vector for this purpose. Note that the dimensions are such that sAt and sBt are both scalars. Then for given A and B, we say that (s,t) is a *Nash equilibrium* for the bimatrix game (A,B) if $sAt = \sup\{s'At : s' \in S\}$ and $sBt = \sup\{sBt' : t' \in T\}$. For a given (A,B), there is no obvious reason to suppose that some pair (s,t) forms a Nash equilibrium for (A,B), but in fact a very nice application of fixed-point theory shows that this is so. For for any (A,B), let $\mathbf{E}(A,B)$ be the set of (s,t) that are Nash equilibria for (A,B).

A4.1. Functions and Correspondences

A *function* is a mathematical object that associates to every point in one set (the *domain* of the function) a single point in a second set (the *range*). The definition of a function specifies that each point in the domain is mapped into a single point in the range. (This is, two points in the domain may be mapped into the same point in the range; but a single point in the domain can't take on two different range values.)

A correspondence generalizes the idea of a function, allowing a point in the domain to be associated with more (or less) than one point in the range. Given sets X and Y, which are called the domain and range of the correspondence, a *correspondence* ϕ is a map that associates to every point $x \in X$ a *subset* $\phi(x)$ of Y.

In general, the domain X and range Y of a correspondence can be very general mathematical objects. In our work, however, X will always be a subset of R^n for some n, and Y will always be a *closed* subset of R^m, for some m. Accordingly, for all the applications we discuss, we can take limits of sequences of points in X or in Y, measure (Euclidean) distances between points in either set, and so on.

Note that a correspondence is nothing more than a fancy function, in the following sense. Let Y denote the set of all subsets of Y, then the correspondence ϕ is a function with domain X and range Y. But to emphasize that the targets are sets, the terminology *correspondence* is used. To write symbolically that ϕ is a correspondence from X to Y—that is, $\phi(x) \subseteq Y$ for each $x \in X$—we write $\phi : X \Rightarrow Y$ and $x \Rightarrow \phi(x)$, using a double arrow. (In comparison, $\phi : X \rightarrow Y$ is used to indicate that ϕ is a function from X to Y.)

In the example of Marshallian demand, two correspondences are worthy of attention. The more obvious, perhaps, is the correspondence that maps price–income pairs (p,y) into the consumer's Marshallian demand, or $\mathbf{D} : R^k_{++} \times R_+ \Rightarrow R^k_+$. But there is also the *budget set* correspondence that defines, for each p and y, the consumer's feasible set: If $\mathbf{B}(p,y)$ denotes $\{x \in R^k_+ : p \cdot x \leq y\}$, then $\mathbf{B} : R^k_{++} \times R_+ \Rightarrow R^k_+$ is a second correspondence.

In the example of equilibria of bimatrix games, $\mathbf{E} : R^{m\times n} \times R^{m\times n} \Rightarrow R^m_+ \times R^n_+$ is the equilibrium correspondence; $\mathbf{E}(A,B)$ is the set of (s,t) that are Nash equilibria for (A,B). Note that (s,t) actually come from a smaller space than the range $R^m_+ \times R^n_+$ written here; they are probability distributions, whose components are both nonnegative (which is indicated) and sum to 1 (which is not).

Language commonly used concerning correspondences includes:

1. If the correspondence ϕ never maps a point in the domain into the empty set, we say that ϕ is *nonempty valued*. So, for example, because u is continuous and the budget sets are compact, **D** is nonempty valued; because of a fixed-point theorem, **E** is nonempty valued.
2. If $\phi(x)$ is convex for every $x \in X$, we say that ϕ is *convex valued*. So, if u is quasi-concave, **D** is convex valued. But (for those who know a smattering of game theory), because of games such as the Battle of the Sexes, **E** is most definitely *not* convex valued.
3. If the correspondence ϕ maps every point in the domain into a set containing a single element, we say that ϕ is *singleton valued*. Therefore, if u is strictly quasi-concave, the Marshallian demand correspondence is singleton valued.
4. If $\phi(x)$ is closed for every $x \in X$, we say that ϕ is *closed valued*. If $\phi(x)$ is compact for every $x \in X$, we say that ϕ is *compact valued*. As in: as part of the proof that **D** is nonempty valued, one must prove that **B** is a compact-valued correspondence. Or, since one can prove that the set of equilibria for any game is a closed set, and since equilibria (being pairs of probability distributions) are clearly bounded, **E** is compact valued.
5. The *graph* of the correspondence $\phi : X \rightrightarrows Y$ is the set of points $\{(x, y) \in X \times Y : y \in \phi(x)\}$.

A property of correspondences that we should spell out more formally is given in the following definition.

Definition A4.1. *The correspondence ϕ is* ***locally bounded*** *if for every $x \in X$, there exists an $\epsilon(x) > 0$ and a bounded set $Y(x) \subseteq Y$ such that $\phi(x') \subseteq Y(x)$ for all x' that are less than ϵ distant from x.*

To paraphrase, the range Y may be unbounded but, for every point x in the domain, we can simultaneously bound $\phi(x')$ for all x' in some (small) neighborhood of x.

Exercise 1. Construct a correspondence that is singleton-valued, and hence compact- and convex-valued, and whose graph is a closed set, but that is not locally bounded. (Hint: Consider the function $f(x) = 1/x$ around $x = 0$.)

A4.2. Continuity of Correspondences

In parallel with the idea of continuity of a function, correspondences have continuity properties. Specifically, three notions of continuity are important: upper semi-continuity, lower semi-continuity, and (full) continuity. Definitions vary from book to book, but for our purposes, the following will do.

Definition A4.2.

a. *The correspondence $\phi : X \Rightarrow Y$ is* ***upper semi-continuous*** *if, whenever $\{x_n\}$ is a sequence in X with limit $x \in X$, and $\{y_n\}$ is a sequence in Y such that $y_n \in \phi(x_n)$ for all n and $\lim_n y_n$ exists, then this limit point is an element of $\phi(x)$.*

b. *The correspondence $\phi : X \Rightarrow Y$ is* ***lower semi-continuous*** *if for every $x \in X$, sequence $\{x_n\}$ from X with limit x, and $y \in \phi(x)$, we can find, for all $n > N$ for some sufficiently large N, $y_n \in \phi(x_n)$ such that $\lim_n y_n = y$.*

c. *The correspondence $\phi : X \Rightarrow Y$ is* ***continuous*** *if it is both upper and lower semi-continuous.*

The meaning of upper semi-continuity can be partially divined from the following alternative characterization.

Proposition A4.3. *The correspondence $\phi : X \Rightarrow Y$ is upper semi-continuous if and only if the graph of ϕ is (relatively) closed in $X \times Y$.*

Proof. This is less of a proposition than a remark, since it is just a matter of recalling the "sequential convergence" definition of a (relatively) closed set. ϕ is upper semi-continuous per Definition A4.2a, and $\{(x_n, y_n)\}$ is a sequence from the graph of ϕ with limit (x, y). Since $\{x_n\}$ has limit x, Definition A4.2a implies that $y \in \phi(x)$, holds. This is just another way of saying that (x, y) is in the graph of ϕ. Therefore, the graph of ϕ is a closed set. On the other hand, suppose the graph of ϕ is a closed set and that $\lim_n x_n = x$, $\lim_n y_n = y$, and $y_n \in \phi(x_n)$ for each n. Then $\{(x_n, y_n)\}$ is a sequence from the graph of ϕ with limit point (x, y), and since the graph of ϕ is closed, this means that (x, y) is in the graph, which is just to say that $y \in \phi(x)$.

The only part of this that requires more than a moment's thought concerns the parenthetical *relatively* in the statement of the proposition. Because X is not necessarily a closed set, when we say that the graph of ϕ is closed, we can't mean closed in the full Euclidean space that contains $X \times Y$, but instead we must mean that it is *relatively* closed within $X \times Y$. That characterization of a relatively closed set that we've used—that it contains all its limit points within the space—shows this. ∎

Here is another very simple result.

Proposition A4.4. *If a correspondence ϕ is upper semi-continuous, then $\phi(x)$ is closed for each x. Therefore, if a correspondence ϕ is upper semi-continuous and locally bounded, it is compact valued.*

Proof. If ϕ is upper semi-continuous, its graph is closed. Since the intersection of two closed sets is closed, and $\{x\} \times Y$ is a closed set, the intersection of the graph of ϕ and $\{x\} \times Y$, or $\{x\} \times \phi(x)$, is closed. Therefore, $\phi(x)$ is closed.[1] And if the correspondence is, in addition, locally bounded, then it is bounded at each x and, therefore, compact valued. ∎

Some examples with pictures may help you understand these definitions.

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[1] Since Y is always a closed set, we don't need in this case to worry about relative closures.

1. Figure A4.1(a) shows the correspondence ϕ defined for $X = Y = R$ as follows:

$$\phi(x) = \begin{cases} \{4-x, 2-x\} & \text{for } x < 2, \\ [2-x, 4-x] & \text{for } 2 \le x \le 3, \text{ and} \\ \{x-3\} & \text{for } x > 3. \end{cases}$$

That is, $\phi(x)$ contains two points for $x < 2$, an entire interval of points for $2 \le x \le 3$, and a single point for $x > 3$.

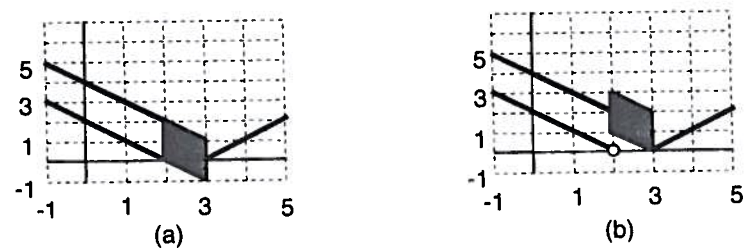

Figure A4.1. Two correspondences

This correspondence is upper semi-continuous. Perhaps the easiest way to see this is that its graph is a closed set. But it is not at all lower semi-continuous. It fails at $x = 2$ and at $x = 3$. For instance, $1 \in \phi(2)$. But as you approach $x = 1$ from below, say along the sequence in which $x_n = 2 - 1/n$, you can't get "close" to the value 1; you can find a sequence approaching the value $y = 2$ and another approaching $y = 0$, but $y = 1$ can't be approached.

2. Now consider the correspondence $\phi' : R \Rightarrow R$ defined by

$$\phi'(x) = \begin{cases} \{4-x, 2-x\} & \text{for } x < 2, \\ [3-x, 5-x] & \text{for } 2 \le x \le 3, \text{ and} \\ \{x-3\} & \text{for } x > 3. \end{cases}$$

This is graphed for you in Figure A4.1b; the open circle indicates a point that is not part of the correspondence. This is just like the previous example, except that for domain values $x \in [2, 3]$, the correspondence has been "shifted up." This is not an upper semi-continuous correspondence: It should be clear to you that the graph is not closed; it lacks the accumulation point $(2, 0)$. (If it had that point, the graph would be closed and the correspondence would be upper semi-continuous.) It also fails to be lower semi-continuous, both at $x = 2$ and at $x = 3$. But imagine changing it at $x = 2$ and $x = 3$ as follows: Suppose the correspondence had $\phi(2) = \emptyset$ and $\varphi(3) = \{0\}$. Then it would be lower semi-continuous. It is lower semi-continuous at $x = 2$ trivially; since $\phi(2) = \emptyset$, the definition has no bite there. And at $x = 3$, no matter how you approach x, there are values in the correspondence along the sequence that approach the one point, 0, in the correspondence at the limit.

Exercise 2. Are either the following correspondences upper semi-continuous? Why or why not? (You should begin by graphing them.)

$$\phi_1(x) = \begin{cases} \{4 - x, 2 - x\} & \text{for } x \leq 2, \\ [2 - x, 4 - x] & \text{for } 2 < x \leq 3, \text{ and} \\ \{x - 3\} & \text{for } x > 3. \end{cases}$$

$$\phi_2(x) = \begin{cases} \{4 - x, 2 - x\} & \text{for } x < 2, \\ [0, 6] & \text{for } x = 2, \\ [3 - x, 5 - x] & \text{for } 2 < x \leq 3, \text{ and} \\ \{x - 3\} & \text{for } x > 3. \end{cases}$$

If either (or both) is not upper semi-continuous, what is the smallest change you can make to make it upper semi-continuous? And is either lower semi-continuous? If not, are there simple changes you can make at the values $x = 2$ and $x = 3$ to make them lower semi-continuous?

A4.3. Singleton-valued Correspondences and Continuity

For reasons that will become apparent next section, the case of singleton-valued correspondences is of special interest. Throughout this section, we deal with a function $f : X \to Y$ and a correspondence $\phi : X \Rightarrow Y$ where $\phi(x) = \{f(x)\}$. We say that ϕ *describes* the function f, f is the function *associated with* the correspondence ϕ, and ϕ is the (singleton-valued) correspondence *associated with* the function f.

Proposition A4.5. *A singleton-valued correspondence ϕ is lower semi-continuous if and only if it describes a continuous function, in which case it is also upper semi-continuous.*

Proof. Suppose ϕ is lower semi-continuous. Take any $x \in X$ and sequence $\{x_n\}$ with limit x. Since $f(x) \in \phi(x)$, lower semi-continuity says that, for some sequences $\{y_n\}$ with $y_n \in \phi(x_n)$, $\lim_n y_n = f(x)$. But the only choice for y_n is $f(x_n)$. Therefore, f is continuous. Conversely, for $x \in X$ and $\{x_n\}$ with limit x, continuity of f ensures that $\lim_n f(x_n) = f(x)$. Therefore, for all $y \in \phi(x)$, and there is but one such y, namely $f(x)$, we can find $y_n \in \phi(x_n)$, namely $f(x_n)$, with limit $f(x)$. This is lower semi-continuity. And, in either case, for any $\{x_n\}$ with limit x, if $y_n \in \phi(x_n)$, then $y_n = f(x_n)$, and the (only) accumulation point of $y_n = \lim_n f(x_n) = f(x) \in \phi(x)$, which is upper semi-continuity. ■

On the other hand, in this special case of singleton-valued correspondences, upper semi-continuity does not ensure continuity of the corresponding function. Consider $X = [0, \infty)$ and $f(x) = 1/x$ for $x > 0$ and $f(0) = 0$. This function is very discontinuous at $x = 0$, but the singleton-valued correspondence associated with this function is upper semi-continuous.

For each θ, maximize (in z) $F(z,\theta)$, subject to $z \in A(\theta)$.

For a parametric constrained maximization problem of this form, we let $Z(\theta)$ be the set of solutions of the problem for parameter value θ—that is, $Z(\cdot)$ is a correspondence with argument θ, giving (as values) sets of z—and we let $f(\theta) = \sup\{F(z,\theta); z \in A(\theta)\}$.[3]

An excellent example of a parametric constrained maximization problem is the consumer's problem (CP). The parameter θ is the vector of prices and income (p, Y), and the (vector) variable z is the consumption bundle x chosen by the consumer. The correspondence of solutions is the Marshallian demand correspondence, and the function f is the indirect utility function. Note that in this case, the objective function $u(x)$ is independent of the parameters (p, y), but the constraint set for x, $\mathbf{B}(p, y)$, does change with changes in (p, y).

Economic theory repeatedly uses the following proposition. The version given here is a bit "fancier" than is standard; see the comments following the statement of the corollary.

Proposition A4.7 (Berge's Theorem, also known as the Theorem of the Maximum). *Consider the parametric constrained-maximization problem*

$$\textit{Maximize } F(z,\theta), \textit{ subject to } z \in A(\theta).$$

Let $Z(\theta)$ be the set of solutions of this problem for the parameter θ, and let $f(\theta) = \sup\{F(z,\theta); z \in A(\theta)\}$. If

a. F is a continuous function in (z,θ),

b. $\theta \Rightarrow A(\theta)$ is lower semi-continuous, and

c. there exists for each θ a set $B(\theta) \subseteq A(\theta)$ such that $Z(\theta) \subseteq B(\theta)$, $\sup\{F(z,\theta) : z \in B(\theta)\} = \sup\{F(z,\theta) : z \in A(\theta)\}$, and $\theta \Rightarrow B(\theta)$ is an upper semi-continuous and locally bounded correspondence.[4]

Then:

d. $Z(\theta)$ is nonempty for all θ, and $\theta \Rightarrow Z(\theta)$ is an upper semi-continuous and locally bounded correspondence; and

e. the function $\theta \to f(\theta)$ is continuous.

Identical conclusions hold if the optimization problem calls for minimizing F rather than maximizing F.

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[3] We write *sup* instead of *max* in the definition of f to cover the case where there is no $x \in A(\theta)$ that is optimal.

[4] We could, in theory, consider a parametric constrained-maximization problem in which, for some values of θ, $A(\theta) = \emptyset$. For those values of θ, $Z(\theta)$ would be empty and $f(\theta)$ would be $-\infty$. But we have just ruled this out, at least as far as this result is concerned. Please note that this assumption doesn't establish that $Z(\theta)$ is nonempty; we'll need to prove this. (It isn't hard.) But it does establish that $f(\theta) \neq -\infty$, because it tells us that $A(\theta)$ is nonempty.

Before giving the proof of Berge's Theorem, we give a corollary that shows why we were interested last section in singleton-valued correspondences.

Corollary A4.8. *In the situation of Proposition A4.7, if in addition you know that* $Z(\theta)$ *is a singleton set* $\{z(\theta)\}$ *for all* θ *in some (relatively) open set of parameter values, then* $z(\theta)$ *is a continuous function over that set of parameter values.*

Proof of the corollary. Berge's Theorem establishes that the solution correspondence Z is upper semi-continuous and locally bounded. Apply Proposition A4.6. ■

In standard statements of this theorem, $B(\theta)$ doesn't appear; it is assumed that $A(\theta)$ is continuous. But the more general version of the result given here permits smoother application of the result in some cases encountered in the text.

Proof of Berge's Theorem. Since B is upper semi-continuous and locally bounded, it is compact valued (Proposition A4.4). Nonemptiness of $Z(\theta)$ for all θ then follows directly from Proposition A2.21, because F is continuous in z and $B(\theta)$ is nonempty and compact for each θ. This also implies that $f(\theta) < \infty$ for all θ.

Next we establish continuity of f. Suppose $\{\theta_n\}$ is a sequence of parameter values with limit θ. Let $\{z_n\}$ be a sequence of corresponding solutions; that is, $z_n \in Z(\theta_n)$ for each n. Therefore, $z_n \in B(\theta_n)$ for each n. Because B is locally bounded, by looking along a subsequence if necessary, we can assume that z_n converges to some z, which by the upper semi-continuity of B lies in $B(\theta)$. Therefore, $f(\theta) \geq F(z,\theta) = \lim_n F(z_n,\theta_n)$ (by continuity of F) $= \lim_n f(\theta_n)$. On the other hand, suppose $z' \in Z(\theta)$. Because A is lower semi-continuous, there exists a sequence $\{z'_n\}$ such that $z'_n \in A(\theta_n)$ and $\lim_n z'_n = z'$. Therefore, $\lim_n f(\theta_n) \geq \lim_n F(z'_n,\theta_n) = F(z',\theta) = f(\theta)$. (The first limit exists by the preceding argument, and the second exists by continuity.) Therefore, $f(\theta) = \lim_n f(\theta_n)$, and f is continuous.

Now we show that Z is upper semi-continuous and locally bounded. By assumption, B is locally bounded. Since $Z(\theta) \subseteq B(\theta)$ by assumption, Z is locally bounded. Suppose $\{\theta_n\}$ and $\{z_n\}$ are sequences of parameters and variables such that $\lim_n \theta_n = \theta$, $\lim_n z_n = z$, and $z_n \in Z(\theta_n)$ for all n. Of course, $z_n \in B(\theta_n)$. Since $\lim_n z_n = z$ and $\lim_n \theta_n = \theta$, it follows that $\lim_n F(z_n,\theta_n) = F(z,\theta)$ by the continuity of F. But since $z_n \in Z(\theta_n)$, $F(z_n,\theta_n) = f(\theta_n)$, and since (as we just showed) f is continuous, $\lim_n f(\theta_n) = f(\theta)$. Therefore, $F(z,\theta) = f(\theta)$, and $z \in Z(\theta)$. ■

A4.5. Why This Terminology?

Why do we use the terminology upper and lower semi-continuity in Definition A4.2? What connects these definitions to how these terms are used for functions?

When we discuss upper and lower semi-continuity of functions, two mathematical concepts are at work: we need a sense of convergence in the domain and the range, and we need a sense of order in the range. Order enters the story when we say stuff like, $\limsup f(x_n) \leq f(x)$, for upper semi-continuity.