Some Properties of Indirect Utility.

- If preferences be represented by a continuous strictly increasing function defined on $\Re^n_+$, then for any price vector $p \in \Re_{++}$ there is a well-defined function $v(p, m)$ which is $\{\max u(x) | px \leq m\}$. (How do we know this?)
- $v(p, m)$ is homogeneous of degree 0 in (p, m). (Prove it.)
- $v(p, m)$ is strictly increasing in m and non-increasing in p. (Prove it.)
- $v(p, m)$ is a quasi-convex function.
- $v(p, m)$ is a continuous function.

Quasi-convexity of indirect utility

- Where A is a convex subset of $\Re^n$, a function $f : A \to \Re$ is quasi-convex if for all x and y in A, and all $\lambda \in [0, 1]$, $f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}$. (Note that f is quasi-convex iff $-f$ is quasi-concave.)
- Show that $v(p, m)$ is quasi-convex.
  - Let x^λ maximize $u(x)$ subject to $(\lambda p + (1 - \lambda)p')x \leq \lambda m + (1 - \lambda)m'$.
  - Rearranging terms, we see that $\lambda(px^\lambda - m) + (1 - \lambda)(p'x^\lambda - m') \leq 0$.
  - So it must be that either $px^\lambda \leq m$ or $p'x^\lambda \leq m'$ (possibly both).
  - Therefore either $v(\lambda p + (1 - \lambda)p', \lambda m + (1 - \lambda)m' \leq v(p, m)$ or $v(\lambda p + (1 - \lambda)p', \lambda m + (1 - \lambda)m' \leq v(p', m')$ (Explain why)
- Thats it. (Explain why)

Continuity, Berge's theorem, a.k.a. Theorem of the Maximum

- Parametric constrained maximization problem: *Maximize $F(x, a)$ subject to $x \in A(a)$.*
  - Let $A(a)$ be a continuous mapping from parameter vectors $a \in \Re^n$ to closed bounded subsets of $\Re^n$ and the function F be a continuous function.
  - Define the correspondence $x(a) = \{x \in A(a) | F(x, a) \geq F(x', a) \text{for all } x' \in A(a)$
  - Define the function $v(a) = \max_{x \in A(a)} F(x, a)$.
- Then the function $v(a)$ is continuous and the correspondence $x(a)$ is upper semi-continous.

The special case of Berge's theorem for demand correspondences

- Maximize $u(x)$ subject to $px \leq m$ where u is a continuous, monotone increasing utility function, where $p >> 0$ and $m > 0$.
- This corresponds to Berge's theorem with the parameter vector being p, m, the function $F(x, a) = u(x)$ and the correspondence $A(p, m) = \{x \in \Re^n_+ | px \leq m\}$ being the budget correspondence.
- The correspondence $A(p, m)$ turns out to be continuous at all $(p, m) >> 0$, so by Berge's theorem, the demand correspondence $x(p, m)$ is upper semi-continuous and the indirect utility $v(p, m)$ is continuous.

Continuous correspondences

- (We will here concern ourselves with correspondences into sets that are non-empty, closed and bounded in $\Re^n$.)
- A correspondence $A(a)$ is upper semi-continuous if it has a closed graph. That is, if the sequence $(a_n, x_n) \to (a, x)$ and if $x_n \in A(a_n)$ for all n then $x \in A(a)$. (See picture on the board.)
- A correspondence is lower semi-continuous if for all $x \in A(a)$ and for any sequence $a_n \to a$, we can find N large enough so that for all $n > N$, there is an $x_n \in A(a_n)$ and $x_n \to x$.
- A correspondence is ~~lower semi-~~continuous if it is both upper and lower semi-continuous.

The budget correspondence is continuous on the set $\Re^{n+1}_{++}$.

- Its upper semi-continuous.
- To see this, note that if $(p_n, m_n) \to (p, x)$ and $(p^n x_n \leq m^n)$ for all n, then $px \leq m$.
- Its lower semi-continuous
- A bit of algebra shows this.