Name ____________________

Answers for Midterm Examination: Economics 210A
October, 1016

Here are my proposed answers for this exam. I won't promise that there are no mistakes (or what generous people might call typos.)

1) *State whether each of the following claims is true or false. If true, give a proof, justifying each claim made in your proof. If false, give a counterexample and prove that your counterexample is a counterexample.*
Let f be a real-valued concave function whose domain is a convex subset of $\Re^n$. Let g be a function from the reals to the reals and define the composite function $h(x) = g(f(x))$.

A) If g is a strictly increasing function, then h is a quasi-concave function.

True.

One way to do this is to prove 1) A concave function must be quasi-concave 2) A monotone increasing function of a quasi-concave function must be quasi-concave. From results 1) and 2), it is immediate that a monotone increasing function of a concave function is quasi-concave.

Proof of 1:

If f is a concave function, it must be that for all $\lambda \in [0,1]$ and all x and x' in the domain of f,

$$f(\lambda x + (1-\lambda)x') \geq \lambda f(x) + (1-)f(x').$$

Since

$$\lambda f(x) + (1-\lambda)f(x') \geq \min\{f(x), f(x')\},$$

it follows that if f is concave,

$$f(\lambda x + (1-\lambda)x') \geq \min\{f(x), f(x')\}.$$

Therefore f os quasi-concave if it is concave.

Proof of 2:

Let $h(x) = g(f(x))$ where h is an increasing function and f is quasi-concave. it must be that

$$\begin{aligned} h\left(\lambda x+(1-\lambda)x'\right) &= g\left(f\left(\lambda x+(1-\lambda)x'\right)\right. & (1)\\ &\geq g\left(\min\{f(x), f(x')\}\right) & (2)\\ &= \min\{g\left(f(x)\right), g(f(x'))\} & (3)\\ &= \min\{h(x), h(x')\}. & (4) \end{aligned}$$

Therefore h is quasi-concave.

Equation 1 follows from the definition of h. Inequality 2 follows from the assumptions that f is quasi-concave and g is and increasing function. Equation 3 follows from the assumption that g is an increasing function, and Equation 4 from the definition of h.

B) If g is an increasing, concave function, then h is a concave function.

True.

Since f is a concave function it must be that

$$f\left(\lambda x+(1-\lambda)x'\right) \geq \lambda f(x) + (1-\lambda)f(x'). \tag{5}$$

Since g is an increasing function, it must be that

$$\begin{aligned} h\left(\lambda x+(1-\lambda)x'\right) &= g\left(f\left(\lambda x+(1-\lambda)x'\right)\right) & (6)\\ &\geq g\left(\lambda f(x)+(1-\lambda)f(x')\right) & (7)\\ &\geq \lambda g\left(f(x)\right)+(1-\lambda)g\left(f(x')\right) & (8)\\ &= \lambda h(x)+(1-\lambda)h(x') & (9) \end{aligned}$$

It follows that $g\left(f\left(\lambda x+(1-\lambda)x'\right)\right) = h\left(\lambda x+(1-\lambda)x'\right) \geq \lambda h(x)+(1-\lambda)h(x')$ which means that h is concave.

Equation 6 follows from the definition of h. The step from Equation 6 to inequality 7 follows from inequality 5 and the assumption that g is an increasing function. The step from 7 to 8 follows from the assumption that g is a concave function. The step from 8 to 9 follows from the definition of h.

2)*A consumer has utility function*

$$u(x_1, x_2) = \min\{v_1(x_1, x_2), v_2(x_1, x_2)\}$$

where v_1 and v_2 are both quasi-concave functions. Is u quasi-concave? If so, prove it. If not, provide a counterexample.

A function $f(x_1, x_2)$ will is quasi-concave if and only if for all (x_1, x_2) in its domain, the upper-contour set $f^+(x_1, x_2) = \{(x_1', x_2)|f(x_1', x_2') \geq f(x_1, x_2)\}$ is a convex set. Since v_1 and v_2 are quasi-concave, it must be that the upper contour sets $v_i^+(x_1, x_2) = \{(x_1', x_2')|v_i(x_1', x_2') \geq v_i(x_1, x_2)\}$ are convex. From the definition of the function u, we see that for every (x_1, x_2) in the domain, the upper contour set of u will be the set $u^+(x_1, x_2) = v_1^+(x_1, x_2) \cap V_2^+(x_1, x_2)$. Since the intersection of convex sets is convex, it follows that u is a quasi-concave function.

3) *Tony consumes two goods. He prefers any bundle in the set*

$$X = \{(x_1, x_2)|x_1 \geq 0 \text{ and } x_2 > 1\}$$

to any bundle that is not in this set. His preferences over bundles in X are represented by the utility function

$$u(x_1, x_2) = \ln(x_1 + 1) + \ln(x_2 - 1).$$

A) For what price-income combinations does Tony consume a bundle in the interior of X?

From the budget constraint and the first order conditions, we find that at an interior solution, in X, Tony consumes

$$x_2 = \frac{m + p_1 + p_2}{2p_2}$$

and

$$x_1 = \frac{m - p_1 - p_2}{2p_1}.$$

Therefore he consumes in the interior of X if and only if $x_2 > 1$, which implies that $2p_2 < m + p_1 + p_2$ or equivalently $m > p_2 - p_1$ and $x_1 > 0$ which implies that $m > p_1 + p_2$.

This occurs when $m > p_2 - p_1$ and $m > p_1 + p_2$. Notice that for $p_1 > 0$, $p_1 + p_2 > p_2 - p_1$. Therefore if $m > p_1 + p_2$, we also have $m > p_2 - p_1$, so $m > p_1 + p_2$ is necessary and sufficient for Tony to choose a bundle in the interior of X.

B) Sketch some indifference curves for Tony. Show an example of a budget from which he chooses a point that is in X but not in its interior.

First show the set X, which is bounded from below by the line $x_2 = 1$ and from the right by the line $x_1 = 0$. The indifference curves are rectangular hyperbolas that are asymptotic to the lines $x_2 = 1$ and $x_1 = -1$. A curve intersects the vertical axis at the point, $(0, x_2)$ has a slope of $-x_2$ at that point. You should be able to draw this and show a budget that has a corner solution at $x_1 = 0$.

C) Find Tony's Marshallian demand functions for goods 1 and 2.

If $m \geq p_1 + p_2$ then

$$x_2 = \frac{m + p_1 + p_2}{p_2}$$

and

$$x_1 = \frac{m - p_1 - p_2}{2p_1}.$$

If $p_2 < m < p_1 + p_2$, then $x_2 = m/p_2$ and $x_1 = 0$. If $m < p_2$, poor Tony won't be able to afford anything in X. Who knows what he will do in that state of misery.

4) Rosemary has utility function

$$u(x_1, x_2) = x_1 + \alpha \ln x_2.$$

where $\alpha > 0$. Find her Marshallian demand function for each good and also find her indirect utility function. (Don't forget to deal with possible corner solutions.)

I think everybody got this right, so I won't do it here.

5) Harry has utility function

$$u(x_1, x_2, x_3) = 3\left(x_1 x_2 x_3\right)^{1/3}$$

.

A) Does Harry have convex preferences? Prove your answer.

We could do this by writing out the Hessian matrix of u and showing that it is negative semi-definite.

An easier way is to remember that a monotone increasing function of a concave function is concave.

Notice that $u(x_1, x_2, x_3)$ is a monotone increasing function of $v(x_1, x_2, x_3) = \ln x_1 + \ln x_2 + \ln x_3$ over the domain where the x_i's are all positive. (In particular, $u(x_1, x_2, x_3) = 3e^{v(x_1,x_2,x_3)}$. But the function v is easily verified to be concave. (It's Hessian is the matrix with $-1/x_i$ on the ith diagonal and 0's off the diagonal.)

B) Find Harry's Marshallian demand function for each of the goods.

Harry's Marshallian demand for good i is

$$x_i(p, m) = \frac{m}{3p_i}.$$

C) Find his indirect utility function.

$$V_i(p, m) = \frac{m}{(p_1 p_2 p_3)^{1/3}}.$$

6) Katy has preferences represented by the utility function

$$U(x_1, x_2, x_3, x_4) = 3\,(x_1 x_2 x_3)^{1/3} + 2x_4^{1/2}.$$

A) Find Katy's Marshallian demand function $h_i(p_1, p_2, p_3, p_4, m)$ for each good i, using the following two-stop procedure. Use the indirect utility function that you found in Problem 3 to find a simple expression for Katy's utility if she buys x_4 units of good 4 and spends his remaining income on goods 1,2, and 3. Solve for the amount of x_4 that maximizes this utility. This should give you her Marshallian demand function for good 4. Knowing how much money is left to spend on the other three goods, you should be able to find Katy's demand functions for each of the other 3 goods as a function of prices and income.

If Katy buys x_4 units of good 4, her utility will be

$$\frac{m - p_4 x_4}{(p_1 p_2 p_3)^{1/3}} + 2x_4^{1/2}.$$

The first order condition for maximizing this with respect to x_4 is

$$\frac{p_4}{(p_1 p_2 p_3)^{1/3}} = x_4^{-1/2}.$$

Thus we have

$$x_4(p, m) = (p_1 p_2 p_3)^{2/3} p_4^{-2}.$$

We also have for $i = 1 - 3$,

$$x_i(p, m) = \frac{m - p_4 x_4}{3p_i} = \frac{1}{3p_i}\left(m - \frac{(p_1 p_2 p_3)^{2/3}}{p_4}\right).$$

B) What is Katy's price elasticity of demand for good 4, and what is her income elasticity of demand for good 4?

Katy's price elasticity of demand for good 4 is -2. Her income elasticity of demand for good 4 is 0.