New York University
Department of Economics

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Homogeneous Functions

For any $\alpha \in \mathbb{R}$, a function $f : \mathbb{R}^n_{++} \to \mathbb{R}$ is *homogeneous of degree* α if $f(\lambda x) = \lambda^\alpha f(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^n_{++}$. A function is *homogeneous* if it is homogeneous of degree α for some $\alpha \in \mathbb{R}$.

A function f is *linearly homogenous* if it is homogeneous of degree 1.

- Along any ray from the origin, a homogeneous function defines a power function. If f is linearly homogeneous, then the function defined along any ray from the origin is a linear function.

Example: Consider a Cobb-Douglas production, $f(x) = \prod_{j=1}^n x_j^{\alpha_j}$, where each $\alpha_j > 0$, and let $\beta = \sum_{j=1}^n \alpha_j$. Then

$$f(\lambda x) = \prod_{j=1}^n (\lambda x_j)^{\alpha_j} = \lambda^\beta \prod_{j=1}^n (x_j)^{\alpha_j} = \lambda^\beta f(x).$$

So f is homogeneous of degree β.

Example: Consider a CES production function $f(x) = \left(\sum_{i=1}^n \alpha_i x_i^\rho\right)^\beta$, where $\beta, \rho > 0$. Then

$$f(\lambda x) = \left(\sum_{i=1}^n \alpha_i (\lambda x_i)^\rho\right)^\beta = \lambda^{\rho\beta} \left(\sum_{i=1}^n \alpha_i x_i^\rho\right)^\beta = \lambda^{\rho\beta} f(x).$$

So f is homogeneous of degree $\rho\beta$.

- If f is homogeneous of degree 0, then $f(\lambda x) = f(x)$. Why?
- If f is homogeneous of degree $\alpha \neq 0$, then $f^{\frac{1}{\alpha}}$ is homogenenous of degree 1. Why?
- Let $f(x) = \left(\prod_{j=1}^n x_j^{\alpha_j}\right)^{\frac{1}{\beta}}$, where each $\alpha_j > 0$ and $\beta = \sum_{j=1}^n \alpha_j$. Then f is linearly homogeneous. Why?
- $f(x) = \min\{x_i : i = 1, ..., n\}$ is linearly homogeneous. Why?

The following theorem relates the value of a homogeneous function to its derivative.

Theorem 1: If $f : \mathbb{R}^n_{++} \to \mathbb{R}$ is continuously differentiable and homogeneous of degree α, then

$$Df(x) \cdot x = \sum_{i=1}^n f_i(x) x_i = \alpha f(x). \qquad \text{(Euler's theorem)}$$

Proof. If f is homogeneous of degree α, then for any $x \in \mathbb{R}^n_{++}$ and any $\lambda > 0$, we have

$$f(\lambda x) = \lambda^\alpha f(x).$$

Then holding x fixed and differentiating both sides with repect to λ, we obtain

$$\begin{aligned}\frac{df(\lambda x)}{d\lambda} &= Df(\lambda x)\cdot x = \sum_{i=1}^{n} f_i(\lambda x)x_i \\ &= \frac{d\left(\lambda^{\alpha} f(x)\right)}{d\lambda} = \alpha\lambda^{\alpha-1}f(x)\end{aligned}$$

Letting $\lambda = 1$, yields the statement to be proved. ■

The next theorem relates the homogeneity of a function to the homogeneity of its partial derivatives.

Theorem 2: If $f : \mathbb{R}^n_{++} \to \mathbb{R}$ is continuously differentiable and homogeneous of degree α, then each partial derivative f_i is homogeneous of degree $\alpha - 1$.

Proof. For fixed $x \in \mathbb{R}^n_{++}$ and $\lambda > 0$, define each $g_i, h_i : (-x_i, \infty) \to \mathbb{R}$ by $g_i(t) = f(\lambda(x + e_i t))$ and $h_i(t) = \lambda^{\alpha} f(x + e_i t)$ Then the homogeneity of f implies

$$g_i(t) = f(\lambda(x + te_i)) = \lambda^{\alpha} f(x + te_i) = h_i(t)$$

and therefore

$$g_i'(t) = h_i'(t) \text{ for all } t \in (-x_i, \infty)$$

But

$$\begin{aligned}g_i'(0) &= Df(\lambda x)\cdot\lambda e_i = \lambda f_i(\lambda x) \\ h_i'(0) &= \lambda^{\alpha} Df(x)\cdot e_i = \lambda^{\alpha} f_i(x)\end{aligned}$$

So

$$f_i(\lambda x) = \lambda^{\alpha-1} f_i(x).$$

■

Example: In the example above, we showed that $f(x) = \prod_{i=1}^{n} x_i^{\alpha_i}$ is homogeneous of degree $\beta = \sum_{i=1}^{n} \alpha_i$. To verify Euler's theorem, observe that, for each $j = 1, ..., n$, we have

$$f_j(x) = \alpha_j x_j^{\alpha_j - 1} \prod_{i \neq j} x_i^{\alpha_i} = \frac{\alpha_j}{x_j} \prod_{i=1}^{n} x_i^{\alpha_i} = \alpha_j \frac{f(x)}{x_j}.$$

Therefore,

$$\sum_{i=1}^{n} f_i(x)x_i = \sum_{i=1}^{n} \alpha_i \frac{f(x)}{x_i} x_i = \sum_{i=1}^{n} \alpha_i f(x) = \beta f(x).$$

To verify Theorem 2, observe that each

$$\begin{aligned}f_i(\lambda x) &= \alpha_i \frac{f(\lambda x)}{\lambda x_i} = \alpha_i \frac{\lambda^{\beta} f(x)}{\lambda x_i} \text{ (since } f \text{ is homogeneous of degree } \beta) \\ &= \lambda^{\beta-1}\alpha_i \frac{f(x)}{x_i} = \lambda^{\beta-1} f_i(x).\end{aligned}$$

Homothetic Functions

A function is *homothetic* if it is a monotonic transformation of a linearly homogeneous function.

- If f is a homogeneous function of degree $\alpha \neq 0$, then f is homothetic. Why?
- Any monotonic transformation of a homothetic function is homothetic. Why?

Example: Let $f : \mathbb{R}^n_{++} \to \mathbb{R}$ be the log-linear function defined by $f(x) = \sum_{i=1}^n \alpha_i \log x_i$. Then f is homothetic since $\exp(f(x)) = \prod_{i=1}^n x_i^{\alpha_i}$ is homogenous of degree $\beta \equiv \sum_{i=1}^n \alpha_i$.

However f is not homogeneous, since for $\lambda < 1 < \mu$, and $e \equiv (1, ..., 1) \in \mathbb{R}^n$, we have

$$f(\lambda e) = \sum_{i=1}^n \alpha_i \log \lambda < 0 < \sum_{i=1}^n \alpha_i \log \mu = f(\mu e)$$

Theorem 3: Suppose $f : \mathbb{R}^n_{++} \to \mathbb{R}$ is homothetic. Then $f(x) = f(z)$ implies $f(\lambda x) = f(\lambda z)$ for all $\lambda > 0$.

Proof. If f is homothetic, then $f = \phi \circ g$ for some increasing $\phi : g(\mathbb{R}^n_{++}) \to \mathbb{R}$ and some linearly homogeneous function $g : \mathbb{R}^n_{++} \to \mathbb{R}$. Therefore $f(x) = f(z)$ if and only if $g(x) = g(z)$. So if $f(x) = f(z)$, then the linear homogeneity of g then implies

$$g(\lambda x) = \lambda g(x) = \lambda g(z) = g(\lambda z)$$

and therefore

$$f(\lambda x) = \phi(g(\lambda x)) = \phi(g(\lambda z)) = f(\lambda z).$$

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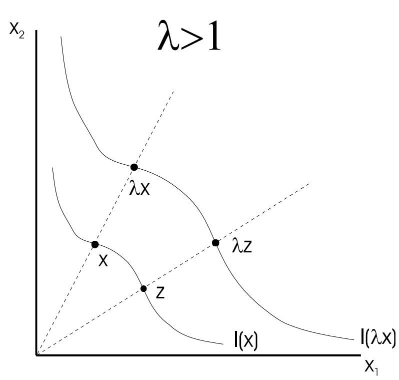
- The level curves curves of a homothetic function are radial translations of each other. Why?



We may use Theorem 2 to show that the gradient of a homothetic function is proportional along any ray from the orgin.

Theorem 4: Suppose $f : \mathbb{R}^n_{++} \to \mathbb{R}$ is homothetic and continuously differentiable. Then for any $x \in \mathbb{R}_{++}$ and $\lambda > 0$, there is a $k > 0$ such that $\nabla f(x) = k \nabla f(\lambda x)$.

Proof. If f is homothetic, then $f = \phi \circ g$ for some increasing $\phi : g(\mathbb{R}^n_{++}) \to \mathbb{R}$ and some linearly homogeneous function $g : \mathbb{R}^n_{++} \to \mathbb{R}$. So, using the chain rule, we have for any $x \in \mathbb{R}^n_{++}$ and $\lambda > 0$,

$$Df(x) = \phi'(g(x))Dg(x)$$

and

$$Df(\lambda x) = \phi'(g(\lambda x))Dg(\lambda x).$$

But if g is homogeneous of degree 1, Theorem 2 implies that each g_i is homogeneous of degree 0 so that we have that $Dg(\lambda x) = Dg(x)$. Therefore,

$$Df(\lambda x) = \phi'(g(\lambda x))Dg(\lambda x) = \phi'(g(\lambda x))Dg(x) = \left(\frac{\phi'(g(\lambda x))}{\phi'(g(x))}\right)\phi'(g(x))Dg(x) = \left(\frac{\phi'(g(\lambda x))}{\phi'(g(z))}\right)Df(x)$$

Then since $\nabla f(x) \equiv Df(x)^T$, letting $k = \frac{\phi'(g(\lambda x))}{\phi'(g(z))}$ proves the theorem. ■

Theorem 4 implies that the slopes of the indifference curves of a homothetic function are parallel along any ray from the origin.

Corollary 1: Suppose $u : \mathbb{R}^n_{++} \to \mathbb{R}$ is a continuously differentiable homothetic utility function. Then for any $x \in \mathbb{R}^2_{++}$ and $\lambda > 0$, we have $MRS_{12}(x) = MRS_{12}(\lambda x)$.

Proof. If u is homothetic, then Theorem 4 implies that $\nabla u(\lambda x) = k\nabla u(x)$. Therefore,

$$MRS_{12}(\lambda x) = \frac{u_1(\lambda x)}{u_2(\lambda x)} = \frac{ku_1(x)}{ku_2(x)} = \frac{u_1(x)}{u_2(x)} = MRS_{12}(x).$$

■

Concavity and Homogeneity

Recall that for any $x \in X$, $P(x) \equiv \{z \in X : f(z) \geq f(x)\}$ is called the *better set* of x. In a previous handout, we established the following property of quasi-concave functions.

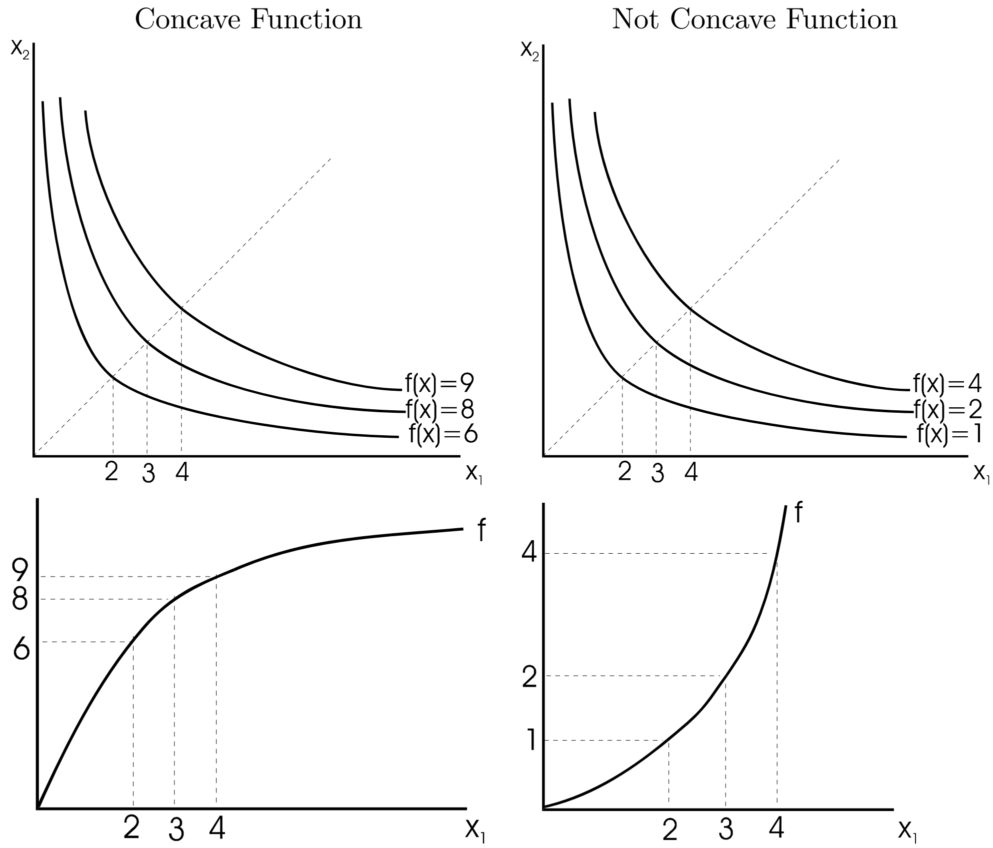
Theorem 04: A function $f : X \to \mathbb{R}$ is quasi-concave if and only if $P(x)$ is a convex set for each $x \in X$.

For an increasing function of two variables, Theorem 04 implies that level sets are concave to the origin. However, it imposes no other additional restrictions on the relation between different level sets. In contrast, concavity requires that an increasing function f becomes less steep with increases in x so that the distance between the level sets of fixed increments in the value of f increases with x. The next proposition, which is proved in the Appendix, provides sufficient conditions for quasi-concave homogeneous functions to satisfy the additional requirement for concavity.

Theorem 5: (a) Suppose $f : \mathbb{R}^n_{++} \to \mathbb{R}$ is quasi-concave and homogeneous of degree $\alpha \in (0, 1]$. Then f is concave.

(b) If f is strictly quasi-concave and homogeneous of degree $\alpha \in (0, 1)$, then f is strictly concave.

The following two figures illustrate the theorem. The figure on the left illustrates level curves of a concave function. Notice that as we increase the vector x along a fixed ray from the origin, the change in the value of the function progressively decreases. The level curves on the right cannot represent a concave function, since as we increase x along a fixed ray from the origin, the change in the value of the function progressively increases.



- If f is homogeneous of degree $\alpha > 1$, then f cannot be a concave function. Why? (Hint: Consider the function $g(\mu) = f(\mu x)$).

We may use Theorem 4 to establish the concavity of an important class of CES functions. The proof will use the following property:

Lemma 1: Suppose $0 < \rho < 1$. Then for any $x, z \in \mathbb{R}$ with $x \neq z$, we have

$$\lambda z^\rho + (1-\lambda)\, x^\rho < (\lambda z + (1-\lambda)\, x)^\rho\,.$$

Proof. Define $h : \mathbb{R}_{++} \to \mathbb{R}$ by $h(t) = t^\rho$. Then, if $\rho < 1$, we have $h''(t) = \rho\,(\rho - 1)\, t^{\rho-2} < 0$, which implies that h is strictly concave and therefore for $x \neq z$, we have

$$(\lambda z + (1-\lambda)\, x)^\rho = h(\lambda z + (1-\lambda)\, x) > \lambda h(z) + (1-\lambda)\, h(x) = \lambda z^\rho + (1-\lambda)\, x^\rho.$$

■

We will also require the following propery, which we established in an earlier handout.

Theorem 05: (a) Any monotonic transformation of a concave function is concave. (b) Any monotonic transformation of a strictly concave function is strictly quasi-concave.

Observation 1: Let $f(x) = \left(\sum \alpha_i x_i^\rho\right)^\beta$ where $\rho, \beta > 0$.

(a) If $\rho = 1$, then f is quasi-concave. If $\rho < 1$, then f is strictly quasi-concave.

(b) If $\rho \leq 1$ and $\beta\rho \leq 1$, then f is concave. If $\rho < 1$ and $\rho\beta < 1$, then f is strictly concave.

Proof. (a) Suppose that $\rho < 1$ and define $g : \mathbb{R}^n_{++} \to \mathbb{R}$ by

$$g(x) \equiv \sum_{i=1}^{n} \alpha_i x_i^\rho$$

Then, for any $x, z \in \mathbb{R}^n_{++}$, we have

$$\begin{aligned}
\lambda g(z) + (1-\lambda)\, g(x) &= \lambda \sum_{i=1}^{n} \alpha_i z_i^\rho + (1-\lambda) \sum_{i=1}^{n} \alpha_i x_i^\rho = \sum_{i=1}^{n} \alpha_i \left(\lambda z_i^\rho + (1-\lambda)\, x_i^\rho\right) \\
&< \sum_{i=1}^{n} \alpha_i (\lambda z_i + (1-\lambda)\, x_i)^\rho \quad \text{(from Lemma 1)} \\
&= g(\lambda z + (1-\lambda)\, x)
\end{aligned}$$

which implies that g is strictly concave. But since f is a monotone increasing transformation of g, it follows from Theorem 05 that f is strictly quasi-concave.

If $\rho = 1$, the inequalilty above is weak, which implies only that g is concave and therefore only that f is quasi-concave.

(b) Observe that

$$f(\lambda x) = \left(\sum_{i=1}^{n} \alpha_i (\lambda x_i)^\rho\right)^\beta = \lambda^{\rho\beta} \left(\sum_{i=1}^{n} \alpha_i x_i^\rho\right)^\beta = \lambda^{\rho\beta} f(x).$$

implies that f is homogeneous of degree $\rho\beta$. Therefore, if $\rho \leq 1$ and $\rho\beta \leq 1$, it follows Theorem 4 and part (a) that f is concave. If $\rho < 1$ and $\rho\beta < 1$, then Theorem 4 and part (a) implies that f is strictly concave. ■

Appendix

Theorem 5: (a) Suppose $f : \mathbb{R}^n_{++} \to \mathbb{R}$ is positive, quasi-concave, and homogeneous of degree $\alpha \in (0, 1]$. Then f is concave. (b) If f is positive, strictly quasi-concave and homogeneous of degree $\alpha \in (0, 1)$, then f is strictly concave.

Proof. (a) Given any $x \in \mathbb{R}^n_{++}$, define $h_x : \mathbb{R}_{++} \to \mathbb{R}$ by $h_x(t) = f(tx)$. Then since the homogeneity of f implies that $h_x(t) = f(tx) = t^\alpha f(x)$, it follows from the restriction that $\alpha \in (0, 1]$ that

$$h_x''(t) = \alpha\,(\alpha - 1)\, t^{\alpha-2} f(x) \leq 0$$

and therefore that h_x is concave. Consequently, for any $t > 0$ and $\lambda \in (0, 1)$, we have

$$\lambda f(tx) + (1-\lambda)\, f(x) = \lambda h_x(t) + (1-\lambda)\, h_x(1) \leq h_x(\lambda t + (1-\lambda)) = f((\lambda t + (1-\lambda))\, x) \qquad (1)$$

Now consider any $x^1, x^2 \in \mathbb{R}^n_{++}$ and choose μ_1,$\mu_2 > 0$ so that $\mu_1^\alpha f(x^1) = f(x^2)$ and $\mu_2^\alpha f(x^2) = f(x^1)$. Then

$$f(x^1) = \mu_2^\alpha f(x^2) = \mu_2^\alpha \mu_1^\alpha f(x^1) = (\mu_2\mu_1)^\alpha f(x)$$

which, since $\alpha \neq 0$, implies that $\mu_2\mu_1 = 1$.

Next observe that relation (1) implies

$$\lambda f(x^2) + (1-\lambda) f(x^1) = \lambda f(\mu_1 x^1) + (1-\lambda) f(x^1) \leq f((\lambda\mu_1 + (1-\lambda)) x^1) \tag{2}$$

and

$$\lambda f(x^2) + (1-\lambda) f(x^1) = \lambda f(x^2) + (1-\lambda) f(\mu_2 x^2) \leq f((\lambda + (1-\lambda)\mu_2) x^2) \tag{3}$$

Now define

$$\beta \equiv \frac{\lambda}{\lambda + (1-\lambda)\mu_2}$$

Then, since $\mu_1\mu_2 = 1$, we have

$$1 - \beta = \frac{\lambda + (1-\lambda)\mu_2 - \lambda}{\lambda + (1-\lambda)\mu_2} = \frac{(1-\lambda)\mu_2}{\lambda + (1-\lambda)\mu_2} = \frac{(1-\lambda)}{\lambda\mu_1 + (1-\lambda)}.$$

Therefore,

$$\begin{aligned}
\lambda f(x^2) + (1-\lambda) f(x^1) &\leq \min\left\{f((\lambda\mu_1 + (1-\lambda)) x^1), f((\lambda + (1-\lambda)\mu_2) x^2)\right\} \text{ (from relations (2) and (3)} \\
&\leq f((1-\beta)(\lambda\mu_1 + (1-\lambda)) x^1 + \beta(\lambda + (1-\lambda)\mu_2) x^2)) \text{ (by quasi-concavity)} \\
&= f((1-\lambda) x^1 + \lambda x^2)
\end{aligned}$$

(b) Suppose $\alpha \in (0,1)$ and f is strictly quasi-concave. Consider any $x^1, x^2 \in \mathbb{R}^n_{++}$ where $x^1 \neq x^2$. There are two cases to consider.

Case I: $x^2 = \mu x^1$ for some $\mu > 0$. Then, defining h_x as above, $\alpha \in (0,1)$ implies $h_x''(t) < 0$ so that each h_x is strictly concave and therefore. Therefore, for any $\lambda \in (0,1)$, we have

$$\begin{aligned}
\lambda f(x^2) + (1-\lambda) f(x^1)\lambda f(\mu x^1) + (1-\lambda) f(x^1) &= \lambda h_{x^1}(\mu) + (1-\lambda) h_{x^1}(1) < h_{x^1}(\lambda\mu + (1-\lambda)(1)) \\
&= f((\lambda\mu + (1-\lambda)) x^1) = f(\lambda x^2 + (1-\lambda) x^1)(1)
\end{aligned}$$

Case II: $x^1 \neq \mu x^2$ for any $\mu > 0$. Define μ_1 and μ_2 as above. Then (2) and (3) imply

$$\begin{aligned}
\lambda f(x^2) + (1-\lambda) f(x^1) &\leq \min\left\{f((\lambda\mu_1 + (1-\lambda)) x^1), f(\left(\lambda + (1-\lambda)\mu^2\right)),\right\} \\
&< f((1-\beta)(\lambda\mu_1 + (1-\lambda)) x^1 + \beta(\lambda + (1-\lambda)\mu_2) x^2)) \text{ (by strict quasi-concavity)} \\
&= f((1-\lambda) x^1 + \lambda x^2).
\end{aligned}$$

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