Final Examination with Answers: Economics 210A

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I asked students to try to answer any 7 of the 8 questions. I intended the exam to have some relatively easy parts and some quite challenging parts. As I prepared the answers, I realized that the exam was longer and more difficult than I expected it to be. Nevertheless, some of the students got almost everything and most students got a lot correct. As a study aid, I have written fairly elaborate answers. It is quite likely that there remain some typos and some mistakes and ambiguities in my answers. I welcome questions and corrections.

Question 1.

A) *State the weak axiom of revealed preference, the strong axiom of revealed preference and the generalized axiom of revealed preference.*

Let x^i be a commodity bundle that is chosen when the price vector is p^i.

Bundle x^i is said to be directly revealed preferred to bundle x if $p^i x^i \geq p^i x$. The WARP states that if one bundle is directly revealed preferred to another, then the second will never be directly revealed preferred to the first.

Another way of stating WARP is: If $p^1 x^1 \geq p^1 x^0$, then $p^0 x^1 > p^0 x^0$.

A bundle x^i is revealed preferred to x if there is a sequence of bundles $x^n, \ldots, x^1$ such that x^t is revealed preferred to x^{t-1} for $t = n, \ldots, 2$.

SARP requires that if x' is revealed preferred to x, then x is not directly revealed preferred to x'.

A bundle x^i that is selected at price vector p^i is said to be revealed strictly preferred to a bundle x if $p^i x^i > p^i x$.

GARP requires that if x' is revealed preferred to x, then x is not strictly revealed preferred to x'.

B) *Give an example of preferences that satisfy the generalized axiom of revealed preferences but do not satisfy the strong axiom of revealed preference. Show that your example satisfies GARP but not WARP.*

Consider preferences on $\Re^2_+$ that are represented by the utility function $x_1 + x_2$. Suppose that $p^1 = p^0 = (1, 1)$, and that $x^1 = (1, 1)$ is chosen at p^1 and $x^0 = (0, 2)$ is chosen at p^1. These choices are consistent with maximizing $x_1 + x_2$ subject to the budget constraint. We see that $p^1 x^1 \geq p^1 x^0$ and $p^0 x^0 \geq p^0 x^1$. Thus we see that this data does not satisfy WARP.

If choices maximize x_1+x_2 subject to a budget constraint, it must be that if (x'_1, x'_2) is directly revealed preferred to (x_1, x_2) then $x'_1 + x'_2 \geq x_1 + x_2$. By the transitivity of the ordering of the real numbers it follows that if (x'_1, x'_2) is revealed preferred to (x_1, x_2) either directly or indirectly, then $x'_1 + x'_2 \geq x_1 + x_2$. Therefore it cannot be that (x'_1, x'_2) is revealed preferred to (x_1, x_2) and $x_1 + x_2 > x'_1 + x'_2$. So GARP is satisfied.

Question 2. *Wilbur is an expected utility maximizer with a von Neumann-Morgenstern utility function $u(x)$ where x is a vector goods. Wilbur is considering moving to one of two cities. He is unsure about his future income and about future prices. The cities are equally attractive to Wilbur in all respects other than the probability distribution of prices and income.*

A) *Wilbur knows the probability distribution of possible price-income combinations in each city, and has von Neumann-Morgenstern utility function $u(x)$ for consumption bundles. How would you go about finding which city Wilbur would prefer?*

Find Wilbur's indirect utility function corresponding to $u(x)$. Wilbur would prefer the city with the higher expected indirect utility.

B) *Suppose that Wilbur consumes just two goods and that his von Neumann Morgenstern utility function is*

$$u(x_1, x_2) = \min\{x_1, \frac{1}{2}x_2\}^{1/2}.$$

Wilbur believes that if he moves to City A, there are two possible outcomes, each of which has a probability of 1/2. One of these outcomes is that his income will be 125 *and prices of goods 1 and 2 will be $p_1 = 1$ and $p_2 = 2$. The other possible outcome is that his income will be* 196 *and prices of goods 1 and 2 will be $p_1 = 2$ and $p_2 = 1$.*

- *Calculate Wilbur's expected utility if he moves to City A. Wilbur believes that in City B, the prices of goods 1 and 2 are certain to be $p_1 = 1$ and $p_2 = 1$.*
- *If in City B he would have an income of M with certainty, for what values of M would he prefer City B to City A? Explain your answer.*

Wilbur's indirect utility function is

$$v(p,m)=\left(\frac{m}{p_1+2p_2}\right)^{1/2}$$

In city A, his expected utility would be

$$\frac{1}{2}v(1,2,125)+\frac{1}{2}v(2,2,196)=\frac{1}{2}5+\frac{1}{2}7=6$$

In city B, his expected utility would be

$$v(1,1,m)=\left(\frac{m}{3}\right)^{1/2}.$$

So he will prefer city B to city A if

$$\left(\frac{m}{3}\right)^{1/2}>6.$$

This is the case if $m>108$.

C) *Wilbur's friend Charlotte has a von Neumann Morgenstern utility function*

$$u(x_1,x_2)=\min\{x_1,\frac{1}{2}x_2\}.$$

If Charlotte and Wilbur face the same prices and have the same incomes, how will their consumptions differ? If Charlotte and Wilbur both believe that the probability distributions over their prices and incomes in Cities A and B are as in Part B, will they ever have different preferences about which city to live in? Explain.

If Wilbur and Charlotte face the same prices and have the same incomes, they will consume the same bundles. This is seen because their vNM utility functions, which are monotone transformations of each other, represent the same preferences over bundles consumed with certainty.

They have different attitudes toward risk, however. Wilbur is more risk averse than Charlotte. Charlotte's expected utility from City A would be $\frac{1}{2}25+\frac{1}{2}49=37$. Her expected utility from City B would be $m/3$. So she would prefer City A to City B if $m<111$ and woud prefer B to A if $m>111$.

Question 3. *Consider a consumer who consumes three goods and has utility function*

$$U(x_1,x_2,x_3)=(x_1+b_1)^{\alpha}(x_2+b_2)^{\beta}(x_3+b_3)^{\gamma}$$

where $b_i \geq 0$ for $i = 1, \ldots 3$ and where $\alpha > 0$, $\beta > 0$, and $\gamma > 0$.

A) *Why can you assume that $\alpha + \beta + \gamma = 1$ without loss of generality?*

The function $V(x_1, x_2, x_3) = U(x_1, x_2, x_3)^{\frac{1}{\alpha+\beta+\gamma}}$ is a monotone increasing transformation of U and thus represents the same preferences. Let $\alpha' = \frac{\alpha}{\alpha+\beta+\gamma}$, $\beta' = \frac{\beta}{\alpha+\beta+\gamma}$, and $\gamma' = \frac{\gamma}{\alpha+\beta+\gamma}$. We see that

$$V(x_1, x_2, x_3) = (x_1 + b_1)^{\alpha'}(x_2 + b_2)^{\beta'}(x_3 + b_3)^{\gamma'}$$

where $\alpha' + \beta' + \gamma' = 1$.

B) *Does this consumer have homothetic preferences? Explain.*

This consumer does not have homothetic preferences. Preferences are homothetic if and only if $U(x) = U(x')$ implies that $U(tx) = U(tx')$ for all scalars $t > 0$. It is easily verified that U does not have this property unless $b_1 = b_2 = b_3 = 0$.

C) *Does this consumer have additively separable preferences? Explain.*

This consumer has additively separable preferences. We see that

$$V(x_1, x_2, x_3) = \log U(x_1, x_2, x_3) = \alpha \ln(x_1 + b_1) + \beta \ln(x_2 + b_2) + \gamma \ln(x_3 + b_3)$$

represents the same preferences as U and V is additively separable.

D) *Find this consumer's Marshallian demand function. Don't forget to account for corner solutions if they exist.*

Since preferences are continuous and strictly convex, the demand function is a well-defined, single valued function. We found in Part A that there is no loss of generality in assuming that $\alpha + \beta + \gamma = 1$.

A shortcut for finding a solution if one exists is to define $z_i = x_i + b_i$. We can rewrite the problem as Maximize

$$U(z_1, z_2, z_3) = (z_1)^{\alpha}(z_2)^{\beta}(z_3)^{\gamma}$$

subject to the constraints that $p_1z_1 + p_2z_2 + p_3z_3 = p_1x_1 + p_2x_2 + p_3x_3 + p_1b_1 + p_2b_2 + p_3z_3 = m + p_1b_1 + p_2b_2 + p_3z_3$ and $z_i \geq b_i$ for $i = 1, \ldots 3$. This states the problem as a standard Cobb-Douglas demand problem in terms of the z_i's, with an inequality constraint. Thus it must be that at an interior maximum,

$$z_1 = \frac{\alpha}{p_1}(m + p_1b_1 + p_2b_2 + p_3b_3)$$

$$z_2 = \frac{\beta}{p_2}(m + p_1b_1 + p_2b_2 + p_3b_3)$$

$$z_3 = \frac{\gamma}{p_3}(m + p_1b_1 + p_2b_2 + p_3b_3)$$

Then

$$x_1 = z_1 - b_1 = \frac{\alpha}{p_1}(m + p_1b_1 + p_2b_2 + p_3b_3) - b_1 \tag{1}$$

$$x_2 = z_2 - b_2 = \frac{\beta}{p_2}(m + p_1b_1 + p_2b_2 + p_3b_3) - b_2 \tag{2}$$

$$x_3 = z_3 - b_3 = \frac{\gamma}{p_3}(m + p_1b_1 + p_2b_2 + p_3b_3) - b_3 \tag{3}$$

These three equations must be satisfied if there is an interior solution where $x_1 > 0$, $x_2 > 0$ and $x_3 > 0$.

I gave full credit to those who found this necessary condition for there to be an interior solution, whether or not they properly characterized the corner solutions.

A full characterization of corner solutions is a little complicated and I did not expect everyone to carry this out in detail during the exam. It is not true, as some exam-takers claimed, that if Equation 3 tells us that $x_3 < 0$, then there will be a solution with $x_3 = 0$ and x_1 and x_2 given by Equations 1 and 2.

But, for your enjoyment, I will sketch a full description of corner solutions and how you would find them. If there is no interior solution, there must be a corner solution. In general, with three goods, there are 6 possible corner solutions. In three of these, two goods are consumed in positive quantities and one is not consumed. Since the demand function is single-valued, we know that exactly one of these corner solutions will obtain. Equations 1-3 don't give us an immediate answer to which of them it is. A corner solution must satisfy the Kuhn-Tucker conditions. These involve a system of 3 inequalities, along with complementary slackness conditions for each of them.

In general, solving a system of 3 Kuhn-Tucker inequalities can be a pain in the neck, because there are 6 possible places to look for a corner solution. It could be that the solution has any two of the variables at interior values and one at a boundary value, or it could be that it has only one of the variables at an interior value and the others at boundary values. For this problem, our search is easier, because there is a fairly simple condition that tells us where a solution could possibly be.

Explaining the answer is a little easier i we use the notation $\alpha_1 = \alpha$, $\alpha_2 = \beta$ and $\alpha_3 = \gamma$. The consumer's maximization problem can be represented as Maximize

$$U(z_1, z_2, z_3)) = \alpha_1 \ln z + 1 + \alpha_2 \ln z_2 + \alpha_3 \ln z_3$$

subject to the constraints

$$p_1 z_1 + p_2 z_2 + p_3 z_3 = m + p_1 b_1 + p_2 b_2 + p_3 z_3$$

and $z_i \geq b_i$ for $i = 1, 2, 3$.

Where the Lagrangean for this problem is

$$L(z_1, z_2, z_3, \lambda) = \sum_{i=1}^{3} \alpha_i \ln z_i + \lambda \left((m + p_1 b_1 + p_2 b_2 + p_3 z_3) - (p_1 z_1 + p_2 z_2 + p_3 z_3) \right),$$

the Kuhn-Tucker conditions require that for each i, either $z_i > b_i$ and the partial derivative of the Lagrangean with respect to z_i is zero or $z_i = b_i$ and the partial derivative of the Lagrangean with respect to z_i is negative.

The interpretation of the Kuhn-Tucker conditions is as follows. If $z_i > b_i$, then the bang-per-buck from i is $z_i/p_i = \lambda$. If $z_i = b_i$, then the bang-per-buck from i is less than λ.

In seeking a solution, one might look first for an interior solution where there is equal bang per buck from all three goods and $z_i \geq b_i$ for all i. If you don't find such a solution, you can simplify your search for a corner solution by calculating b_i/p_i for each of the i's. This is the bang per buck at the boundary. Suppose that we have $b_i/p_i > b_j/p_j > b_k/p_k$. Then there are only two possibilities for a corner solution. One of them has $z_k = b_k$, $z_i > b_i$, and $z_j > b_j$. The other has $z_j = b_j$, $z_k = b_k$ and $z_i > b_i$. (I will leave it as an exercise for you to show that this is the case.)

Since strict convexity implies that there is exactly one solution, it is easy find out which one is the equilibrium by checking which one satisfies the Kuhn-Tucker conditions.

Having found solutions for the z_i's, one can then find the x_i's since $x_i = z_i - b_i$.

Question 4. *For the consumer described in the previous question:*

A) *Find this consumer's indirect utility function and verify that Roy's identity holds in this case.*

(I didn't expect you to work this out for the corner solutions. You got full credit if you did it correctly for the interior solutions.)

Let the direct utility function be $x_1^\alpha x_2^\beta x_3^\gamma$ where $\alpha+\beta+\gamma=1$. If there is an interior solution, then the indirect utility function is

$$V(p_1, p_2, p_3, M) = (M + p_1 b_1 + p_2 b_2 + p_3 b_3)\, \alpha^\alpha \beta^\beta \gamma^\gamma p_1^{-\alpha} p_2^{-\beta} p_3^{-\gamma}$$

According to Roy's law, where $x_i(p,m)$ is the Marshallian demand function, we must have

$$x_i(p,m) = -\left(\frac{\partial V(p_1,p_2,p_3,M)}{\partial p_i}\right) \div \left(\frac{\partial V(p_1,p_2,p_3,M)}{\partial M}\right). \tag{4}$$

Let's verify this for good 1. Very similar calculations do the trick for goods 2 and 3.

$$\frac{\partial V}{\partial p_1} = b_1 \alpha^\alpha \beta^\beta \gamma^\gamma p_1^{-\alpha} p_2^{-\beta} p_3^{-\gamma} - \frac{\alpha}{p_1} b_1 \alpha^\alpha \beta^\beta \gamma^\gamma p_1^{-\alpha} p_2^{-\beta} p_3^{-\gamma} (M + p_1 b_1 + p_2 b_2 + p_3 b_3) \tag{5}$$

$$\frac{\partial V}{\partial M} = \alpha^\alpha \beta^\beta \gamma^\gamma p_1^{-\alpha} p_2^{-\beta} p_3^{-\gamma} \tag{6}$$

Substituting Equations 5 and 6 into the right side of Equation 4 and cancelling, we have

$$x_i(p,m) = \frac{\alpha}{p_1}(m + p_1 b_1 + p_2 b_2 + p_3 b_2) - b_1.$$

Looking back at the answer to Question 3, we see that this is the same answer we found by calculating Marshallian demand directly.

For the masochists among you, we could go on to calculate indirect utilities at the corner solutions.

At a corner solution, where $x_1 = 0$ $x_2 > 0$ and $x_3 > 0$, the indirect utility must be

$$(M + p_2 b_2 + p_3 b_3)\, b_1^\alpha \left(\frac{\beta}{\beta+\gamma}\right)^\beta \left(\frac{\gamma}{\beta+\gamma}\right)^\gamma p_2^{-\beta} p_3^{-\gamma}.$$

At a corner solution where $x_1 = 0$, $x_2 = 0$ and $x_3 > 0$, indirect utility must be

$$(M + p_3 b_3)\, b_1^{\alpha} b_2^{\beta} p_3^{-\gamma}$$

.

B) *Find this consumer's expenditure function.*

A fairly quick way to find this is to use the fact that where V is the indirect utility function,

$$V\,(p, e(p,u)) = u.$$

Using the indirect utility function that we found previously, we then have

$$(e(p,u) + p_1 b_1 + p_2 b_2 + p_3 b_3)\, \alpha^{\alpha} \beta^{\beta} \gamma^{\gamma} p_1^{-\alpha} p_2^{-\beta} p_3^{-\gamma} = u. \tag{7}$$

From Equation 7 it follows that

$$e(p,u) = u p_1^{\alpha} p_2^{\beta} p_3^{\gamma} \alpha^{-\alpha} \beta^{-\beta} \gamma^{-\gamma} - p_1 b_1 - p_2 b_2 - p_3 b_3 \tag{8}$$

C) *Find this consumer's Hicksian demand function and the substitution matrix.*

The Hicksian demand functions are the partial derivatives of the expenditure function.

$$h_1(p,u) = u\frac{\alpha}{p_1} p_1^{\alpha} p_2^{\beta} p_3^{\gamma} \alpha^{-\alpha} \beta^{-\beta} \gamma^{-\gamma} - b_1$$

$$h_2(p,u) = u\frac{\alpha}{p_2} p_1^{\alpha} p_2^{\beta} p_3^{\gamma} \alpha^{-\alpha} \beta^{-\beta} \gamma^{-\gamma} - b_2$$

$$h_3(p,u) = u\frac{\alpha}{p_3} p_1^{\alpha} p_2^{\beta} p_3^{\gamma} \alpha^{-\alpha} \beta^{-\beta} \gamma^{-\gamma} - b_3$$

The substitution matrix is the Hessian of the expenditure function. the ijth element of this matrix is the partial derivative of $h_i(p,u)$ with respect to p_j.

Question 5. *Consider an economy with two types of people. While their utility functions are all of the form described in Question 3, they have different values of the b_i's. Type 1 has $b_1 = b_2 = b_3 = 10$ and Type 2 has $b_1 = 10$, and $b_2 = b_3 = 0$. Persons of Type 1 have incomes M_1 and persons of type 2 have incomes M_2.*

A) *If there are 100 people of each type, write an expression for the aggregate demand for good 1 as a function of the prices and incomes.*

At interior solutions it must be that the demand functions of Type 1's are given by

$$x_1^1(p,m) = \frac{\alpha\left(M_1 + 10(p_1+p_2+p_3)\right)}{p_1} - b_1.$$

and Type 2's by

$$x_1^2(p,m) = \frac{\alpha(M_2 + 10p_1)}{p_1} - b_1.$$

Then aggregate demand for good 1 is

$$100\left(x_1^1(p,m) + x_1^2(p,m)\right) = 100\left(\frac{M_1 + M_2 + 20p_1 + 10p_2 + p_3}{p_1} - b_1.\right) \tag{9}$$

B) *Suppose that initially prices and income are such that everybody buys positive amounts of all goods. Income is redistributed from type 2's to type 1's and after the income redistribution everyone is still buying positive amounts of all goods. What can you say about the change in aggregate demand for good 1? Explain your answer.*

There will be no change in aggregate demand for good 1. We see this from Equation 9, since aggregate demand for Good 1 is determined by aggregate income.

C) *Suppose that the prices are $p_1 = p_2 = p_3 = 1$. Find incomes M_1 and M_2 such that a transfer of income from type 1's to type 2's must change the aggregate demand for good 1. Explain your answer.*

This will happen if enough income is transferred from Consumer 1 to consumer 2 so that Consumer 1 is at a corner solution, consuming no Good 1. If this is the case, further transfers from 1 to 2 will increase consumer 2's consumption of Good 1, but will not change Consumer 1's. Therefore such transfers will increase aggregate demand for Good 1.

Consumer 1 will demand a positive amount of good 1 only if $\alpha(M_1+30) > 10$, which is equivalent to $M_1 > \frac{10}{\alpha} - 30$. So if $M_1 < \frac{10}{\alpha} - 30$, transfers from 1 to 2 will increase aggregate demand for Good 1.

Question 6. *Harry consumes just one commmodity and he will live for T periods. His current preferences over consumption streams are represented by a utility function of the form*

$$U(x_1, \ldots, x_T) = \sum_{t=1}^{T} \beta_t u(x_t)$$

where x_t is the amount of the commodity that he will consume in year t and where the function $u(\cdot)$ is increasing, strictly concave and twice continuously differentiable. Harry knows that his income stream will be $(w_1, \ldots, w_T)$ where w_t is the income that he will receive in period t. Harry is able to borrow or save at the constant interest r. At time 1, Harry is able to commit himself to any time path of consumption that satisfies his budget constraint. His budget constraint is that the present value of his lifetime consumption does not exceed the present value of his lifetime income stream.

A) *Suppose that for some α where $0 < \alpha < 1$, and for all $t = 1, \ldots T$, $\beta_t = \alpha^{t-1}$. At what interest rate will Harry will choose to consume the same amount of goods in every period of his life? Explain your answer. Does this interest rate depend on the time path of his income stream? At this interest rate, what can you say about the way in which his borrowing and saving behavior depend on the time path of his income stream?*

Harry's intertemporal budget constraint is

$$\sum_{t=1}^{T} x_t \frac{1}{(1+r)^t} = \sum_{t=1} w_t \frac{1}{(1+r)^t}.$$

Harry would choose a consumption profile such that his marginal rate of substitution between consumption in period $t+1$ and t is equal to their relative prices. That is

$$\frac{\alpha^t u'(x_{t+1})}{\alpha}^{t-1} u'(x_t) = (1+r).$$

If Harry chooses constant consumption, it must be hat $u'(x_{t+1}) = u'(x_t)$ and therefore $\alpha = \frac{1}{1+r}$. His time path of consumption does not depend on the time path of his income but only on the present value of income. Where PV is the present value of his income and c is his constant consumption, it must be that

$$c(1 + r + r^2 \ldots + r^T) = PV$$

Since

$$1 + r + r^2 + \ldots r^T = \frac{1 - r^{T+1}}{1 - r}$$

it follows that

$$c = (1 - r)\frac{PV}{1 - r^{T+1}}$$

He will spend less than his income in periods where

$$w_t > (1 - r)\frac{PV}{1 - r^{T+1}}$$

and spend more than his income when this inequality is reversed.

B) *Suppose that* $\beta_2 = \beta_1 = 1$ *and that for* $t = 3, \ldots T$, $\beta_t = \alpha^{t-2}$. *Suppose that at time 1, Harry can commit himself to a time path of future consumption. Qualitatively, how does his time path of consumption depend on the interest rate? For example, at what if any interest rates* $r > 0$ *is his consumption first increasing, then constant, at what interest rates is his consumption always increasing, at what interest rates is his consumption first increasing, then decreasing, etc, etc.*

If he can commit himself to a plan for future consumption and the interest rate is constant at r, Harry will choose a consumption path such that

$$\frac{\beta_{t+1} u'(x_{t+1})}{\beta_t u'(x_t)} = \frac{1}{1 + r}.$$

If his preferences are convex, this means that x_{t+1} is greater or less than x_t depending on whether $\frac{\beta_{t+1}}{\beta_t}$ is greater or less than $\frac{1}{1+r}$. In this example, Harry would increase his consumption between periods 1 and 2. His consumption in period 3 would be less than his consumption in period 2 and would be increasing or decreasing over the remaining course of his life, depending on whether $1 + r > \alpha$ or $1 + r < \alpha$.

C) *Suppose that* $T = 3$ *and Harry's utility function is*

$$U(x_1, x_2, x_3) = \sqrt{x_1} + \sqrt{x_2} + \alpha\sqrt{x_3}.$$

Harry earns income $W > 0$ in period 1, while $w_t = 0$ for $t > 1$. Suppose also that

$$\frac{1}{1+r} = \alpha.$$

If at time 1, Harry can choose his consumption for each period, subject to his budget constraint, solve for his choice of x_1, x_2, and x_3 as a function of the parameters W and r.

Harry's budget constraint is

$$x_1 + \frac{x_2}{1+r} + \frac{x+3}{(1+r)^2} = W$$

Setting marginal rates of substitution equal to price ratios, we find that $x_3 = x_2 = (1+r)^2 x_1$. Together with the budget constraint, this implies that

$$x_1 \left(1 + (1+r) + 1\right) = W$$

and hence

$$x_1 = \frac{W}{3+r},$$

and

$$x_2 = x_3 = \frac{W(1+r)^2}{3+r}.$$

D *Suppose that Harry can save money in period 1 but he must leave the choice of allocation between periods 2 and 3 to his future self. Harry is aware of this and knows that in Period 2 his utility function for consumptions periods 2 and 3 will be*

$$U(x_2, x_3) = \sqrt{x_2} + \sqrt{x_3}.$$

He also knows that the interest rate will continue to satisfy the equation

$$\frac{1}{1+r} = \alpha.$$

If Harry consumes x_1 in period 1, what consumptions will his period 2 self choose for periods 2 and 3? Write down an expression for Harry's utility as a function of x_1, taking into account the fact that he knows that his period 2 self will determine the division of income between his period 2 self and his *period 3 self. Find the optimal choice of x_1 for Harry. Is this the same as the amount of x_1 that he would choose in Part C above?*

Harry's budget in Period 2 will be

$$x_2 + \frac{1}{1+r}x_3 = W - x_1.$$

Setting his marginal rate of substitution equal to the price ratio, we see that

$$x_3 = (1+r)^2 x_2.$$

Substituting into the budget, we find that $x_2(1+1+r) = W - x_1$. Hence

$$x_2 = \frac{W - x_1}{2+r}$$

and

$$x_3 = (1+r)x_2.$$

Then Harry's lifetime utility evaluated in period 1 if he chooses x_1 will be

$$x_1^{1/2} + \left(\frac{W - x_1}{2+r}\right)^{1/2} + \left(\frac{(W - x_1)(1+r)}{2+r}\right)^{1/2}. \tag{10}$$

Expression 10 simplifies to

$$x_1^{1/2} + (W - x_1)^{1/2}\left(\frac{1 + (1+r)^{1/2}}{(2+r)^{1/2}}\right) \tag{11}$$

To find Harry's optimal choice of x_1, set the derivative of Expression 10 with respect to x_1 equal to zero. This implies that

$$x_1^{-1/2} = (W - x_1)^{-1/2}\left(\frac{1 + (1+r)^{1/2}}{(2+r)^{1/2}}\right) \tag{12}$$

and hence

$$\frac{W - x_1}{x_1} = \frac{2+r}{1 + (1+r)^{1/2}} \tag{13}$$

Looking back at the answer to Part C, we find that when Harry can commit future consumptions, he chooses x_1 such that

$$\frac{W - x_1}{x_1} = 2 + r. \tag{14}$$

Comparing Equation 13 with Equation 14, we see that Harry chooses to save less and spend more in the case where he can not commit than in the case where he can commit future behavior.

Question 7. *A pure exchange economy has 1000 Type 1 consumers and 1000 Type 2 consumers. There are two commodities. Everyone has a utility function of the form*

$$U(x_1, x_2) = (x_1^\alpha + x_2^\alpha)^{\frac{1}{\alpha}}.$$

Type 1's have an initial endowment of ω_1 units of good 1 and zero units of good 2. Type 2's have an initial endowment of ω_2 units of good 2 and zero units of good 1. Suppose that there is competitive trading where good 1 is the numeraire, the price of good 2 is p and each consumer has wealth equal to the value of his or her initial endowment.

A) *Suppose that $\alpha = -1$. At the price p, how much of good 2 will Type 2's demand for themselves and how much will they be willing to sell? Does the amount that they are willing to sell increase or decrease as p increases?*

The budget equation for Type 2's is

$$x_1 + px_2 = p\omega_2. \tag{15}$$

Setting p equal to the ratio of the marginal utility of good 2 to the marginal utility of good 1, we have

$$p = \frac{x_2^{\alpha-1}}{x_1^{\alpha-1}}. \tag{16}$$

Where $\alpha = -1$, this implies that

$$p = \frac{x_1^2}{x_2^2}$$

and hence

$$x_1 = \sqrt{p}x_2. \tag{17}$$

Let $x_2^2(p)$ be the amount that a Type 2 will demand for her own consumption when the price is p. Substituting from Equation 17 into Equation 15, we have

$$x_2^2 = \frac{p\omega_2}{p + \sqrt{p}}. \tag{18}$$

The amount that a Type 2 will be willing to sell will be

$$\omega_2 - x_2^2(p) = \omega_2 \left(\frac{\sqrt{p}}{p + \sqrt{p}} \right) = \omega_2 \left(\frac{1}{1 + \sqrt{p}} \right) \tag{19}$$

We see from Equation 19 that Type 2's will offer to sell less of good 2 when its price rises.

B) *At the price p, how much of good 2 will Type 1's demand?*

Type 1's have budget

$$x + 1 + px_2 = \omega_1$$

and their demand function for good 2 is

$$x_2^1(p) = \frac{\omega_1}{p + \sqrt{p}}.$$

C) *Find the competitive equilibrium price p.*

There are (at least) two ways to solve this.

Brute force Method

Aggregate demand for Good 2 equals aggregate supply when

$$1000 \left(\frac{\omega_1}{p + \sqrt{p}} + \frac{\sqrt{p}\omega_2}{1 + \sqrt{p}} \right) = 1000\omega_2 \tag{20}$$

Simplifying Equation 20, we find that

$$p = \left(\frac{w_1}{w_2} \right)^2.$$

Quick method

All consumers have identical homothetic preferences. So if they all face the same prices, they must all consume the two goods in the same ratio. Since supply must equal demand in both markets, the ratio in which each consumer consumes the two goods must be the same as the ratio in which the goods are available, that is ω_2/ω_1.

The equilibrium price p must equal the marginal rate of substitution of a consumer who consumes the goods in this ratio. So we must have

$$p = \left(\frac{\omega_2}{\omega_1}\right)^{-2} = \left(\frac{\omega_1}{\omega_2}\right)^2.$$

D) *Suppose that a natural catastrophe reduces the endowment of each Type 2 by 50% and the price moves to the new equilibrium price. What happens to the value of the endowment of type 2's? Are the type 2's better off or worse off after the catastrophe than they were before? What about the type 1's?*

It is easy to see that Type 1's are made worse off. Their income does not change and the price of good 2 has increases.

It is less easy to see whether type 2's are better off. Their income doubles, but the cost of Good 2 quadruples. (In fact, when I wrote the question, I didn't realize that the answer was as difficult as it is.)

One way to check this out is to note that the indirect utility function for type 2's when good 1 is numeraire is

$$V(p, m) = \frac{m}{(1+\sqrt{p})^2}. \tag{21}$$

In equilibrium, Type 2's have income

$$m = p\omega_2 = \frac{\omega_1^2}{\omega_2}. \tag{22}$$

$$(1+\sqrt{p})^2 = \left(1 + \frac{\omega_1}{\omega_2}\right)^2. \tag{23}$$

Let us define $F(\omega_1, \omega_2$ to be the utility of Type 2's in competitive equilibrium when endowments are such that type 1's have ω_1 units of good 1 and no good 2 and type 2's have ω_2 units of good 2 and no good 1.

Substituting from equations 22 and 23 into equation 21 and arranging terms, we have

$$F(\omega_1, \omega_2) = \frac{\omega_1^2 \omega_2}{(\omega_1 + \omega_2)^2} \tag{24}$$

Now we can check whether reducing the endowment of good 2 by 1/2 benefits or harms Type 2's. Calculations are just a bit easier if we look at the effect of doubling ω_2.

We have

$$F(\omega_1, 2\omega_2) \geq F(\omega_1, \omega_2)$$

if and only if

$$\frac{2\omega_2}{(\omega_1 + 2\omega_2)^2} \geq \frac{\omega_2}{(\omega + 1 + \omega_2)^2}.$$

This is the case if and only if

$$2\omega_2(\omega_1 + \omega_2)^2 \geq \omega_2(\omega_1 + 2\omega_2)^2,$$

which is equivalent to

$$2\omega_1^2 + 4\omega_1\omega_2 + 2\omega_2^2 > \omega_1^2 + 4\omega_1\omega_2 + 4\omega_2^2.$$

But this implies that

$$\omega_1^2 \geq 2\omega_2^2,$$

or equivalently

$$\omega_1 \geq \sqrt{2}\omega_1.$$

Since we showed that doubling the endowments of ω_2 makes type 2's better off if $\omega_1 > \sqrt{2}\omega_1$, it follows that a natural disaster that reduces ω_2 by 50% benefits Type 2's if $\omega_1 < \sqrt{2}\omega_2$ and harms Type 2's if the inequality is reversed.

Question 8.

A) *A consumer consumes n goods and his utility function is $u(x)$ where x is his consumption vector. Define this consumer's expenditure function.*

The consumer's expenditure function $e(p, u)$ is the minimum income with which the consumer can achieve utility u when the price vector is p.

B)*If the utility function $u(\cdot)$ is homogeneous of degree k, what special structure does the expenditure function have?*

If the utility function is homogeneous of degree k, the expenditure function is homogeneous of degree $1/k$.

C) *Assuming that the expenditure function corresponding to utility function $u(x)$ is well-defined, show that it is a concave function.*

We show that $e(p, u)$ is concave in p. Let p and p' be two different price vectors. Then $e(p, u) = ph(p, u)$ and $e(p', u) = p'h(p', u)$. Let $p(\lambda) = \lambda p + (1 - \lambda)p'$ where $0 \leq \lambda \leq 1$. Then $e(p(\lambda), u) = p(\lambda)h\left(p(\lambda)\right), u)$.

Since $h(p, u)$ is the cheapest bundle at prices p with utility u, it must be that

$$ph(p(\lambda)) \geq ph(p) = e(p, u). \tag{25}$$

Likewise, since $h(p., u)$ is the cheapest bundle at prices p' with utility u, it must be that

$$p'h(p(\lambda)) \geq p'h(p') = e(p', u). \tag{26}$$

Now

$$\begin{aligned} e(p(\lambda), u) &= p(\lambda)h(p(\lambda) \\ &= \lambda ph(p(\lambda)) + (1 - \lambda)p'h(p(\lambda)) \end{aligned} \tag{27}$$

Then it follows from Inequalities 25 and 26 and Equation 27, that

$$e(p(\lambda, u) \geq \lambda e(p, u) + (1 - \lambda)e(p', u). \tag{28}$$

Therefore $e(p, u)$ is concave in p.