Calculus conditions for concave functions (of a single variable).

- Recall that a real-valued function f is concave if and only if its domain is a convex set $A \subset \Re_n$ and for all x_1 and x_2 in A and for all $\lambda \in [0, 1]$,

$$f\left(\lambda x_1 + (1-\lambda)x_2\right) \geq \lambda f(x_1) + (1-\lambda)f(x_2)$$

- If $A \subset \Re$, this implies that for all x_1 and x_2 in A and for all $\lambda \in [0, 1]$,

$$f(x_2) \leq f(x_1) + (x_2 - x_1)f'(x_1).$$

- Draw some pictures.–See slide for "rooftop theorem"

From rooftops to second derivatives.

- The rooftop theorem tells us that if f is concave, $f(x_2) \leq f(x_1) + (x_2 - x_1)f'(x_1)$ for all x_1 and x_2 in A.
- Rearranging terms, we have

$$f(x_2) - f(x_1) \leq (x_2 - x_1)f'(x_1). \tag{1}$$

- The rooftop theorem also tells us that if f is concave, $f(x_1) \leq f(x_2) + (x_1 - x_2)f'(x_2)$ for all x_1 and x_2 in A.
- Rearranging terms, we have $f(x_1) - f(x_2) \leq (x_1 - x_2)f'(x_2)$.
- Multiply both sides by -1, above implies

$$f(x_2) - f(x_1) \geq (x_2 - x_1)f'(x_2) \tag{2}$$

- Let $x_2 > x_1$. Then Inequalities 1 and 2 imply that $f'(x_2) \leq f'(x_1)$.

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Maxima for differentiable concave functions

- From elementary calculus we know that $f : \Re \rightarrow \Re$ and if $f'(x)$ exists and x is in the interior of its domain, then a necessary condition for x to be a local maximum of f on A is that $f'(x) = 0$.
- We also know that a necessary and sufficient condition for x to be an interior local max is that $f'(x) = 0$ and $f''(x) < 0$ and $f''(x) < 0$.
- Show that if f is a concave function and has a local max at x, then it has a global max at x.
- So we know that if f is a concave function such that $f''(x)$ exists everywhere, the $f'(x) = 0$ is necessary and sufficient for x to be a global maximum on A.
- Is $f'(x) = 0$ and $f''(x) \leq 0$ sufficient for x to be a maximum?

Going to Higher Dimensions

- Where $f : \Re^n_+ \to \Re$, and for all x and $y \in \Re^n_+$, let us define the function

$$g(t) = f\left(x + t(y - x)\right)$$

  for all $t \in [0, 1]$.
- If f is a concave function on $\Re^n_+$, then g is a concave function on the real interval $[0, 1]$.
- So if f is concave and twice differentiable, then $g''(0) \leq 0$.
- Let's find out more about $g''(0)$.

Quadratic Forms Emerge

- Applying the chain rule,

$$g'(t) = \sum_{i=1}^{n} (y_i - x_i) f_i \left(x + t(y - x)\right).$$

- Then

$$g''(t) = \sum_{i=1}^{n} (y_i - x_i) \frac{d}{dt} f_i \left(x + t(y - x)\right).$$

- So

$$g''(0) = \sum_{i=1}^{n} (y_i - x_i) \sum_{j=1}^{n} (y_j - x_j) f_{ij} \left(x + t(y - x)\right).$$

What did we learn?

- If f is a concave function, then it must be that the quadratic form

$$g''(0) = \sum_{i=1}^{n}(y_i - x_i)\sum_{j=1}^{n}(y_j - x_j)f_{ij}\left(x + t(y - x)\right) \leq 0$$

  for all x and y in $\Re^n_+$.
- But this will be true if and only if the quadratic form $x'Mx$ is negative semidefinite, where M is the matrix of second order partial derivatives of f. That is $M_{ij} = f_{ij}$.

An example

- Let
$$f(x_1, x_2) = (x_1 + x_2) - \frac{1}{2}\left(x_1^2 + x_2^2\right) + cx_1x_2.$$
- Then $f_{11}(x_1, x_2) = f_{22}(x_1, x_2) = -1$ and $f_{12}(x_1, x_2) = f_{21}(x_1, x_2) = c$ for all x_1, x_2.
- The matrix
$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} -1 & c \\ c & -1 \end{pmatrix}$$
is negative semidefinite if and only if $|c| \leq 1$.