is concave, and
(*c*) f itself is concave.

Conversely, suppose that f is concave on U. Let $\mathbf{z}$ be an arbitrary point in U and let $\mathbf{v}$ be an arbitrary displacement vector in $\mathbf{R}^n$. We want to show that $\mathbf{v}^T D^2 f(\mathbf{z})\mathbf{v} \leq 0$. Since U is open, there is a $t_0 > 0$ such that $\mathbf{y} = \mathbf{z} + t_0\mathbf{v}$ is in U. Since f is concave, $g_{\mathbf{z},\mathbf{y}}$ is concave and $g''_{\mathbf{z},\mathbf{y}}(0) \leq 0$. By the previous paragraph,

$$\begin{aligned} 0 \geq g''_{\mathbf{z},\mathbf{y}}(0) &= (\mathbf{y}-\mathbf{z})^T \cdot D^2 f(\mathbf{z}) \cdot (\mathbf{y}-\mathbf{z}) \\ &= (t_0\mathbf{v})^T \cdot D^2 f(\mathbf{z}) \cdot (t_0\mathbf{v}) \\ &= (t_0^2)[\mathbf{v}^T \cdot D^2 f(\mathbf{z}) \cdot \mathbf{v}]. \end{aligned}$$

Therefore, $\mathbf{v}^T \cdot D^2 f(\mathbf{z}) \cdot \mathbf{v} \leq 0$, and $D^2 f(\mathbf{z})$ is negative semidefinite for all $\mathbf{z}$ in U. ■

Example 21.2 The Hessian of the function $f(x, y) = x^4 + x^2y^2 + y^4 - 3x - 8y$ is

$$D^2 f(x,y) = \begin{pmatrix} 12x^2 + 2y^2 & 4xy \\ 4xy & 2x^2 + 12y^2 \end{pmatrix}.$$

For $(x, y) \neq (0, 0)$, the two leading principal minors, $12x^2 + 2y^2$ and $24x^4 + 132x^2y^2 + 24y^4$, are both positive, so f is a convex function on all $\mathbf{R}^n$.

Example 21.3 A commonly used simple utility or production function is $F(x, y) = xy$. Its Hessian is

$$D^2 F(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

whose second order principal minor is $\det D^2F(x, y) = -1$. Since this second order principal minor is negative, D^2F is indefinite and F is neither concave nor convex.

Example 21.4 Consider the monotonic transformation of the function F in the previous example by the function $g(z) = z^{1/4}$: $G(x, y) = x^{1/4}y^{1/4}$, defined only on the positive quadrant $\mathbf{R}^2_+$. The Hessian of G is

$$D^2 G(x,y) = \begin{pmatrix} -\frac{3}{16}x^{-7/4}y^{1/4} & \frac{1}{16}x^{-3/4}y^{-3/4} \\ \frac{1}{16}x^{-3/4}y^{-3/4} & -\frac{3}{16}x^{1/4}y^{-7/4} \end{pmatrix}.$$

For $x > 0, y > 0$, the first order leading principal minor is negative and the second order leading principal minor, $x^{-3/2}y^{-3/2}/128$, is positive. Therefore, $D^2G(x, y)$ is negative definite on $\mathbf{R}^2_+$ and G is a concave function on $\mathbf{R}^2_+$.

Example 21.5 Now, consider the general Cobb-Douglas function on $\mathbf{R}^2_+$: $U(x, y) = x^a y^b$. Its Hessian is

$$D^2 U(x,y) = \begin{pmatrix} a(a-1)x^{a-2}y^b & abx^{a-1}y^{b-1} \\ abx^{a-1}y^{b-1} & b(b-1)x^a y^{b-2} \end{pmatrix},$$

whose determinant is

$$\det D^2U(x,y) = ab(1-a-b)x^{2a-2}y^{2b-2}.$$

In order for U to be concave on $\mathbf{R}^2_+$, we need $a(a-1) < 0$ and $ab(1-a-b) > 0$; that is, we need $0 < a < 1$, $0 < b < 1$, and $a + b \leq 1$. In summary, a Cobb-Douglas production function on $\mathbf{R}^2_+$ is concave if and only if it exhibits constant or decreasing returns to scale.

Remark These four examples illustrate some relationships among the various classes of functions we have been studying. Examples 21.3 and 21.5 show that a function can be homogeneous or homothetic and not be concave or convex. Example 21.2, along with Exercise 20.13, shows that a function can be convex or concave and not be homogeneous or homothetic. Examples 21.4 and 21.5 show that a function can be both concave (or convex) *and* homogeneous (or homothetic). Finally, the last three examples clearly show that concavity is a *cardinal property*; a monotonic transformation of a concave function need not be concave.

EXERCISES

21.1 Prove that every linear function is homogeneous, concave, *and* convex.

21.2 Which of the following functions on $\mathbf{R}^n$ are concave or convex? At least *attempt* the first order test of Theorem 21.3, before settling down with the second order test of Theorem 21.5.

a) $f(x) = 3e^x + 5x^4 - \ln x$; *b)* $f(x,y) = -3x^2 + 2xy - y^2 + 3x - 4y + 1$;
c) $f(x,y,z) = 3e^x + 5y^4 - \ln z$; *d)* $f(x,y,z) = Ax^a y^b z^c$, $a, b, c > 0$.

21.3 Prove that a quadratic form on $\mathbf{R}^n$ is concave if and only if it is negative semidefinite. Prove that it is convex if and only if it is positive semidefinite. What can be said about the more general "quadratic function" $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{b}\cdot\mathbf{x} + c$?

21.4 Prove that every homogeneous function on $\mathbf{R}^1_+$ is either concave or convex.

21.5 Prove that a function of n variables is concave if and only if the set **below** its graph in $\mathbf{R}^{n+1}$ is a convex set. By Theorem 21.1, one need only prove this statement for functions of one variable.

21.6 Interpret inequalities (4) and (5) geometrically.

21.7 Suppose that f is concave and that g is the affine function $x \mapsto ax + b$, with $a \geq 0$. Prove that $g \circ f$ is concave.

21.8 Let f and g be functions on $\mathbf{R}^1$. What assumptions on f and g guarantee that the composite $f \circ g$ is a concave function?

21.9 For what concave functions f is $1/f$ a convex function?

21.10 The product of two homogeneous functions is always homogeneous. Give conditions under which the product of two concave functions is concave.

21.2 PROPERTIES OF CONCAVE FUNCTIONS

The three properties that make concave functions so valuable in economics are that their critical points are automatically global maxima, that the weighted sum of concave functions is a concave function, and that the level sets of a concave function have just the right shapes for consumption and production theory.

As shown in Chapter 17, in using calculus to find the interior maximum of a function f, one first finds the critical points of f by setting its first derivatives equal to zero and solving the corresponding equations. Then, one uses a second derivative test to separate the maxima from the minima and saddles, and one evaluates the function at all the local maxima to decide which of these local maxima is the global maximum. However, one never needs these extra steps for a *concave* function. A critical point of a concave function is automatically a maximum, and in fact a global maximum.

Theorem 21.6 Let f be a concave (convex) function on an open, convex subset U of $\mathbf{R}^n$. If $\mathbf{x}_0$ is a critical point of f, that is, $Df(\mathbf{x}_0) = \mathbf{0}$, then $\mathbf{x}_0 \in U$ is a *global maximizer (minimizer)* of f on U.

Proof To prove that a critical point of a concave function is automatically a global maximizer, we simply refer to Theorem 21.3 and Corollary 21.4. If f is concave and $Df(\mathbf{x}_0) = \mathbf{0}$, then by inequalities (6) and (8), $f(\mathbf{y}) - f(\mathbf{x}_0) \leq 0$ for all $\mathbf{y}$ in U. In other words, for all $\mathbf{y} \in U$, $f(\mathbf{y}) \leq f(\mathbf{x}_0)$, or $\mathbf{x}_0$ is a global maximizer of f on U. ■

In fact, an even stronger result than Theorem 21.6 holds for concave functions. In the discussion above the statement of Theorem 21.6, we were speaking only of interior maxima. But frequently, the global maximum occurs on the *boundary* on the convex domain U. Corollary 21.4 immediately gives the following condition for a global maximum of a concave function, even if the maximizer is on the boundary of the domain. We leave its proof as an exercise. Note that this theorem includes Theorem 21.6 as a special case.

Theorem 21.7 Let f be a C^1 function defined on a convex subset U of $\mathbf{R}^n$. If f is a concave function and if $\mathbf{x}_0$ is a point in U which satisfies $Df(\mathbf{x}_0)(\mathbf{y}-\mathbf{x}_0) \leq 0$ for all $\mathbf{y} \in U$, then $\mathbf{x}_0$ is *global maximizer* of f on U. If f is a convex function and if $\mathbf{x}_0$ is a point in U which satisfies $Df(\mathbf{x}_0)(\mathbf{y}-\mathbf{x}_0) \geq 0$ for all $\mathbf{y} \in U$, then $\mathbf{x}_0$ is a *global minimizer* of f on U.

Example 21.6 If f is a C^1 *increasing*, concave function of one variable on the interval $[a,b]$, then $f'(b)(x-b) \leq 0$ for all $x \in [a,b]$. (Why?) By Theorem 21.7, b is the global maximizer of f on $[a,b]$.

Example 21.7 Consider the concave function $U(x, y) = x^{1/4}y^{1/4}$ on the (convex) triangle

$$B = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 2\}.$$

By symmetry, we would expect that $(x_0, y_0) = (1,1)$ is the maximizer of U on B. To prove this, use Theorem 21.7. Let (x, y) be an arbitrary point in B. Then,

$$\begin{aligned}\frac{\partial U}{\partial x}(1, 1)(x - 1) + \frac{\partial U}{\partial y}(1, 1)(y - 1) &= \frac{1}{4}(x - 1) + \frac{1}{4}(y - 1) \\ &= \frac{1}{4}(x + y - 2) \\ &\leq 0\end{aligned}$$

since $x + y - 2 \leq 0$ for (x, y) in the constraint set B. By Theorem 21.7, (1,1) is the global maximizer of U on B.

The property that critical points of concave functions are global maximizers is an important one in economic theory. For example, many economic principles, such as marginal rate of substitution equals the price ratio, or marginal revenue equals marginal cost are simply the first order *necessary* conditions of the corresponding maximization problem. Ideally, an economist would like such a rule to also be a *sufficient* condition guaranteeing that utility or profit is being maximized so that it can provide a guideline for economic behavior. This situation does occur when the objective function is concave. Furthermore, an economist, who wants to analyze how the maximizer in a parameterized problem depends on the parameters involved, will usually apply the implicit function theorem to the equations of the first order necessary conditions for maximization. The only situation in which it can be guaranteed that the solution to these perturbed equations is indeed a maximum for all values of the parameters occurs when the objective function is concave.

Example 21.8 Consider the problem of maximizing profit for a firm whose production function is $y = g(\mathbf{x})$, where y denotes output and $\mathbf{x}$ denotes the input bundle. If p denotes the price of the output and w_i is the cost per unit of input i, then the firm's profit function is

$$\Pi(\mathbf{x}) = pg(\mathbf{x}) - (w_1x_1 + \cdots + w_nx_n). \tag{9}$$

As can easily be checked, Π will be a concave function provided that the production function is a concave function. (Exercise.) In this case, the first order condition

$$p\frac{\partial g}{\partial x_i} = w_i \quad \text{for } i = 1, 2, \ldots, n, \tag{10}$$

which says marginal revenue product equals the factor price for each input, is both necessary and sufficient for an interior profit maximizer. If one wants to study the effect of changes in w_i or p on the optimal input bundle, one would apply the comparative statics analysis to system (10). Since profit is concave for all p and $\mathbf{w}$, the solution to system (10) will automatically be the *optimal* input for all choices of p and $\mathbf{w}$.

A second valuable property of concave functions is that they behave well under addition and scalar multiplication by positive numbers, as the following theorem indicates. Its proof follows directly from the definition (1) of a concave function and is left as an exercise.

Theorem 21.8 Let $f_1, \ldots, f_k$ be concave (convex) functions, each defined on the same convex subset U of $\mathbf{R}^n$. Let $a_1, \ldots, a_k$ be positive numbers. Then, $a_1f_1 + \cdots + a_kf_k$ is a concave (convex) function on U.

One can use Theorem 21.8 and the fact that linear functions are concave to deduce immediately that, if the production function g in Example 21.8 is concave, so is the corresponding profit function Π. One can sometimes use Theorem 21.8 to prove that a sum of functions is concave by showing that each summand is itself concave. For example, $f(x_1, \ldots, x_n) = a_1x_1^{k_1} + \cdots + a_nx_n^{k_n}$ is concave on $\mathbf{R}^n_+$ provided each $a_i \geq 0$ and each k_i lies between 0 and 1. (See Exercise 21.4.)

Such a summation occurs in social welfare theory. In an economy with m consumers with utility functions $u_1, \ldots, u_m$ respectively, one measure of the social welfare of any allocation of resources is the sum $a_1u_1 + \cdots + a_mu_m$, where the a_i's are any set of positive weights. If the u_i's are all concave, the corresponding social welfare function will be concave. In this case, the set of maximizers of the various social welfare functions will be the set of Pareto optimal allocations.

A third advantage of concave functions is that their level sets have just the right shapes: they bound convex subsets "from below."

Theorem 21.9 Let f be a function defined on a convex set U in $\mathbf{R}^n$. If f is concave, then for every $\mathbf{x}_0$ in U, the set

$$C^+_{\mathbf{x}_0} \equiv \{\mathbf{x} \in U : f(\mathbf{x}) \geq f(\mathbf{x}_0)\}$$

is a *convex* set. If f is convex, then for every $\mathbf{x}_0$ in U, the set

$$C^-_{\mathbf{x}_0} \equiv \{\mathbf{x} \in U : f(\mathbf{x}) \leq f(\mathbf{x}_0)\}$$

is a *convex* set.

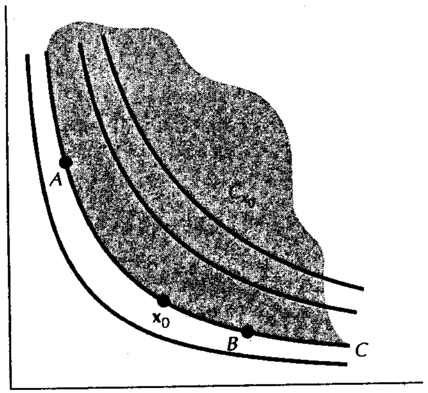


Figure 21.6 *The set "above" the level curve C is a convex set for a concave utility function.*

Proof Let $\mathbf{x}$ and $\mathbf{y}$ be two points in $C^+_{\mathbf{x}_0}$ so that $f(\mathbf{x}) \geq f(\mathbf{x}_0)$ and $f(\mathbf{y}) \geq f(\mathbf{x}_0)$. Then,

$$\begin{aligned} f(t\mathbf{x} + (1-t)\mathbf{y}) &\geq tf(\mathbf{x}) + (1-t)f(\mathbf{y}) \\ &\geq tf(\mathbf{x}_0) + (1-t)f(\mathbf{x}_0) \\ &= f(\mathbf{x}_0). \end{aligned}$$

So, $t\mathbf{x} + (1-t)\mathbf{y}$ is in $C^+_{\mathbf{x}_0}$ and $C^+_{\mathbf{x}_0}$ is a convex set. ■

The property that the set above any level set of a concave function is a convex set is a natural requirement for utility and production functions. For example, consider an indifference curve C of the concave utility function U pictured in Figure 21.6. Two bundles A and B have been labeled on curve C. By Theorem 21.9, the shaded region in Figure 21.6, which represents all bundles which are preferred to bundles A and B, is a convex set. In particular, the set of convex combinations of A and B — the bundles that can be formed by mixing the contents of bundles A and B — all lie on the line segment joining A to B and therefore lie in the shaded area. Thus, concave utility functions have the desirable property that given any two bundles A and B of "goods", a consumer with a concave utility function, will always prefer a mixture of bundles A and B to either A or B. An elementary microeconomics text might phrase this property as: a consumer would prefer a bundle containing a mixture of soda and chips to a bundle of all soda and no chips and to a bundle of no soda and all chips.

A more important advantage of the shape of the indifference curve in Figure 21.6 is that it displays a *diminishing marginal rate of substitution.* As one moves left to right along the indifference curve C increasing consumption of good one, the consumer is willing to give up more and more units of good one to gain an additional unit of good two. This property — a central axiom of consumer

theory — is a property of concave utility functions because each level set forms the boundary of a convex region.

Concave Functions in Economics

We have just described three properties of concave functions that make them especially useful in economic models. In addition, there are some functions which arise in economic models that are naturally concave. For example, consider the **expenditure function** $e(\mathbf{p}, u)$, which describes the minimal amount of income necessary to achieve utility u at the prices $\mathbf{p}$. It can be described analytically as

$$e(\mathbf{p}, u) = \min\{p_1x_1 + \ldots + p_nx_n : u(\mathbf{x}) \geq u\},$$

and will be studied in more detail in Section 22.1.

Theorem 21.10 The expenditure function is concave and homogeneous of degree one in $\mathbf{p}$.

Proof Let $(\mathbf{p}, \mathbf{x})$ and $(\mathbf{p}', \mathbf{x}')$ be two price-consumption combinations that minimize expenditure at utility level u. Let $\mathbf{p}'' = t\mathbf{p} + (1-t)\mathbf{p}'$ for any t between 0 and 1; and let $\mathbf{x}''$ be the corresponding expenditure minimizing bundle. Then,

$$e(\mathbf{p}'', u) = \mathbf{p}'' \cdot \mathbf{x}'' = t\mathbf{p} \cdot \mathbf{x}'' + (1-t)\mathbf{p}' \cdot \mathbf{x}''. \qquad (11)$$

But $\mathbf{x}''$ is not necessarily the cheapest way to achieve utility u at prices $\mathbf{p}$ or $\mathbf{p}'$. Therefore,

$$\mathbf{p} \cdot \mathbf{x}'' \geq e(\mathbf{p}, u) \quad \text{and} \quad \mathbf{p}' \cdot \mathbf{x}'' \geq e(\mathbf{p}', u) \qquad (12)$$

Combining (11) and (12) yields the concavity of $e(\mathbf{p}, u)$ in $\mathbf{p}$:

$$e(\mathbf{p}'', u) \geq te(\mathbf{p}, u) + (1-t)e(\mathbf{p}', u).$$

To see that $e(t\mathbf{p}, u) = te(\mathbf{p}, u)$, notice that from the above definition of the expenditure function, the $\mathbf{x}$ which minimizes $\mathbf{p} \cdot \mathbf{x}$ subject to $u(\mathbf{x}) \geq u$ will also minimize $t\mathbf{p} \cdot \mathbf{x}$ with the same constraint. ■

In fact, all that the above proof requires is that we were minimizing a *linear* objective function on a constraint set and that the function under consideration is just the minimum value. A number of other economic functions arise this way. For example, the cost function $c(\mathbf{w}, y)$ corresponding to a given production function g can be considered the minimum cost needed to produce output y, when input prices are given by $\mathbf{w}$:

$$c(\mathbf{w}, y) = \min\{w_1x_1 + \cdots + w_nx_n : g(\mathbf{x}) = y\}.$$

The same argument as in the proof of Theorem 21.10 shows that $c(\mathbf{w}, y)$ is concave and homogeneous in factor prices $\mathbf{w}$. Finally, consider the optimal profit function $\pi(p, \mathbf{w})$ which is the maximum profit that can be achieved when the price of the output is p and the cost of the inputs is $\mathbf{w}$. Write π as

$$\pi(p, \mathbf{w}) = \max_{y,\mathbf{x}}\{py - \mathbf{w} \cdot \mathbf{x} : y \leq g(\mathbf{x})\}. \tag{13}$$

Then, $\pi(p, \mathbf{w})$ is *convex* and homogeneous of degree one in $(p, \mathbf{w})$. (Exercise.)

These three functions illustrate a general phenomenon about optimizing linear objective functions, which we state in the following theorem. The proof we leave as an exercise.

> **Theorem 21.11** Consider the problem of maximizing the linear objective function $\mathbf{a} \cdot \mathbf{x}$ with respect to $\mathbf{x}$ on a given constraint set. The value of the optimal objective function will be a convex and homogeneous of degree one function of the parameter $\mathbf{a}$. For a minimization problem with a linear objective function, the optimal value of the objective function will be a concave and homogeneous of degree one function of $\mathbf{a}$.

Finally, concave utility functions play a major role in expected utility theory because, as K. Arrow first observed, in such models the level of risk aversion of a consumer is measured by the concavity of the consumer's utility function.

EXERCISES

21.11 Prove that the profit function (9) in Example 21.8 is concave if the production function $y = g(\mathbf{x})$ is concave.

21.12 Suppose that a one-product monopolist faces an inverse demand function $p = F(q)$ and a cost function $q \mapsto C(q)$.
a) Write out the expression for its profit as a function of q.
b) What assumptions on F and C yield a concave profit function?

21.13 Prove Theorem 21.7. Show that it implies Theorem 21.6.

21.14 Write out the proof of Theorem 21.2 for convex functions.

21.15 Prove Theorem 21.8 directly from the definition (1) of a concave function.

21.16 Prove that $\pi(p, \mathbf{w})$ in (13) is *convex* and homogeneous of degree one in $(p, \mathbf{w})$.

21.17 Prove Theorem 21.11.

21.3 QUASICONCAVE AND QUASICONVEX FUNCTIONS

Concave functions pose the same dilemma that homogeneous functions did in Section 20.2. They have many desirable properties for production and utility functions. However, as Examples 21.3, 21.4, and 21.5 clearly indicate, concavity

is a *cardinal* property. It depends on the numbers which the function assigns to the level sets, not just on the shape of the level sets. In other words, a monotonic transformation of a concave function need *not* be concave.

However, concave functions have one fundamental ordinal property, as Theorem 21.9 indicates. Their level sets bound convex sets from below. This property leads to the highly desirable condition of diminishing marginal rate of substitution for indifference curves.

Just as we did for homogeneous functions, we give a name to the class of functions which have the desired ordinal property that concave functions have. A somewhat natural name for a function that is the ordinal version of a concave function is a quasiconcave function.

Definition A function f defined on a convex subset U of $\mathbf{R}^n$ is **quasiconcave** if for every real number a,

$$C_a^+ \equiv \{\mathbf{x} \in U : f(\mathbf{x}) \geq a\}$$

is a convex set. Similarly, f is **quasiconvex** if for every real number a,

$$C_a^- \equiv \{\mathbf{x} \in U : f(\mathbf{x}) \leq a\}$$

is a convex set.

We present some alternative definitions of quasiconcave and quasiconvex in the following theorem, whose proof is left as an exercise.

> **Theorem 21.12** Let f be a function defined on a convex set U in $\mathbf{R}^n$. Then, the following statements are equivalent to each other:
>
> (*a*) f is a quasiconcave function on U.
> (*b*) For all $\mathbf{x}, \mathbf{y} \in U$ and all $t \in [0, 1]$,
> $$f(\mathbf{x}) \geq f(\mathbf{y}) \quad \text{implies} \quad f(t\mathbf{x} + (1-t)\mathbf{y}) \geq f(\mathbf{y}).$$
> (*c*) For all $\mathbf{x}, \mathbf{y} \in U$ and all $t \in [0, 1]$,
> $$f(t\mathbf{x} + (1-t)\mathbf{y}) \geq \min\{f(\mathbf{x}), f(\mathbf{y})\}.$$

By Theorem 21.9, every concave function is quasiconcave and every convex function is quasiconvex. Furthermore, any monotonic transformation of a concave function is a quasiconcave function. In particular, since every Cobb-Douglas function is a monotonic transformation of a Cobb-Douglas function with decreasing returns to scale, every Cobb-Douglas function of two variables is quasiconcave.

> **Theorem 21.13** Every Cobb-Douglas function $F(x, y) = Ax^a y^b$ with A, a, and b all positive is quasiconcave.

Example 21.9 Consider the Leontief or fixed-coefficient production function $Q(x, y) = \min\{ax, by\}$ with $a, b > 0$. The level sets of Q are drawn in Figure 21.7. Certainly, the region above and to the right of any of this function's L-shaped level sets is a convex set. Therefore, Q is quasiconcave.

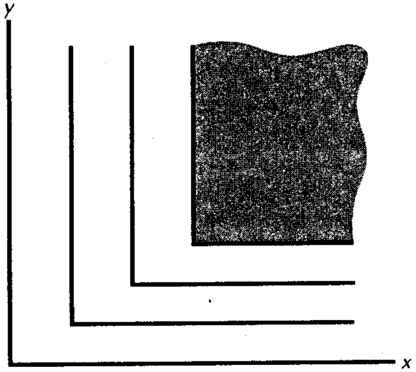


Figure 21.7 *Fixed-coefficient production function.*

Example 21.10 Consider the constant elasticity of substitution (CES) production function

$$Q(x, y) = (a_1 x_1^r + a_2 x_2^r)^{1/r}, \quad \text{where } 0 < r < 1.$$

By Theorem 21.8 and Exercise 21.4, $(a_1 x_1^r + a_2 x_2^r)$ is concave. Since $g(z) = z^{1/r}$ is a monotonic transformation, Q is a monotonic transformation of a concave function and therefore is quasiconcave.

Example 21.11 Let $y = f(x)$ be any increasing function on $\mathbf{R}^1$, as in Figure 21.8. For any x^*, $\{x : f(x) \geq f(x^*)\}$ is just the interval $[x^*, \infty)$, a convex subset of $\mathbf{R}^1$. So, f is quasiconcave. On the other hand, $\{x : f(x) \leq f(x^*)\}$ is the convex set $(-\infty, x^*]$. Therefore, an increasing function on $\mathbf{R}^1$ is both quasiconcave and quasiconvex. The same argument applies to a decreasing function.

Example 21.12 Any function on $\mathbf{R}^1$ which rises monotonically until it reaches a global maximum and then monotonically falls, such as $y = -x^2$ or the bell-shaped probability density function $y = ke^{-x^2}$, is a quasiconcave function, as Figure 21.9 indicates. For any x_1 as in Figure 21.9, there is a x_2 such that $f(x_1) = f(x_2)$. Then, $\{x : f(x) \geq f(x_1)\}$ is the convex interval $[x_1, x_2]$.

Remark Notice that we take a slightly different tack in going from concave to quasiconcave than we did in going from homogeneous to homothetic. In the latter case, we simply defined a homothetic function as any function which has

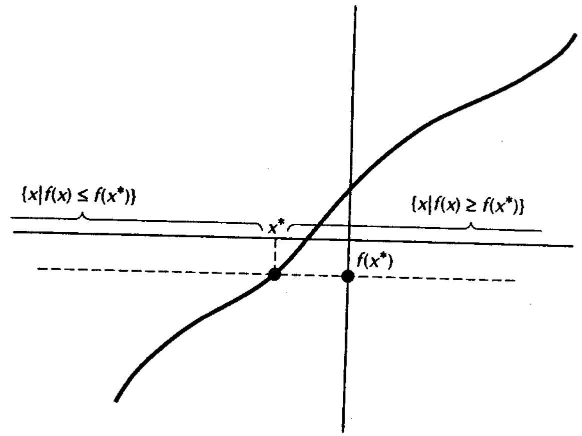


Figure 21.8 *An increasing function on $\mathbf{R}^1$ is both quasiconcave and quasiconvex.*

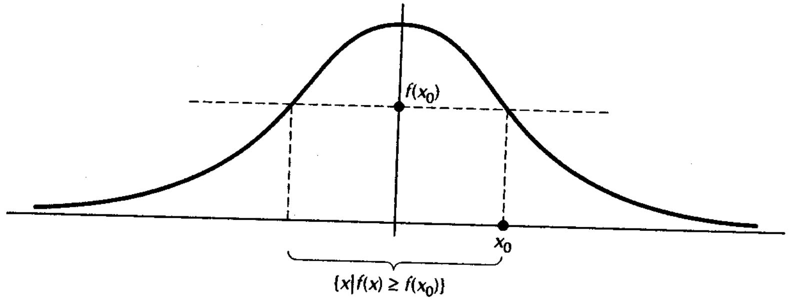


Figure 21.9 *These bell-shaped functions are quasiconcave.*

the same level sets as a homogeneous function. In the former case, we defined a quasiconcave function as any function which has the desired ordinal property of concave functions. It is natural to ask if indeed any quasiconcave function is equivalent to some concave function by a monotonic transformation. K. Arrow and A. Enthoven considered this question in their path-breaking treatise on quasiconcave functions and provided a concrete example of a quasiconcave function which is *not* a monotonic transformation of any concave function. (See the Notes at the end of this chapter.)

Calculus Criteria

We now work toward developing calculus criteria for quasiconcavity. Analogous to Theorem 21.3 for concave functions, there is a necessary and sufficient first derivative test for quasiconcavity, that provides a useful technique for proving theorems about quasiconcave functions. Like Theorem 21.3, it is too unwieldy to use for checking whether or not any specific function is quasiconcave, so we will develop a simpler *second* order test in the next section. Since we have first order conditions for the other special classes of functions, we present this first order condition for quasiconcave functions.

Theorem 21.14 Suppose that F is a C^1 function on an open convex subset U of $\mathbf{R}^n$. Then, F is quasiconcave on U if and only if

$$F(\mathbf{y}) \geq F(\mathbf{x}) \text{ implies that } DF(\mathbf{x})(\mathbf{y}-\mathbf{x}) \geq 0; \tag{14}$$

F is quasiconvex on U if and only if

$$F(\mathbf{y}) \leq F(\mathbf{x}) \text{ implies that } DF(\mathbf{x})(\mathbf{y}-\mathbf{x}) \leq 0.$$

Proof Suppose that F is quasiconcave on U and that $F(\mathbf{y}) \geq F(\mathbf{x})$ for some $\mathbf{x}, \mathbf{y} \in U$. Then, for all $t \in [0, 1]$,

$$F(\mathbf{x} + t(\mathbf{y}-\mathbf{x})) \geq F(\mathbf{x}).$$

Since

$$\frac{F(\mathbf{x} + t(\mathbf{y}-\mathbf{x})) - F(\mathbf{x})}{t} \geq 0$$

for all $t \in (0, 1)$, we let $t \to 0$ to obtain

$$DF(\mathbf{x})(\mathbf{y}-\mathbf{x}) \geq 0.$$

The proof of the converse follows the same type of argument, but is a bit more intricate and can be found in the Appendix to this chapter. ■

Remark By Theorem 21.9, all concave functions are quasiconcave. This fact can also be seen by comparing the corresponding first order conditions (6) and (14). As Examples 21.8 and 21.9 illustrate, not every quasiconcave function is concave. In fact, quasiconcave functions fail to have two of the three important properties of concave functions that we highlighted earlier. First of all, a critical point of a quasiconcave function need not be a maximum, let alone a global maximum. For example, the function $y = x^3$ in $\mathbf{R}^1$ is quasiconcave by Example 21.8; its critical point $x = 0$ is certainly not any kind of a maximum. Secondly, the sum of quasiconcave functions need not be quasiconcave. For example, $f_1(x) = x^3$ and $f_2(x) = -x$ are both monotone functions on $\mathbf{R}^1$ (and therefore quasiconcave). However, $f_3 = x^3 - x$ is neither quasiconcave nor quasiconvex. (Check.)

However, a quasiconcave function that is also homogeneous of degree one *is* concave, as the following Theorem indicates.

Theorem 21.15 Suppose that F is a real-valued, positive function defined on a convex cone C in $\mathbf{R}^n$. If F is homogeneous of degree one and quasiconcave on C, it is concave on C.

The proof of Theorem 21.15 is straightforward, but a bit long, and is presented in the Appendix of this chapter.

EXERCISES

21.18 For each of the following functions on $\mathbf{R}^1$, determine whether it is quasiconcave, quasiconvex, both, or neither:

a) e^x,	*b)* $\ln x$,	*c)* $x^3 + x$,	*d)* $x^3 - x$,
e) $x^4 - x^2$,	*f)* $x^4 + x^2$,	*g)* $3x^3 - 5x^2 + 7x$,	*h)* $\sin x$.

21.19 Write out a careful proof of Theorem 21.12. The proof is straightforward but should help one's understanding of quasiconcave functions and reinforce one's ability to prove theorems.

21.20 Write out the corresponding theorem for quasiconvex functions.

21.21 Prove that a quasiconcave function cannot have a strict interior minimum.

21.22 Let $f_1, \ldots, f_k$ be concave functions of one variable. Let $g(z)$ be a monotonic transformation. Prove that $F(x_1, \ldots, x_k) = g(f_1(x_1) + \cdots + f_k(x_k))$ is a quasiconcave function.

21.4 PSEUDOCONCAVE FUNCTIONS

In order to develop the most useful calculus criterion for quasiconcavity and quasiconvexity, namely the appropriate second order condition, we introduce one more class of functions, a class that forms a link between concave functions and quasiconcave functions. This class of functions was specifically defined by O. Mangasarian (see Chapter Notes) so that it is very close to the class of quasiconcave functions, yet it retains the important property of concave functions that critical points are automatically global maxima.

Definition Let U be an open convex subset of $\mathbf{R}^n$. A C^1 function $F: U \to \mathbf{R}$ is **pseudoconcave** at $\mathbf{x}^* \in U$ if

$$DF(\mathbf{x}^*)(\mathbf{y}-\mathbf{x}^*) \leq 0 \quad \text{implies} \quad F(\mathbf{y}) \leq F(\mathbf{x}^*) \tag{15}$$

for all $\mathbf{y} \in U$. The function F is pseudoconcave on U if (15) holds for all $\mathbf{x}^* \in U$. To define a **pseudoconvex function** on U, one simply reverses the first inequality in (15).

As Corollary 21.4 points out, the first order criterion (21.3) for concavity clearly implies the defining condition (15) for pseudoconcavity; thus a C^1 concave function is always pseudoconcave. Furthermore, condition (15) is exactly the condition one uses to prove that a critical point of a concave function is automatically a *global max*. Since we will refer to it in later chapters, we state this observation as a theorem.

Theorem 21.16 Let U be a convex subset of $\mathbf{R^n}$, and let $F: U \to \mathbf{R}$ be a C^1 pseudoconcave function. If $\mathbf{x}^* \in U$ has the property $DF(\mathbf{x}^*)(\mathbf{y} - \mathbf{x}^*) \leq 0$ for all $\mathbf{y} \in U$, for example, $DF(\mathbf{x}^*) = \mathbf{0}$, then $\mathbf{x}^*$ is a global max of F on U. An analogous result holds for pseudoconvex functions.

To see how pseudoconcavity relates to quasiconcavity at the other end of the link, note that the contrapositive of the first order criterion (14) for quasiconcavity,

$$DF(\mathbf{x})(\mathbf{y} - \mathbf{x}) < 0 \quad \text{implies} \quad F(\mathbf{y}) < F(\mathbf{x}), \tag{16}$$

is very nearly the definition (15) of a pseudoconcave function; one merely has to change the $<$ signs in (16) to $\leq$ signs to obtain (15). The following theorem makes precise the close relationship between pseudoconcave and quasiconcave functions. Its proof is straightforward. However, we delay the proof until the Appendix of this chapter so as not to break the flow of our presentation at this point. Try to write out the proof before reading the Appendix.

Theorem 21.17 Let U be a convex subset of $\mathbf{R^n}$. Let $F: U \to \mathbf{R}$ be a C^1 function. Then,

(a) if F is pseudoconcave on U, F is quasiconcave on U, and
(b) if U is open and if $\nabla F(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in U$, then F is pseudoconcave on U if and only if F is quasiconcave on U.

The primary reason for introducing pseudoconcave functions is the fact that there is a straightforward second derivative test for such functions, a test that is also the most efficient check that a given function is *quasiconcave*. This second order condition arises from a constrained maximization approach to pseudoconcave functions, which is summarized in the following theorem.

Theorem 21.18 Let U be an open convex subset of $\mathbf{R^n}$. Let $F: U \to \mathbf{R}$ be a C^1 function on U. Then, F is pseudoconcave on U if and only if for each $\mathbf{x}^*$ in U, $\mathbf{x}^*$ itself is the solution to the constrained maximization problem

$$\begin{aligned} &\text{maximize} \quad F(\mathbf{x}) \\ &\text{subject to} \quad C_{\mathbf{x}^*} \equiv \{\mathbf{y} \in U : DF(\mathbf{x}^*)(\mathbf{y} - \mathbf{x}^*) \leq 0\}. \end{aligned} \tag{17}$$

The proof of Theorem 21.18 follows immediately from noticing that the defining condition (15) of pseudoconcavity is equivalent to the maximization statement of Theorem 21.18. Details are left as an exercise.

Chapters 18 and 19 developed first and second order necessary and sufficient conditions for constrained maximization problems like (17). For any given $\mathbf{x}^*$, the Lagrangian function for Problem (17) is

$$\begin{aligned} L(\mathbf{x}, \lambda) &= F(\mathbf{x}) - \lambda\, DF(\mathbf{x}^*) \cdot (\mathbf{x} - \mathbf{x}^*) \\ &= F(\mathbf{x}) - \lambda \sum_i \frac{\partial F}{\partial x_i}(\mathbf{x}^*)(x_i - x_i^*). \end{aligned}$$

Since the constraint in (17) is a linear one, we do not need a multiplier for the objective function by Theorem 19.12. Since

$$\frac{\partial L}{\partial x_i}(\mathbf{x}^*, 1) = \frac{\partial F}{\partial x_i}(\mathbf{x}^*) - 1 \cdot \frac{\partial F}{\partial x_i}(\mathbf{x}^*) = 0,$$

$\mathbf{x} = \mathbf{x}^*, \lambda = 1$ is a solution of the first order condition for Problem (17). The corresponding second order sufficient condition that $\mathbf{x}^*$ must satisfy to be a solution of (17) involves the bordered Hessian H of F. This matrix H is formed by bordering the usual Hessian $D^2F(\mathbf{x})$ above and to the left by the first order partial derivatives $DF(\mathbf{x})$ of F:

$$H = \begin{pmatrix} 0 & F'_{x_1} & F'_{x_2} & \cdots & F'_{x_n} \\ F'_{x_1} & F''_{x_1x_1} & F''_{x_1x_2} & \cdots & F''_{x_1x_n} \\ F'_{x_2} & F''_{x_2x_1} & F''_{x_2x_2} & \cdots & F''_{x_2x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F'_{x_n} & F''_{x_nx_1} & F''_{x_nx_2} & \cdots & F''_{x_nx_n} \end{pmatrix}. \tag{18}$$

To carry out the second order test, one calculates the $(n - 1)$ largest *leading principal minors* of H, beginning with the top leftmost 3×3 submatrix and continuing to the full $(n + 1) \times (n + 1)$ matrix H, as described in Section 16.3. We want these last $(n - 1)$ leading principal minors of H to alternate in sign with the smallest one, the leading 3×3 minor, which is positive. The corresponding sufficient condition for pseudoconvexity requires that the last $(n - 1)$ leading principal minors of H all be negative. In both cases, these sign patterns must hold for all $\mathbf{x}$ in the domain of F. The following theorem summarizes this test for pseudoconcavity.

Theorem 21.19 Let F be a C^2 function on an open convex subset W in $\mathbf{R}^n$. Consider the bordered Hessian H in (18).

(a) If the largest $(n-1)$ leading principal minors of H alternate in sign, *for all* $\mathbf{x} \in$ W, with the smallest of these — the third order leading principal minor — positive, then F is pseudoconcave, and therefore quasiconcave, on W.
(b) If these largest $(n-1)$ leading principal minors are all negative for all $\mathbf{x} \in$ W, then F is pseudoconvex, and therefore quasiconvex, on W.

Remark The condition of Theorem 21.19 is a sufficient condition but not a necessary one. There are a number of necessary conditions in the literature. One such necessary condition is that we replace all the strong inequalities regarding the leading principal minors of H in Theorem 21.19 by weak inequalities and apply this test to *all* of the principal minors of H which include the first row and column and which are at least 3×3 in size, not just the *leading* principal minors. See the Notes at the end of this chapter for references.

To better understand Theorem 21.19, we write out its two-dimensional version — a version which requires computing the sign of only one determinant, since the corresponding bordered Hessian is itself 3×3. We present a special proof for this two-dimensional version in the Appendix of this chapter, a proof which does not use the constrained optimization approach (17) to pseudoconcavity. To simplify matters, we focus directly on quasiconcave monotone functions.

Theorem 21.20 Let F be a C^2 function on a convex set W in $\mathbf{R}^2$. Suppose that F is monotone in that $F'_x > 0$ and $F'_y > 0$ on W. If the determinant

$$\begin{vmatrix} 0 & F'_x & F'_y \\ F'_x & F''_{xx} & F''_{xy} \\ F'_y & F''_{xy} & F''_{yy} \end{vmatrix} \tag{19}$$

is > 0 for all $(x, y) \in W$, then F is quasiconcave on W. If the determinant (19) is negative for all $(x, y) \in W$, then F is quasiconvex on W.

Conversely, if F is quasiconcave on W, then the determinant (19) is ≥ 0; if F is quasiconvex on W, then the determinant (19) is ≤ 0 for all $(x, y) \in W$.

Remark As mentioned in our discussion on bordered matrices in Chapter 16, some texts "border the Hessian" on the right and below by $DF(\mathbf{x})$ rather than on the left and above as we did in (18):

$$\begin{pmatrix} F''_{x_1x_1} & F''_{x_1x_2} & \cdots & F''_{x_1x_n} & F'_{x_1} \\ F''_{x_2x_1} & F''_{x_2x_2} & \cdots & F''_{x_2x_n} & F'_{x_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F''_{x_nx_1} & F''_{x_nx_2} & \cdots & F''_{x_nx_n} & F'_{x_n} \\ F'_{x_1} & F'_{x_2} & \cdots & F'_{x_n} & 0 \end{pmatrix}.$$

In this case, one applies the test of Theorem 21.19 to the last $(n-1)$ *trailing* principal minors, which are formed by deleting the *first* k rows and columns from the bordered Hessian. As in Theorem 21.19, we need the bottom-right hand 3×3 principal minor to be positive and all the larger trailing principal minors to alternate in sign in order to guarantee that F is quasiconcave.

Example 21.13 Theorem 21.13 implies that the Cobb-Douglas function $U(x, y) = x^a y^b$ is quasiconcave on $\mathbf{R}^2_+$ for $a, b > 0$ since it is a monotone transformation of a concave function. Let's use Theorem 21.20 to prove the quasiconcavity of U. The bordered Hessian (19) is

$$\begin{pmatrix} 0 & ax^{a-1}y^b & bx^a y^{b-1} \\ ax^{a-1}y^b & a(a-1)x^{a-2}y^b & abx^{a-1}y^{b-1} \\ bx^a y^{b-1} & abx^{a-1}y^{b-1} & b(b-1)x^a y^{b-2} \end{pmatrix},$$

whose determinant is

$$(ab + ab^2 + a^2b)x^{3a-2}y^{3b-2},$$

which is always positive for $x > 0$, $y > 0$, $a > 0$, and $b > 0$. By Theorem 21.20, U is pseudoconcave, and therefore quasiconcave.

EXERCISES

21.23 We have three practical ways of checking for the quasiconcavity of a given function. We can show that it has the same level sets as a concave function. We can use the method of the proof of Theorem 21.20 in the Appendix to show that its level sets are graphs of convex functions on $\mathbf{R}^{n-1}$. We can use the bordered Hessian test of Theorem 21.19. For each of the following functions, determine whether it is quasiconcave, quasiconvex, neither, or both:

a) $f(x, y) = ye^{-x}$ on $\mathbf{R}^2$;
b) $f(x, y) = ye^{-x}$ on $\mathbf{R}^2_+$;
c) $f(x, y) = (2x + 3y)^3$ on $\mathbf{R}^2$;
d) $f(x, y, z) = (e^x + 5y^4 + |z|)^{1/2}$;
e) $f(x, y) = (y - x^4)^{1/3}$;
f) $f(x, y) = \dfrac{y}{x^2 + 1}$ on $\mathbf{R}^2_+$;
g) $f(x, y) = \dfrac{y}{x^2 + 1}$ on $\mathbf{R}^2$;
h) $f(x, y) = yx^{-2}$ on $\mathbf{R}^2_+$;
i) $ke^{\mathbf{x}^T A \mathbf{x}}$, where A is a positive definite matrix and k is a positive constant.

21.24 Prove Theorem 21.18.

21.25 Let $L: \mathbf{R}^m \to \mathbf{R}^n$ be a linear function. Let $f: \mathbf{R}^n \to \mathbf{R}$. Show that if f is quasiconcave, so is $f \circ L$ and that if f is pseudoconcave, so is $f \circ L$.

21.5 CONCAVE PROGRAMMING

As we have seen throughout this chapter, not only do concave and quasiconcave functions arise naturally in economics, but such functions also provide much more structure in the analysis of the optimization problems that lie at the heart of economic theory. In particular, the first order necessary conditions that characterize the solution of the general differentiable optimization problem are also sufficient conditions when the functions involved are concave.

Unconstrained Problems

We begin by repeating the statement of Theorem 21.6 for the *unconstrained* concave programming problem. Recall by Theorem 21.16 that Theorem 21.6 holds even if f is pseudoconcave, but not necessarily if f is quasiconcave.

Theorem 21.21 Let U be a convex subset of $\mathbf{R}^n$. Let $f: U \to \mathbf{R}$ be a C^1 concave (convex) function on U. Then, $\mathbf{x}^*$ is a global max of f on U *if and only if* $Df(\mathbf{x}^*)(\mathbf{x}-\mathbf{x}^*) \le 0$ for all $\mathbf{x} \in U$. In particular, if U is open, or if $\mathbf{x}^*$ is an interior point of U, then $\mathbf{x}^*$ is a global max (min) of f on U *if and only if* $Df(\mathbf{x}^*) = \mathbf{0}$.

Constrained Problems

For constrained problems, we need some concavity or convexity hypotheses on the constraint functions too.

Theorem 21.22 Let U be a convex open subset of $\mathbf{R}^n$. Let $f: U \to \mathbf{R}$ be a C^1 pseudoconcave function on U, for example, f quasiconcave with nonvanishing gradient. Let $g_1, \dots, g_k: U \to \mathbf{R}$ be C^1 quasiconvex functions. Consider the programming problem

$$\begin{aligned} &\text{maximize} \quad f(x) \\ &\text{subject to} \quad \mathbf{x} \in C_\mathbf{b} \equiv \{\mathbf{x} \in U : g_i(\mathbf{x}) \le b_i,\ i = 1, \dots, k\}. \end{aligned} \tag{20}$$

Suppose that one of the constraint qualifications in Theorem 19.12 holds. Form the Lagrangian

$$L(\mathbf{x}, \lambda_1, \dots, \lambda_k) \equiv f(\mathbf{x}) - \sum_{i=1}^{k} \lambda_i [g_i(\mathbf{x}) - b_i]. \tag{21}$$

If there exist $\mathbf{x}^*$ and λ^* such that

$$\frac{\partial L}{\partial x_j}(\mathbf{x}^*, \lambda^*) = 0, \qquad \text{for } j = 1, \dots, n, \tag{22}$$

$$\text{and} \quad \lambda_i^* \ge 0, \quad g_i(\mathbf{x}^*) \le b_i, \quad \lambda_i^* \cdot (g_i(\mathbf{x}^*) - b_i) = 0, \quad \text{for } i = 1, \dots, k. \tag{23}$$

Then, $\mathbf{x}^*$ is a *global* max of f on the constraint set $C_\mathbf{b}$.

Proof Write condition (22) as

$$Df(\mathbf{x}^*) - \sum_{i=1}^{k} \lambda_i^* Dg_i(\mathbf{x}^*) = \mathbf{0}. \tag{24}$$

Let $\mathbf{x}$ be an arbitrary point in the constraint set. For each binding constraint g_i, $g_i(\mathbf{x}) \le g_i(\mathbf{x}^*)$. Since g_i is quasiconvex,

$$Dg_i(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \le 0, \tag{25}$$

by Theorem 21.14. Since $\lambda_i^* = 0$ for the nonbinding constraints g_i by (23),

$$\lambda_i^* Dg_i(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \le 0,$$

for all i and all $\mathbf{x} \in C_\mathbf{b}$. By (24),

$$Df(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) \le 0, \tag{26}$$

for all $\mathbf{x} \in C_\mathbf{b}$. Since f is pseudoconcave, (26) implies that $f(\mathbf{x}) \le f(\mathbf{x}^*)$ and, therefore, that $\mathbf{x}^*$ is a global max of f on $C_\mathbf{b}$. ■

Remark

(1) We really only needed the *binding* inequality constraints to be quasiconvex in the proof of Theorem 21.21.

(2) The most natural constraint qualifications for problem (20) are either that the g_i's are linear or that the g_i's are convex functions with $g_i(\mathbf{z}^*) < b_i$ for some $\mathbf{z}^* \in U$ and all i. See Theorem 19.12.

(3) As in the statement of Theorem 21.7, the sufficient condition (22) for a global max of Problem (20) can be weakened to

$$D_\mathbf{x} L(\mathbf{x}^*, \lambda^*)(\mathbf{x} - \mathbf{x}^*) \le \mathbf{0} \quad \text{for all } \mathbf{x} \in C_\mathbf{b}. \tag{27}$$

As the following theorem indicates, both the set of maximizers and the maximal value function for problem (20) have nice properties in concave programming problems.

Theorem 21.23 Let $f, g_1, \dots, g_k$ be as in the hypothesis of Theorem 21.22.

(*a*) For any fixed $\mathbf{b} = (b_1, \dots, b_k) \in \mathbf{R}^k$, let $Z(\mathbf{b})$ denote the set of $\mathbf{x} \in C_\mathbf{b}$ that are global maximizers of f on $C_\mathbf{b}$. Then, $Z(\mathbf{b})$ is a convex set.

(*b*) For any $\mathbf{b} \in \mathbf{R}^k$, let $V(\mathbf{b})$ denote the maximal value of the objective function f in problem (20). If f is concave and the g_i's are convex, then $\mathbf{b} \mapsto V(\mathbf{b})$ is a *concave function* of $\mathbf{b}$.

Proof

(*a*) Suppose that $\mathbf{x}^1$ and $\mathbf{x}^2$ are in $Z(\mathbf{b})$, and let

$$\mathbf{x}^3 = t\mathbf{x}^1 + (1-t)\mathbf{x}^2 \in \ell(\mathbf{x}^1, \mathbf{x}^2).$$

Since the g_i's are quasiconvex functions, $C_{\mathbf{b}}$ is a convex set and $\mathbf{x}^3 \in C_{\mathbf{b}}$. Since f is quasiconcave,

$$f(\mathbf{x}^3) \geq \min\{f(\mathbf{x}^1), f(\mathbf{x}^2)\}.$$

Since $f(\mathbf{x}^1) = f(\mathbf{x}^2) = \max\{f(\mathbf{x}) : \mathbf{x} \in C_{\mathbf{b}}\}$, $f(\mathbf{x}^3) = f(\mathbf{x}^2)$ and $\mathbf{x}^3$ is in $Z(\mathbf{b})$, also. Therefore, $Z(\mathbf{b})$ is a convex set.

(*b*) Let $\mathbf{b}^3 = t\,\mathbf{b}^1 + (1-t)\,\mathbf{b}^2$, and let $\mathbf{x}^i \in Z(\mathbf{b}^i)$ for $i = 1, 2, 3$. Then, for $j = 1, \ldots, k$,

$$\begin{aligned} g_j(t\mathbf{x}^1 + (1-t)\mathbf{x}^2) &\leq t g_j(\mathbf{x}^1) + (1-t)\, g_j(\mathbf{x}^2) \quad \text{(convexity of } g_j) \\ &\leq t\, b_j^1 + (1-t)\, b_j^2 \quad \text{since} \quad (g_j(\mathbf{x}^i) \leq b_j^i,\ i = 1, 2) \\ &= b_j^3, \end{aligned}$$

and so $t\mathbf{x}^1 + (1-t)\mathbf{x}^2$ is in $C_{\mathbf{b}^3}$. Therefore,

$$\begin{aligned} V(\mathbf{b}^3) &= f(\mathbf{x}^3) \\ &\geq f(t\mathbf{x}^1 + (1-t)\mathbf{x}^2) \qquad (\text{since } \mathbf{x}^3 \in Z(\mathbf{b}^3)) \\ &\geq t f(\mathbf{x}^1) + (1-t) f(\mathbf{x}^2) \quad (\text{concavity of } f) \\ &= tV(\mathbf{b}_1) + (1-t)V(\mathbf{b}^2). \ \blacksquare \end{aligned}$$

Saddle Point Approach

In order to *compute* maxima of a constrained optimization problem like (20), one often considers the corresponding *saddle point problem*, especially when the functions involved are concave.

Definition Let U be a convex subset of $\mathbf{R}^n$. Consider the Lagrangian function (21) for the programming problem (20), as a function of $\mathbf{x}$ and λ. Then, $(\mathbf{x}^*, \lambda^*)$ is **saddle point** of L if

$$L(\mathbf{x}, \lambda^*) \leq L(\mathbf{x}^*, \lambda^*) \leq L(\mathbf{x}^*, \lambda) \tag{28}$$

for all $\lambda \geq \mathbf{0}$ and all $\mathbf{x} \in U$. Usually, $U = \mathbf{R}^n$ or $U = \mathbf{R}^n_+$, the positive orthant of $\mathbf{R}^n$. In the latter case, we say that $(\mathbf{x}^*, \lambda^*)$ is a **nonnegative saddle point** of L.

Theorem 21.24 If $(\mathbf{x}^*, \lambda^*)$ is a (nonnegative) saddle point for L in Problem (20), then $\mathbf{x}^*$ maximizes f on $C_{\mathbf{b}}$ $(C_{\mathbf{b}} \cap \mathbf{R}^n_+)$.

Proof We first show that $\mathbf{x}^*$ is in $C_{\mathbf{b}}$. The right side of (28) implies that

$$\sum_{i=1}^{k} (\lambda_i - \lambda_i^*)(g_i(\mathbf{x}^*) - b_i) \leq 0, \tag{29}$$

for all $\lambda_i \geq 0$. For any fixed h, plug $\lambda_h = \lambda_h^* + 1$ and $\lambda_i = \lambda_i^*$ for all $i \neq h$ into (29). Then, (29) becomes $g_h(\mathbf{x}^*) - b_h \leq 0$. Therefore, $\mathbf{x}^* \in C_{\mathbf{b}}$.

It follows that $\sum_i \lambda_i^*(g_i(\mathbf{x}^*) - b_i) \leq 0$. On the other hand, setting each $\lambda_h = 0$ in (29) yields $\sum_i \lambda_i^*(g_i(\mathbf{x}^*) - b_i) \geq 0$, and thus

$$\sum_i \lambda_i^*(g_i(\mathbf{x}^*) - b_i) = 0 \quad \text{and each} \quad \lambda_i^*(g_i(\mathbf{x}^*) - b_i) = 0. \tag{30}$$

If $\mathbf{x} \in C_{\mathbf{b}}$, then, since each $\lambda_i^*(g_i(\mathbf{x}) - b_i) \leq 0$,

$$\begin{aligned} f(\mathbf{x}) &\leq f(\mathbf{x}) - \sum_i \lambda_i^*(g_i(\mathbf{x}) - b_i) \\ &\leq f(\mathbf{x}^*) - \sum_i \lambda_i^*(g_i(\mathbf{x}^*) - b_i) \qquad \text{by (28)} \\ &= f(\mathbf{x}^*) \qquad \text{by (30).} \ \blacksquare \end{aligned}$$

Notice that there were no concavity hypotheses in the statement of Theorem 21.24. In concave programming, solutions of saddle point problems are more or less equivalent to solutions of programming problems, as the following theorem, due to Kuhn and Tucker, points out.

Theorem 21.25 Suppose that $U = \mathbf{R}^n_+$ or that U is an open convex subset of $\mathbf{R}^n$. Suppose that f is a C^1 concave function and that $g_1, \ldots, g_k$ are C^1 convex functions on U. Suppose that $\mathbf{x}^*$ maximizes f on the constraint set $C_{\mathbf{b}}$ as defined in (20). Suppose further that one of the constraint qualifications in Theorem 19.12 holds. Then, there exists $\lambda^* \geq \mathbf{0}$ such that $(\mathbf{x}^*, \lambda^*)$ is a saddle point of the Lagrangian (21).

Proof First, work with the case where U is an *open* subset of $\mathbf{R}^n$, for example, $\mathbf{R}^n$ itself. By the usual first order condition, there exists $\lambda^* \geq \mathbf{0}$ such that $\lambda_i^* \cdot (g_i(\mathbf{x}^*) - b_i) = 0$ for $i = 1, \ldots, k$ and

$$D_{\mathbf{x}}L(\mathbf{x}^*, \lambda^*) = Df(\mathbf{x}^*) - \sum_i \lambda_i^* Dg_i(\mathbf{x}^*) = \mathbf{0}. \tag{31}$$

By Theorem 21.8, the function $\mathbf{x} \mapsto L(\mathbf{x}, \lambda^*)$ is a concave function of $\mathbf{x}$. By the first derivative criterion for concavity in Theorem 21.3 and by (31), for any $\mathbf{x} \in C_{\mathbf{b}}$,

$$L(\mathbf{x}, \lambda^*) - L(\mathbf{x}^*, \lambda^*) \le D_{\mathbf{x}}L(\mathbf{x}^*, \lambda^*)(\mathbf{x} - \mathbf{x}^*) = \mathbf{0}. \tag{32}$$

On the other hand, for any $\lambda \ge \mathbf{0}$ in $\mathbf{R}^k$,

$$\begin{aligned} L(\mathbf{x}^*, \lambda^*) &= f(\mathbf{x}^*) - \sum_i \lambda_i^*(g_i(\mathbf{x}^*) - b_i) \\ &= f(\mathbf{x}^*) \qquad (\text{since each } \lambda_i^*(g_i(\mathbf{x}^*) - b_i) = 0) \\ &\le f(\mathbf{x}^*) - \sum_i \lambda_i (g_i(\mathbf{x}^*) - b_i) \\ &= L(\mathbf{x}^*, \lambda). \end{aligned}$$

Now, suppose that $U = \mathbf{R}^n_+$, so that we are looking for a *nonnegative* saddle point. The function (21) is now the Kuhn-Tucker Lagrangian of problem (20), as discussed in Section 18.6. By the first order conditions for a constrained max at $\mathbf{x}^*$ in Section 18.6, there is a $\lambda^* \ge \mathbf{0}$ such that

$$\frac{\partial L}{\partial x_i}(\mathbf{x}^*, \lambda^*) \le 0, \quad x_i^* \ge 0, \quad \text{and} \quad x_i^* \frac{\partial L}{\partial x_i}(\mathbf{x}^*, \lambda^*) = 0. \tag{33}$$

Conditions (33) replace the above equation (31). Now,

$$D_{\mathbf{x}}L(\mathbf{x}^*, \lambda^*)(\mathbf{x} - \mathbf{x}^*) = \sum_i \frac{\partial L}{\partial x_i}(\mathbf{x}^*, \lambda^*)x_i - \frac{\partial L}{\partial x_i}(\mathbf{x}^*, \lambda^*)x_i^* \le 0$$

for $x_i \ge 0$, by (33). The rest of the above proof goes through after one replaces "$= \mathbf{0}$" at the end of (32) by "$\le \mathbf{0}$." ■

To give some indication of the interest in the saddle point approach in economics, we return to the *activity analysis* models of a firm's behavior. In these models, a firm has n production processes; $x_i \ge 0$ represents the level of activity of process i, for $i = 1, \ldots, n$. For each *activity vector* $\mathbf{x} = (x_1, \ldots, x_n)$, $f(\mathbf{x})$ denotes the firm's profit when it runs process i at level x_i and $g_j(\mathbf{x})$ denotes the amount of resource j required at activity level $\mathbf{x}$. Let b_j denote the amount of resource currently available. The firm's optimization problem is to choose $\mathbf{x}$ to maximize $f(\mathbf{x})$ subject to $g_j(\mathbf{x}) \le b_j$ for $j = 1, \ldots, k$ and $\mathbf{x} \ge \mathbf{0}$.

If we let U be the positive orthant of $\mathbf{R}^n$, the (Kuhn-Tucker) Lagrangian for this problem is

$$L(\mathbf{x}, \lambda) = f(\mathbf{x}) + \sum_{j=1}^{k} \lambda_j(b_j - g_j(\mathbf{x})). \tag{34}$$

By the discussion in Section 19.1, the multiplier λ_j can be considered as the shadow price or internal valuation of factor j. Thus, the Lagrangian function (34) can be considered the combined value of the firm's output $f(\mathbf{x})$ and the unused balance of its resources $\sum_j \lambda_j(b_j - g_j(\mathbf{x}))$. The existence of a saddle point $(\mathbf{x}^*, \lambda^*)$ expresses an equilibrium between the value of the output and the value of these unused resources. It is a basic step in the equilibrium theory for production economies. It is especially important in studying firms that deal with activities like investment, fisheries, or timber, in which decisions must be made about whether to use resources or to let them continue to grow at their natural rate.

EXERCISES

21.26 Use the problem of maximizing Cobb-Douglas utility $U(x, y) = xy$ on the budget set $2x + 2y \le 8$ to show that one cannot replace the hypothesis of the concavity of f in Theorem 21.25 by the weaker hypothesis that f is quasiconcave or pseudoconcave.

21.27 Suppose that $(\mathbf{x}, \mathbf{a}) \mapsto f(\mathbf{x}, \mathbf{a})$ is a concave function of $\mathbf{x} \in \mathbf{R}^n$ and of the parameter $\mathbf{a} \in \mathbf{R}^m$, and that $(\mathbf{x}, \mathbf{a}) \mapsto g_i(\mathbf{x}, \mathbf{a})$ are convex functions of $\mathbf{x} \in \mathbf{R}^n$ and $\mathbf{a} \in \mathbf{R}^m$ for $i = 1, \ldots, k$. Let $C_{\mathbf{a}} = \{\mathbf{x} \in \mathbf{R}^n : g_i(\mathbf{x}, \mathbf{a}) \le 0,\ i = 1, \ldots, k\}$. Let $Z(\mathbf{a})$ denote the set of maximizers of $f(\cdot, \mathbf{a})$ on $C_{\mathbf{a}}$; and let $V(\mathbf{a}) = f(Z(\mathbf{a}), \mathbf{a})$. Show that V is a concave function of $\mathbf{a}$.

21.28 In the previous exercise, drop the dependence of g_i on $\mathbf{a}$ and the convexity hypothesis on the g_i's. Suppose only that each $\mathbf{a} \mapsto f(\mathbf{x}, \mathbf{a})$ is a *convex* function of $\mathbf{a}$. Show that the maximum value function $\mathbf{a} \mapsto V(\mathbf{a})$ is a convex function of $\mathbf{a}$.

21.6 APPENDIX

This section presents proofs that were omitted in the earlier sections.

Proof of the Sufficiency Test of Theorem 21.14

Theorem 21.14 Suppose that F is a C^1 function on an open convex subset U of $\mathbf{R}^n$. If for all $\mathbf{x}, \mathbf{y} \in U$,

$$F(\mathbf{y}) \ge F(\mathbf{x}) \quad \text{implies that} \quad DF(\mathbf{x})(\mathbf{y} - \mathbf{x}) \ge 0, \tag{35}$$

then F is quasiconcave on U.

Proof Choose $\mathbf{x}_0$ and $\mathbf{x}_1$ in U with $\mathbf{x}_0 \ne \mathbf{x}_1$ and $F(\mathbf{x}_1) \ge F(\mathbf{x}_0)$. Let $\mathbf{x}_t = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0)$ parameterize the line from $\mathbf{x}_0$ to $\mathbf{x}_1$. We want to prove that $F(\mathbf{x}_t) \ge F(\mathbf{x}_0)$ for all $t \in [0, 1]$.

To reach a contradiction, suppose that there is a $t^* \in (0, 1)$ such that $F(\mathbf{x}_1) \ge F(\mathbf{x}_0) > F(\mathbf{x}_{t^*})$. Let $J = [t_1, t_2]$ be a (connected) interval in $(0, 1)$ with

$t^* \in J$, with $F(\mathbf{x}_0) \geq F(\mathbf{x}_t)$ for all $t \in J$, and $F(\mathbf{x}_{t_1}) = F(\mathbf{x}_{t_2}) = F(\mathbf{x}_0)$. We first claim that

$$DF(\mathbf{x}_t)(\mathbf{x}_1 - x_0) = 0 \quad \text{for all} \quad t \in J, \tag{36}$$

because if $t \in J$, $F(\mathbf{x}_1) \geq F(\mathbf{x}_0) \geq F(\mathbf{x}_t)$. By (35),

$$DF(\mathbf{x}_t)(\mathbf{x}_0 - \mathbf{x}_t) \geq 0 \quad \text{and} \quad DF(\mathbf{x}_t)(\mathbf{x}_1 - \mathbf{x}_t) \geq 0. \tag{37}$$

By definition,

$$\mathbf{x}_0 - \mathbf{x}_t = -t(\mathbf{x}_1 - \mathbf{x}_0) \quad \text{and} \quad \mathbf{x}_1 - \mathbf{x}_t = (1-t)(\mathbf{x}_1 - \mathbf{x}_0).$$

Plugging these equalities into (37) yields

$$-tDF(\mathbf{x}_t)(\mathbf{x}_1 - \mathbf{x}_0) \geq 0 \quad \text{and} \quad (1-t)DF(\mathbf{x}_t)(\mathbf{x}_1 - \mathbf{x}_0) \geq 0.$$

Since t and $1-t$ are positive, $DF(\mathbf{x}_t)(\mathbf{x}_1 - \mathbf{x}_0) = 0$; this proves claim (36).

On the other hand,

$$\begin{aligned} 0 &< F(\mathbf{x}_0) - F(\mathbf{x}_{t^*}) \\ &= F(\mathbf{x}_{t_1}) - F(\mathbf{x}_{t^*}) \\ &= DF(\mathbf{x}_{t_3})(\mathbf{x}_{t_1} - \mathbf{x}_{t^*}) \quad \text{for some } t_3 \in (t_1, t^*) \\ &\qquad \text{(by the Mean Value Theorem of Section 30.1 and the Chain Rule)} \\ &= (t^* - t_1)DF(\mathbf{x}_{t_3})(\mathbf{x}_1 - \mathbf{x}_0), \end{aligned}$$

since $\mathbf{x}_{t_1} - \mathbf{x}_{t^*} = (t_1 - t^*)(\mathbf{x}_1 - \mathbf{x}_0)$. This contradiction to claim (36) implies that there is no t^* with $F(\mathbf{x}_{t^*}) < F(\mathbf{x}_0)$. Since $F(\mathbf{x}_t) \geq F(\mathbf{x}_0)$ for all $t \in [0, 1]$, F is quasiconcave by Theorem 21.12. ■

Proof of Theorem 21.15

Theorem 21.15 Suppose that F is a real-valued, positive function defined on a convex cone C in $\mathbf{R}^n$. If F is homogeneous of degree one and quasiconcave on C, it is concave on C.

Proof We will show that the subgraph of F, that is, the set below the graph of F in $\mathbf{R}^{n+1}$,

$$G_F \equiv \{(\mathbf{x}, y) \in C \times \mathbf{R}_+ : y \leq F(\mathbf{x})\},$$

is a convex set. (See Exercise 21.5.) We first show that

$$G_F^+ \equiv \{(\mathbf{x}, y) \in G_F : 0 < y \leq F(\mathbf{x})\}$$

is a convex set. Let $(\mathbf{x}, y)$ and $(\mathbf{x}', y')$ be points in G_F^+, so that $0 < y \leq F(\mathbf{x})$ and $0 < y' \leq F(\mathbf{x}')$. Since F is homogeneous of degree one, $y > 0$, and $(\mathbf{x}, y) \in G_F^+$,

$$F\left(\frac{\mathbf{x}}{y}\right) = \frac{1}{y} \cdot F(\mathbf{x}) \geq \frac{1}{y} \cdot y = 1.$$

Similarly, $F\left(\frac{\mathbf{x}'}{y'}\right) \geq 1$. Therefore,

$$\left(\frac{\mathbf{x}}{y}, 1\right) \quad \text{and} \quad \left(\frac{\mathbf{x}'}{y'}, 1\right) \quad \text{are in} \quad G_F^+.$$

Let $\lambda \in [0, 1]$, and define

$$\theta \equiv \frac{\lambda y}{\lambda y + (1-\lambda)y'}.$$

Then, θ is also in $[0, 1]$. Since F is quasiconcave,

$$F\left(\theta\left(\frac{\mathbf{x}}{y}\right) + (1-\theta)\left(\frac{\mathbf{x}'}{y'}\right)\right) \geq 1,$$

that is,

$$\left(\theta\left(\frac{\mathbf{x}}{y}\right) + (1-\theta)\left(\frac{\mathbf{x}'}{y}\right), 1\right) \quad \text{is in} \quad G_F^+.$$

By the definition of θ,

$$\left(\theta\left(\frac{\mathbf{x}}{y}\right) + (1-\theta)\left(\frac{\mathbf{x}'}{y}\right), 1\right) = \left(\frac{\lambda\mathbf{x} + (1-\lambda)\mathbf{x}'}{\lambda y + (1-\lambda)y'}, 1\right).$$

Once again by the homogeneity of F,

$$(\lambda\mathbf{x} + (1-\lambda)\mathbf{x}', \lambda y + (1-\lambda)y') \quad \text{is in} \quad G_F^+;$$

that is, $\lambda(\mathbf{x}, y) + (1-\lambda)(\mathbf{x}', y')$ is in G_F^+, and G_F^+ is a convex set.

To see that G_F is convex, let $\gamma = \{(\mathbf{x}_t, y_t) : 0 \leq t \leq 1\}$ be a line segment with its endpoints in G_F. If the endpoints of γ lie in G_F^+, then γ lies in G_F^+ by the argument of the preceding paragraph. If both endpoints of γ lie below $y = 0$, then so does the segment; in this case, the segment γ lies below the graph of F because the graph of the positive function F lies above the $\{y = 0\}$-hyperplane. If one endpoint of γ lies above $\{y = 0\}$ and one below, break γ into two segments, one below $\{y = 0\}$ and one above $\{y = 0\}$, and apply the above arguments to each piece separately. ■

Proof of Theorem 21.17

Theorem 21.17 Let U be a convex subset of $\mathbf{R}^n$. Let $F: U \to \mathbf{R}$ be a C^1 function. Then,

(*a*) if F is pseudoconcave on U, F is quasiconcave on U, and
(*b*) if U is open and if $\nabla F(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in U$, then F is pseudoconcave on U if and only if F is quasiconcave on U.

Proof

(*a*) Suppose that F is pseudoconcave on U. Let $\mathbf{y}_0$ and $\mathbf{y}_1$ be two points in U with $F(\mathbf{y}_1) \geq F(\mathbf{y}_0)$. Let $\mathbf{y}_t \equiv \mathbf{y}_0 + t(\mathbf{y}_1 - \mathbf{y}_0)$ for $0 \leq t \leq 1$, so that the line segment is parameterized from $\mathbf{y}_0$ to $\mathbf{y}_1$. Let $g(t) = F(\mathbf{y}_t)$.

We claim that $F(\mathbf{y}_t) \geq F(\mathbf{y}_0)$ for all $t \in [0, 1]$. This claim holds automatically if the minimum value of g on $[0, 1]$ occurs at $t = 0$ or $t = 1$. We can assume then that the minimum value of g on $[0, 1]$ occurs at some t^* in the open interval $(0, 1)$. In this case,

$$0 = g'(t^*) = DF(\mathbf{y}_0 + t^*(\mathbf{y}_1 - \mathbf{y}_0)) \cdot (\mathbf{y}_1 - \mathbf{y}_0)$$

by the usual first order condition for a minimum and the Chain Rule; then,

$$0 = DF(\mathbf{y}_0 + t^*(\mathbf{y}_1 - \mathbf{y}_0)) \cdot (-t^*(\mathbf{y}_1 - \mathbf{y}_0)).$$

Applying the definition (15) of pseudoconcavity with $\mathbf{x}^* = \mathbf{y}_0 + t^*(\mathbf{y}_1 - \mathbf{y}_0)$ and $\mathbf{y} = \mathbf{y}_0$, we conclude that

$$F(\mathbf{y}_0 + t^*(\mathbf{y}_1 - \mathbf{y}_0)) \geq F(\mathbf{y}_0);$$

this proves the claim at the beginning of this paragraph. By Theorem 21.12, F is quasiconcave on U.

(*b*) Suppose, now, that F is quasiconcave on U, that U is open, and $\nabla F(\mathbf{x})$ is never zero for $\mathbf{x} \in U$. To prove that F is pseudoconcave, we assume that $DF(\mathbf{x}^*)(\mathbf{y} - \mathbf{x}^*) \leq 0$, as in the hypothesis of (15), and prove that $F(\mathbf{y}) \leq F(\mathbf{x}^*)$. If $DF(\mathbf{x}^*)(\mathbf{y} - \mathbf{x}^*) < 0$, then $F(\mathbf{y}) < F(\mathbf{x}^*)$ by (16). We need only rule out the case

$$DF(\mathbf{x}^*)(\mathbf{y} - \mathbf{x}^*) = 0 \quad \text{and} \quad F(\mathbf{y}) > F(\mathbf{x}^*). \tag{38}$$

We will show that, under the hypotheses of this theorem and under assumption (38), we can perturb $\mathbf{y}$ to $\mathbf{y}'$ so that

$$DF(\mathbf{x}^*)(\mathbf{y}' - \mathbf{x}^*) < 0 \quad \text{and} \quad F(\mathbf{y}') > F(\mathbf{x}^*); \tag{39}$$

this contradicts our assumption that F is quasiconcave. Let $\mathbf{v}$ be the nonzero vector $-\nabla F(\mathbf{x}^*)$. For all $t > 0$,

$$\begin{aligned} DF(\mathbf{x}^*)(\mathbf{y} + t\mathbf{v} - \mathbf{x}^*) &= DF(\mathbf{x}^*)(t\mathbf{v} + \mathbf{y} - \mathbf{x}^*) \\ &= tDF(\mathbf{x}^*)(\mathbf{v}) + DF(\mathbf{x}^*)(\mathbf{y} - \mathbf{x}^*) \\ &= -t\|\nabla F(\mathbf{x}^*)\|^2 + 0 \\ &< 0. \end{aligned}$$

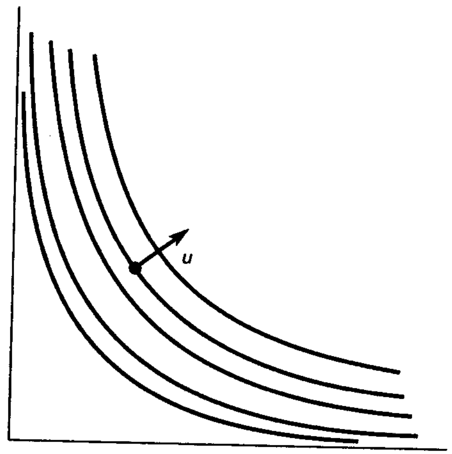
Since F is continuous at $\mathbf{y}$, we can choose nonzero t small enough so that

$$F(\mathbf{y} + t\mathbf{v}) > F(\mathbf{x}^*) \quad \text{and} \quad DF(\mathbf{x}^*)(\mathbf{y} + t\mathbf{v} - \mathbf{x}^*) < 0;$$

that is, $\mathbf{y}' = \mathbf{y} + t\mathbf{v}$ satisfies (39)—a contradiction to the characterization (16) of quasiconcavity. This contradiction proves that (38) cannot hold and therefore that F is pseudoconcave. ■

Proof of Theorem 21.20

We now prove the bordered Hessian test for pseudoconcavity and quasiconcavity for functions of two variables, a special case of Theorem 21.19. For simplicity, we will work with C^2 utility functions U in the plane which are quasiconcave and monotone. The former means that indifference curves bound convex sets from below; the latter means that utility is strictly increasing as the amount of either good increases. In fact, we will write this monotonicity assumption as $U'_x > 0$ and $U'_y > 0$. Quasiconcavity and monotonicity imply that the level sets of U are as in Figure 21.10.



Indifference curves of a monotone, quasiconcave utility function. **Figure 21.10**

Theorem 21.20 Let U be a C^2 function on a convex set W in $\mathbf{R}^2$. Suppose that U is monotone in that $U'_x > 0$ and $U'_y > 0$ on W. If the determinant

$$\begin{vmatrix} 0 & U'_x & U'_y \\ U'_x & U''_{xx} & U''_{xy} \\ U'_y & U''_{xy} & U''_{yy} \end{vmatrix}. \tag{40}$$

is > 0 on W, then U is quasiconcave on W. Conversely, if U is quasiconcave on W, then the determinant (40) is ≥ 0.

Proof Think of each level curve as a graph of a function $y = g(x)$. (We can do this by the monotonicity assumption $U'_y > 0$. See Exercise 21.29.) Since U is quasiconcave, the set above the graph of g (i.e., above the level set of U) is a convex set. Therefore, the quasiconcavity of U implies the convexity of g as a function of one variable (Exercise 21.5), which in turn implies that $g''(x) \geq 0$. Now, by Theorem 15.1, $g'(x)$ is the marginal rate of substitution $-U'_x(x, g(x))/U'_y(x, g(x))$. Therefore,

$$\begin{aligned} 0 \leq g''(x) &= \frac{d}{dx}\left(-\frac{U'_x(x, g(x))}{U'_y(x, g(x))}\right) \\ &= -\frac{(U''_{xx} + U''_{xy}g'(x))U'_y - (U''_{yx} + U''_{yy}g'(x))U'_x}{(U'_y)^2} \\ &= \frac{-(U'_y)^2 U''_{xx} - (U'_x)^2 U''_{yy} + 2U'_x U'_y U''_{xy}}{(U'_y)^3} \end{aligned} \tag{41}$$

using $g'(x) = -U'_x/U'_y$. The numerator in expression (41) is simply the determinant (40).

Conversely, one can follow the above steps in reverse order to conclude that if the determinant (40) is positive, then $g''(x) > 0$, g is a convex function, the set above the graph of g is a convex set, and finally that the set above each level curve of U is a convex set. So, U is quasiconcave. ■

EXERCISES

21.29 Why does the assumption $\partial U/\partial y > 0$ imply that we can work with each level curve of $U(x, y)$ as a graph of a function $y = g(x)$?

NOTES

(*a*) One of the earliest papers on quasiconcave functions is K. Arrow and A. Enthoven, "Quasiconcave programming," *Econometrica* 29 (1961) 779–800. It includes a concrete example of a quasiconcave function which is *not* a monotonic transformation of any concave function.

(*b*) One of the earliest papers on pseudoconcave functions is O. Mangasarian, "Pseudoconvex functions," *Society for Industrial and Applied Mathematics Journal on Control* 3 (1965), 281–290. Mangasarian defined pseudoconcave functions as a class that retains some of the more interesting properties of quasiconcave functions and of concave functions.

(*c*) Papers on the relationship between concave, pseudoconcave and quasiconcave functions include Arrow and Enthoven, "Quasiconcave programming," J. Ferland, "Mathematical programming problems with quasiconvex objective functions," *Mathematical Programming* 3 (1972) 296–301; and J-P. Crouzeix and J. Ferland, "Criteria for quasiconvexity and pseudoconvexity: relationships and comparisons," *Mathematical Programming* 23 (1982) 193–205. These papers also give necessary conditions for a function to be pseudoconcave.