Microeconomic Foundations I:
Choice and Competitive Markets

Student's Guide

Chapter 1: Choice, Preference, and Utility

This chapter discusses the basic microeconomic models of consumer choice, preference, and utility. It is very abstract, consisting primarily of proofs of mathematical (deductive) propositions. If you are rusty at reading (and constructing) mathematical proofs, it may be painful. If you are rusty, or if you haven't done anything like this before, please take it slow. Be sure to follow the details of the proofs one step at a time; it helps to have a pad and pen or pencil by your side, so you can follow along, make notes, finish arguments, and so forth. This recommendation extends to the entire book and, indeed, to any book or article you are reading that is mathematical in character. But it comes with a complementary recommendation: If you read carefully and slowly for details, you may lose the "plot line," which is just as important. So my suggestion is to read this sort of thing at least twice, with pad and pen or pencil each time: First, read to get the big picture. What is the framework? What are the results? How do the results tie together? And then go back and read for the details: How is each step done?

In a few places, I leave the proofs of propositions for you to complete; unless you are very confident in your ability to do this, you should write out proofs and have them checked by someone—a peer, a TA, your instructor—who is well versed in this skill. Constructing mathematical proofs is a skill you learn best—and perhaps only—by doing. (The proof of Proposition 1.19 will arrive in Chapter 2. Try it if you wish, but it takes considerable mathematical sophistication.)

On pedagogical grounds, it would be nice to begin with something more concrete. But this is the logical starting point for consumer theory, which in turn is the logical starting point of microeconomics. Persevere until Chapter 3, and you'll get to an application—the theory of the consumer—that isn't quite so abstract.

Summary of the Chapter

The chapter is about the standard economic model of consumer choice.

1. A set of *objects of choice*, X, is given.

2. A *choice function* c is given that, for each nonempty subset A of X, tells us the *set* of objects $c(A)$ the consumer would be content to have. We require that $c(A) \subseteq A$. We allow for the possibility that $c(A) = \emptyset$. The consumer gets only one element out of A; if $c(A)$ contains more than one element, the interpretation is that the consumer would be equally happy with any one. A more general formulation would have as domain of c some collection $\mathcal{A}$ of nonempty subsets of X, but for the balance of this chaper, we simplify by assuming that $c(A)$ is defined for *all* the nonempty subsets A of X, and we let $\mathcal{A}$ denote this domain.

3. The *preferences* of the consumer are specified by a binary relation $\succeq$, where $x \succeq y$ (for x and y from X) is read "x is as good as or better than y" or as "x is weakly preferred to y." Choice is generated by the preferences $\succeq$ if, for all A,

$$c(A) = \{x \in A : x \succeq y \text{ for all } y \in A\}. \tag{1.2}$$

   (Equation numbers are out of order here, so that they conform to the numbers in the text.)

4. A *utility function* for the consumer is a real-valued function $u : X \to R$, with the interpretation that the consumer regards items of higher utility as better. In accordance with this interpretation, we say that u represents the preference relation $\succeq$ if

$$x \succeq y \quad \text{if and only if} \quad u(x) \geq u(y). \tag{1.3}$$

   And the choice function c is generated by utility maximization with the utility function u if, for all A,

$$c(A) = \{x \in A : u(x) \geq u(y) \text{ for all } y \in A.\} \tag{1.1}$$

Most economic models have consumers who are utility maximizers or, at least, preference driven. The point of the chapter, then, is to say when choice behavior, given by a choice function c, is generated by preferences or by utility maximization for some utility function. The basic answers, in the context of a finite set X, are given by Definition 1.1 and Proposition 1.2:

Definition 1.1.

*a. A choice function c satisfies **finite nonemptiness** if $c(A)$ is nonempty for every finite $A \in \mathcal{A}$.*

b. *A choice function c satisfies* ***choice coherence*** *if, for every pair x and y from X and A and B from $\mathcal{A}$, if $x, y \in A \cap B$, $x \in c(A)$, and $y \notin c(A)$, then $y \notin c(B)$.*

c. *A preference relation on X is* ***complete*** *if for every pair x and y from X, either $x \succeq y$ or $y \succeq x$ (or both).*

d. *A preference relation on X is* ***transitive*** *if $x \succeq y$ and $y \succeq z$ implies that $x \succeq z$.*

Proposition 1.2. *Suppose that X is finite.*

a. *If a choice function c satisfies finite nonemptiness and choice coherence, then there exist both a utility function $u : X \to R$ and a complete and transitive preference relation $\succeq$ that produce choices according to c via the formulas (1.1) and (1.2), respectively.*

b. *If a preference relation $\succeq$ on X is complete and transitive, then the choice function it produces via formula (1.2) satisfies finite nonemptiness and choice coherence, and there exists a utility function $u : X \to R$ such that*

$$x \succeq y \text{ if and only if } u(x) \geq u(y). \tag{1.3}$$

c. *Given any utility function $u : X \to R$, the choice function it produces via formula (1.1) satisfies finite nonemptiness and choice coherence, the preference relation it produces via (1.3) is complete and transitive, and the choice function produced by that preference relation via (1.2) is precisely the choice function produced directly from u via (1.1).*

In words, choice behavior (for a finite X) that satisfies finite nonemptiness and choice coherence is equivalent to preference-maximization (that is, formula (1.2)) for complete and transitive preferences, both of which are equivalent to utility-maximization (via formulas (1.1) and (1.3)). Whether expressed in terms of choice, preference, or utility, this conglomerate (with the two pairs of assumptions) is the standard model of consumer choice in microeconomics.

The chapter goes on to prove and generalize Proposition 1.2, and to provide complements to it. This includes the following:

1. Those results that extend automatically to infinite sets X are extended.

2. For a complete and transitive preference relation $\succeq$, strict preference $\succ$ is defined by $x \succ y$ if $x \succeq y$ and not $y \succeq x$, and indifference $\sim$ is defined by $x \sim y$ if $x \succeq y$ and $y \succeq x$. Properties of these two relations are derived, and the derivation of $\succeq$ from $\succ$ is discussed.

3. Define the *no better than x* set $\text{NBT}(x) := \{y \in X : x \succeq y\}$. If $\succeq$ is complete and transitive, then $x \succeq y$ if and only if $\text{NBT}(y) \subseteq \text{NBT}(x)$, with strict set inclusion if and only if $x \succ y$. The no-better-than sets are used in many ways; in particular, they allow the construction of utility functions that represent $\succeq$.

4. Utility representations for infinite sets X are considered. Necessary and sufficient conditions on a complete and transitive binary relation on (infinite) X to have a

utility representation are provided; then the important special case of *continuous preferences* when X is a (nice) subset of finite dimensional Euclidean space (R^k) is discussed in detail.

5. The phenomenon of $c(A) = \emptyset$ for infinite sets A is discussed.
6. The relationship between two different utility functions for the same preferences is given, making in particular the point that utility numbers (in this chapter) have only ordinal and not cardinal significance.
7. A number of comments, extensions, variations, and criticisms of the standard model are provided.

Solutions to Starred Problems

■ 1.1. In case you had problems producing a counterexample, consider the four bottles x = California Red for \$20, x' = French white for \$20, x'' = California Red for \$25, and x''' =French red for \$30. With this list of bottles, you can produce a counterexample to the choice coherence axiom, using either two wine lists of three bottles apiece, or one having three bottles and another having two.

Here is one example:

From the wine list $L_1 = \{x, x', x''\}$, my friend's choice algorithm proceeds as follows: Two California bottles and one French, so take a California bottle. Two California reds, so take a red. Take the most expensive California Red, which is x''. Thus $c(\{x, x', x''\}) = \{x''\}$.

From the wine list $L_2 = \{x', x'', x'''\}$, he reasons: One California and two French, so take one of the French. Of the two bottles of French wine, one is white and one red. He must invoke his tie-breaking rule, which leads him to choose white. Since there is only one bottle of French white on the list, he chooses that: $c(\{x', x'', x'''\}) = \{x'\}$.

This constitutes a violation of choice coherence. Both x' and x'' are available on both wine lists, and x' but not x'' is chosen from the second list while x'' and not x' is chosen from the first.

■ 1.3. It is easy to see that $\succeq^*$ is complete: For any x and y, since $\succeq_{\text{Larry}}$ is complete, either $x \succeq_{\text{Larry}} y$ or $y \succeq_{\text{Larry}} x$, which immediately imply $x \succeq^* y$ and $y \succeq^* x$, respectively.

To see that $\succeq^*$ is not transitive, suppose that X has three elements, x, y, and z. Suppose Larry ranks the three $x \succ_{\text{Larry}} y \succ_{\text{Larry}} z$, while Moe ranks them $y \succ_{\text{Moe}} z \succ_{\text{Moe}} x$. Then $z \succeq^* x$, because Moe likes z at least as much as x. And $x \succeq^* y$, because Larry likes x at least as much as y. But it is not true that $z \succeq^* y$, because both Larry and Moe think that y is strictly better than z.

■ 1.6. (a) Suppose that $x \succeq y$ and $y \succeq z$ for $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k)$, and

$z = (z_1, \ldots, z_k)$. Then $x_i \geq y_i$ and $y_i \geq z_i$ for each i. Hence, by transitivity of $\geq$ for real numbers, $x_i \geq z_i$ for all i, and thus $x \succeq z$. This shows that $\succeq$ is transitive.

On the other hand, for $k = 2$, $x = (1, 2)$, and $y = (2, 1)$, neither $x \succeq y$ nor $y \succeq x$; $\succeq$ is not complete.

(b) $x \succ y$ if $x \succeq y$ and not $y \succeq x$, which is $x_i \geq y_i$ for all i, and not $y_i \geq x_i$ for all i, which is $x_i \geq y_i$ for all i, and $x_i > y_i$ for some i.

This is asymmetric: If $x \succ y$, then $x_i > y_i$ for some i. Thus neither $y \succeq x$ nor $y \succ x$ are possible.

But this is not negatively transitive: Take $x = (2, 2)$, $z = (1, 1)$, and $y = (3, 0)$. We have $x \succ z$, but neither $x \succ y$ nor $y \succ z$ is true.

(c) $x \sim y$ if $x \succeq y$ and $y \succeq x$, which is $x_i \geq y_i$ and $y_i \geq x_i$ for all i, which is $x_i = y_i$ for all i. Thus we have

$$x \sim y \text{ if and only if } x = y.$$

This is clearly reflexive, symmetric, and (trivially) transitive.

■ 1.7. As suggested in the hint, the first thing to do is to characterize what *not* $y \succeq x$ means. For $y \succeq x$, two (or more) of y's three components must be at least as large as the corresponding components of x. For this to fail, two (or more) of those three components must be strictly less than the corresponding components of x. Thus

not $y \succeq x$ is equivalent to

two or more of x's components strictly exceed y's corresponding components.

(If you couldn't solve this problem because you didn't get this far, assume the characterization above and try parts (a) and (b) again.) If you aren't satisfied with this verbal argument, a very formal argument can be given, but it is gruesome. Here it is:

(i) $y \succeq x$ by definition is $[y_1 \geq x_1$ and $y_2 \geq x_2]$ or $[y_1 \geq x_1$ and $y_3 \geq x_3]$ or $[y_2 \geq x_2$ and $y_3 \geq x_3]$.

(ii) *Not [a or b or c]* is *[not a] and [not b] and [not c]*, so *not* $y \succeq x$ is

[not $[y_1 \geq x_1$ and $y_2 \geq x_2]]$ and [not $[y_1 \geq x_1$ and $y_3]]$ and [not $[y_2 \geq x_2$ and $y_3 \geq x_3]]$.

(iii) The negation of *a and b* is *not a or not b*, and the negation of $a \geq b$ is $b > a$, for real numbers a and b, so *not* $y \succeq x$ is

$[x_1 > y_1$ or $x_2 > y_2]$ and $[x_1 > y_1$ or $x_3 > y_3]$ and $[x_2 > y_2$ or $x_3 \geq y_3]$.

(iv) This, in turn, has the form $[\alpha$ or $\beta]$ and $[\alpha$ or $\gamma]$ and $[\beta$ or $\gamma]$. Either by considering the two cases α and not α or by constructing a Venn diagram, you can show that this is the same as [α and β] or [α and γ] or [β and γ]; i.e., two of the three must be true. Translating this back to the components of x and y, this is the desired conclusion.

As promised, this is rather gruesome, and you probably came to the correct conclusion without all these details, but if you are a freak for mathematical rigor, the mess just previous should make you happy.

(a) With this result, however obtained, the problem is easy. First, to show that $\succeq$ is complete, take any x and y. If it is not the case that $y \succeq x$, then x strictly exceeds y in at least two components, and so $x \succeq y$. Thus $\succeq$ is complete.

To show that $\succeq$ is not transitive, an example will do: $(2,2,1) \succeq (1,1,3)$, and $(1,1,3) \succeq (3,0,2)$, but it is not true that $(2,2,1) \succeq (3,0,2)$.

(b) Since not $y \succeq x$ implies $x \succeq y$ (see just previously), $x \succ y$ is equivalent to not $y \succeq x$, which we saw means that x strictly exceeds y in at least two components. This is clearly asymmetric: If x exceeds y in at least two components, then y can strictly exceed x in at most one. But negative transitivity fails, and the same example as we used before will work: $(2,2,1) \succ (1,1,3)$, but if the third bundle is $(3,0,2)$, then neither $(2,2,1) \succ (3,0,2)$ nor $(3,0,2) \succ (1,1,3)$.

■ 1.10. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. If $x_1 \neq y_1$, then either $x_1 > y_1$, in which case $x \succeq y$, or $y_1 > x_1$, in which case $y \succeq x$. And if $x_1 = y_1$, then either $x_2 \geq y_2$, implying $x \succeq y$, or $y_2 \geq x_2$, implying $y \succeq x$. Thus $\succeq$ is complete.

Suppose that $x = (x_1, x_2) \succeq y = (y_1, y_2)$, and $y \succeq z = (z_1, z_2)$. Since $x \succeq y$, either $x_1 > y_1$ or [$x_1 = y_1$ and $x_2 \geq y_2$]. Similarly, $y \succeq z$ implies either $y_1 > z_1$ or [$y_1 = z_1$ and $y_2 \geq z_2$]. It is boring, but the easiest way to proceed is to take all four = two-by-two cases seriatum:

Case 1: $x_1 > y_1$ and $y_1 > z_1$. In this case $x_1 > z_1$, so $x \succeq z$.

Case 2: $x_1 > y_1$ and [$y_1 = z_1$ and $y_2 \geq z_2$]. In this case $x_1 > z_1$, so $x \succeq z$.

Case 3: [$x_1 = y_1$ and $x_2 \geq y_2$] and $y_1 > z_1$. In this case $x_1 > z_1$, so $x \succeq z$.

Case 4: [$x_1 = y_1$ and $x_2 \geq y_2$] and [$y_1 = z_1$ and $y_2 \geq z_2$]. In this case $x_1 = z_1$ and $x_2 \geq z_2$, so $x \succeq z$.

In all four possible cases, we conclude $x \succeq z$, so $\succeq$ is transitive.

To show that there is no numerical representation, we could prove that no countable set X^* as is required for a numerical representation can be found. This is relatively easy to do: For every real number $r \in [0,1]$, consider the points $x_r = (r, 0.8)$ and $y_r = (r, 0.2)$. By the definition of the preference relation, $x_r \succ y_r$, so if a set X^* existed, it would have to contain a point x_r^* that lies between these two, perhaps tied with x_r. The set of candidates for x_r^* is $\{(r,q) : 0.8 \geq q > 0.2\}$. But this implies that for any two different real numbers r and r', $x_r^* \neq x_{r'}^*$, and since there are uncountably many r, the set X^* must have an uncountable number of elements.

An alternative proof is a bit more direct. Assume that u is a numerical representation for $\succeq$. Since for every $r \in [0,1]$, $x_r = (r, 0.8) \succ (r, 0.2) = y_r$, it follows that $u(x_r) > u(y_r)$. But then, since the rationals are dense in the real line, it follows that for every $r \in [0,1]$, there is a rational number q_r in the open interval $(u(y_r), u(x_r))$. This would constitute a one-to-one map from the the unit interval $[0,1]$ onto the rational numbers, which of course cannot be, since there are uncountably many elements of $[0,1]$ and only countably many rationals.

■ 1.11. The first step is to show that *c holds if and only if d holds.* Note that if $\succeq$ is complete and transitive, then for all pairs x and y, either $x \succeq y$ or $y \succ x$, but never both. Therefore, the sets $\text{NBT}(x) = \{y \in R^k_+ : x \succeq y\}$ and $\text{SBT}(x) = \{y \in R^k_+ : y \succ x\}$ are complements, and NWT(x) and SWT(x) are complements. Hence, the sets NBT(x) and NWT(x) are both closed if and only if their complements, SBT(x) and SWT(x) are (relatively) open.

Next I'll show that *c and d imply the original definition*: (This is by far the longest step.) Suppose $x \succ y$. I need to produce a w such that $x \succ w \succ y$. Here is one way to do it: Look at all convex combinations of x and y, $ax + (1-a)y$, for $a \in [0,1]$. Since c holds for $\succeq$, the set NWT(x) is closed, and therefore $NWT(x) \cap \{ax + (1-a)y : a \in [0,1]\}$ is a closed set. This intersection contains $a = 1$ and does not include $a = 0$ and, being closed, it contains its infimum (in terms of a); that is, if we let $a^* = \inf\{a \in [0,1] : ax + (1-a)y \succeq x\}$, we know that $a^*x + (1-a^*)y \succeq x$, $a^* > 0$, and, for all $a \in [0, a^*)$, $x \succ ax + (1-a)y$. Since the set NBT(x) is also closed, and $a^*x + (1-a^*)y$ can be approached by points all in NBT(x) (namely, $ax + (1-a)y$ for $a < a^*$), we know that $x \succeq a^*x + (1-a^*)y$; we conclude that $a^*x + (1-a^*)y \sim x \succ y$. Let z denote $a^*x + (1-a^*)y$. Now repeating the argument (but with inequalities reversed), we can find a $b^* < 1$ such that $b^*z + (1-b^*)y \sim y$ and $bz + (1-b)y \succ y$ for all $b \in (b^*, 1]$. Let $w = bz + (1-b)y$ for any $b \in (b^*, 1)$; note that w is a convex combination of x and y with weight less than a^* on x, so putting everything together, we know that $x \succ w \succ z$.

Now that I have w, the rest of this step is easy. I know that $x \in \text{SBT}(w)$ and this set is open, there is an open neighborhood of x, say all x' within $\epsilon_1 > 0$ of x, that is in SBT(w). And I know that $y \in \text{SWT}(w)$ and this set is open, so for some $\epsilon_2 > 0$, every y' within ϵ_2 of y is in SWT(w). But then letting $\epsilon = \min\{\epsilon_1, \epsilon_2\}$, for all x' within ϵ of x and y' within ϵ of y, $x' \succ w \succ y'$, and by transitivity, $x' \succ y'$.

Now to show that *the definition implies b*: Suppose $\{x_n\}$ is a sequence with limit x and $x \succ y$. Per the definition, we can find $\epsilon > 0$ so that all x' within ϵ of x and all y' within ϵ of y, $x' \succ y'$. Taking $y' = y$, this tells us that all x' within ϵ of x satisfy $x' \succ y$. But since the sequence has limit x, for all sufficiently large n, x_n will be within ϵ of x. The other half is similar.

Next, I'll show that *b implies a*. Suppose that b holds, $\{x_n\}$ is a sequence with limit x, and $x_n \succeq y$ for all n If it is not true that $x \succeq y$, then $y \succ x$ must be true. But if b holds, then $y \succ x$ and $\lim_n x_n = x$, then it must be that $y \succ x_n$ for all sufficiently large n. To the contrary, the assumption is that $x_n \succeq y$ for all n. We've derived a contradiction to the assumption that $x \not\succeq y$. The argument for the other half is similar.

To conclude, I have to show that *a implies c and d*. It should be clear that showing *a implies c* is the way to go: To prove that NBT(x) is closed, I'll show that it contains all its limit points: Suppose $\{y_n\}$ is a sequence drawn from NBT(x) with limit y. Then $x \succeq y_n$ for each n. But then a tells us that $x \succeq y$, and $y \in$ NBT(x). The other half is similar.

▪ 1.13. The proposition concerns the choice function c, so the first step is to note that since c satisfies nonemptiness, choice coherence, and Assumption 1.16, Proposition 1.17 tells us the $c \equiv c_{\succeq_c}$. So to show that $c(A) \neq \emptyset$ for every compact A, we must show that, for each compact A, $c_{\succeq_c}(A) = \{x \in A : x \succeq_c y \text{ for all } y \in A\}$ is nonempty. (We know, of course, that $\succeq_c$ is complete and transitive, and the problem tells us to assume that $\succeq_c$ is continuous.)

So suppose, by way of contradiction, that for some compact set A, $\{x \in A : x \succeq_c y$ for all $y \in A\}$ is empty. That is, for every $x \in A$, there is some $y \in A$ such that $x \not\succeq_c y$, or $y \succ_c x$. This means that for every $x \in A$, there is some $y \in A$ such that $x \in \text{SWT}(y) = \{z \in X : y \succ_c z\}$. So if we look at the union $\cup_{y \in A} \text{SWT}(y)$, this union contains all of A; the sets SWT(y) for $y \in A$ constitute a *cover* of A.

But Proposition 1.14 tells us that, if preferences $\succeq$ are continuous, then the strictly worse than y sets SWT(y) are all (relatively) open. Among the characterizations of compactness—in some sense, the basic characterization—is that for any compact set, every open cover of the set has a finite subcover. That is, for some finite collection $\{y_1, \ldots, y_n\}$ from A, A is a subset of the union of the SWT(y_j). We know that $c_{\succeq_c}(\{y_1, \ldots, y_n\})$ is nonempty from the finite nonemptiness property, so there is some $k = 1, \ldots, n$ such that $y_k \succeq_c y_j$ for $j = 1, \ldots, n$. But this $y_k \in A$, so it must be in some SWT(y_j) for some j. This is a contradiction; $y_k \in \text{SWT}(y_j)$ means that $y_j \succ_c y_k$. But y_k was chosen to satisfy $y_k \succeq_c y_j$ for all j. We have the desired contradition; there must be some $x \in A$ such that $x \succeq_c y$ for all $y \in A$, and $c(A) = c_{\succeq_c}(A)$ is nonempty. (Where in this proof did I use the very necessary assumption that A is nonempty?)

▪ 1.16. (a) The idea here is simple, once you see the trick. For any set X, consider the weak preferences $\succeq^0$ given by $x \succeq^0 y$ for all x and y in X. That is, everything is weakly preferred to everything, thus the consumer is indifferent among all options. For these preferences, $c_{\succeq^0}(A) = A$ for all A. And as long as we see the consumer making a single choice from any set A, we have no evidence against the theory. If we can't infer something about strict preference from the observable data, the theory has no implications.

One way we learn about strict preferences from the data is if the data purport to show, for each A, the *entire* set $c_{\succeq}(A)$. Then if $x \notin c_{\succeq}(A)$, we infer that $y \succ x$ for every $y \in c_{\succeq}(A)$. This is the case we deal with in part (b) of this problem. Another way is coming in Chapter 4; as foreshadowing for this, I describe this alternative, although my description may not make sense to you until you get to Chapter 4: In Chapter 4, we will make inferences about strict preferences under a joint hypothesis that the consumer's observed choices are preference-driven *and* the consumer is locally insatiable. Therefore if there is a ball of positive diameter around the point x, all of which is contained in A, and if what is chosen from A is some distance from x, then we know that there is something in A that is strictly preferred to x (local insatiability) which was not chosen, hence what was chosen is at least as good as something that is strictly preferred to x, and thus the thing chosen must be strictly better than x.

(b) Suppose $X = \{x, y, z\}$, and we observe the following choices out of the three two-element sets:

$$c(\{x, y\}) = \{x\}, \quad c(\{y, z\}) = \{y\}, \quad \text{and} \quad c(\{x, z\}) = \{z\}.$$

(If you didn't get this far, see if you can finish the argument from here. You have to show (a) that there are no direct violations of choice coherence in these data, and (b) these data are inconsistent with preference–driven choice; i.e., there is no complete and transitive $\succeq$ that, if used to choose, would produce these data.)

The argument that there is no direct violation of choice coherence in these data is: We never have two distinct sets A and B and two distinct elements w and v such that both w and v are in both A and B. Since the *if* part of Houthakker's axiom is never satisfied by these data, the axiom has no content for these data.

Nonetheless, if these data could be explained by some complete and transitive preferences $\succeq$, $c(\{x, y\}) = \{x\}$ would mean that $x \succ y$ must be true, $c(\{y, z\}) = \{y\}$ would imply $y \succ z$, and $c(\{x, z\})$ would imply $z \succ x$. Since strict preference is transitive (Proposition 1.9), this would imply $x \succ x$, which violates the asymmetry of $\succ$. Hence these data are inconsistent with our standard model of choice driven by complete and transitive weak preferences.

(c) Fixing X and c, define (for $A \subseteq X$) $b(A) = A \setminus c(A)$. That is, $b(A)$ is the set of "bad" (really, less than best) elements out of A.

Suppose the data are consistent with choice according to some complete and transitive $\succeq$. Then $x \succeq^r y$ implies $x \succeq y$: if $x \succeq^r y$, then for some k, $x \in c(A_k)$ while $y \in A_k$. But if c is consistent with choice according to $\succeq$, $x \in c(A_k)$ implies that $x \succeq z$ for all $z \in A_k$, and this includes y.

Moreover, $x \succ^r y$ implies $x \succ y$: $x \succ^r y$ implies that, for some A_k containing both x and y, $x \in c(A_k)$ but $y \notin c(A_k)$. The former implies that $x \succeq z$ for all $z \in A_k$. Now if $x \not\succ y$, then $y \succeq x$, and by transitivity of $\succeq$, $y \succeq z$ for all $z \in A_k$, which would imply $y \in c(A_k)$, a contradiction.

So suppose the data violate SGARP. This means there is some set $\{x_1, \ldots, x_m\}$ that $x_i \succeq^r x_{i+1}$ for $i = 1, \ldots, m-1$ and $x_m \succeq^r x_1$. But then by the previous two paragraphs, $x_i \succeq x_{i+1}$ for $i = 1, \ldots, m-1$, and so by transitivity of $\succeq$, $x_1 \succeq x_m$, which contradicts $x_m \succ x_1$. Any violation of SGARP rules out the possibility that $c(\cdot)$ can be rationalized by a complete and transitive $\succeq$, which is the first half of Proposition 1.23.

The second half of the proposition, that no violations of SGARP means that the data are consistent with some $\succeq$, is a good deal harder. Because it is easy to get lost in the details of the proof, I will take it in steps, by first proving and then applying an abstract lemmas.

The lemma concerns a finite set of objects K, on which is defined a pair of binary relations, P and I. A binary relation (in case you don't know) is a mathematical object that concerns pairs of elements of a given set. For k and k' from K, we write kPk' if k stands in relation P to k', and we write *not* kPk' if not. Examples of binary relations are weak preference, strict preference, and indifference. But there are many others, such as: If K is the set of all students in a class, we might define a binary relation B by kBk' if k is the brother of k'. Note that order is important; it is certainly possible that kBk' and not $k'Bk$ (if, for example, k' is k's sister). Or, to take another example, in the binary relation $\succ$, order is crucial. In fact, $\succ$ is asymmetric, meaning that $x \succ y$ implies that *not* $y \succ x$.

The binary relations P and I on the finite set K have the following properties:

Property 1: *kPk' implies not kIk'.* (By contraposition, the reverse is true as well.)

Property 2: *I is reflexive. That is, for all $k \in K$, kIk.* (Note that this, together with property 1, implies that for no k is it true that kPk.)

Property 3: *I is symmetric. That is, for all k and $k' \in K$, kIk' implies $k'Ik$.*

Property 4: *(a) Both I and P are transitive; (b) kPk' and $k'Ik''$ implies kPk''; and (c) kIk' and $k'Pk''$ implies kPk''.*

(If you need a concrete example to think about, think of P as something like revealed strict preference and I as revealed indifference, or see below.)

Lemma G1.1. *Suppose that binary relations P and I on a finite set K satisfy properties 1 through 4. Then there exists a function $V : K \to R$ such that kPk' implies $V(k) > V(k')$ and kIk' implies $V(k) = V(k')$.*

This is like numerical representation of P and I, except that the implications run one way only.

Proof of the lemma. For each $k \in K$, let

$$\mathbf{W}(k) = \{k'' \in K : kPk''\},$$

and let $V(k)$ be the number of elements in $\mathbf{W}(k)$.

Suppose kIk'. Then $k'Ik$ (by property 3) and so if $k'' \in \mathbf{W}(k)$, kPk'' and hence $k'Pk''$ (by property 4(c)). Thus $k'' \in \mathbf{W}(k')$. By the symmetric argument, if kIk', then $\mathbf{W}(k) =$

$\mathbf{W}(k')$, and therefore $V(k) = V(k')$.

Suppose kPk'. Then $k' \in \mathbf{W}(k)$ by definition. We know (see the parenthetical remark in property 2) that not $k'Pk$, and thus $k \notin \mathbf{W}(k')$. Moreover, for all $k'' \in \mathbf{W}(k')$, $k'Pk''$ and thus by property 4(a), kPk''. So $\mathbf{W}(k')$ is a strict subset of $\mathbf{W}(k)$, and $V(k) > V(k')$. ∎

Now we return to the proof of the second half of Proposition 1.23. Recall where we are: We know $c(A_k)$ for a finite number of sets A_k, $k = 1, 2, \ldots, n$, and we know that these data admit no violations of SGARP.

To apply the lemma, we let $K = \{1, 2, \ldots, n\}$, and we define:

(1) kIk', if there is a finite sequence $k = k_1, \ldots, k_m = k'$ such that $c(A_{k_i}) \cap c(A_{k_{i+1}}) \neq \emptyset$, for $i = 1, \ldots, m-1$.

(2) kPk', if there is a finite sequence $k = k_1, \ldots, k_m = k'$ such that $c(A_{k_{i+1}}) \cap A_{k_i} \neq \emptyset$, for $i = 1, \ldots, m-1$, and $c(A_{k_{i+1}}) \cap b(A_{k_i}) \neq \emptyset$ for at least one i.

We must show that properties 1 through 4 hold in this case:

It is easiest to begin with reflexivity and symmetry of I; i.e., properties 2 and 3. The symmetry of I is clear, because the definition of I is symmetric in k and k'. As for reflexivity, take $m = 1$ (so there is a single element in the sequence) and apply the definition trivially.

For property 1, suppose kPk' and kIk'. If kPk', there is a finite sequence $k = k_1, \ldots, k_m = k'$ such that $c(A_{k_{i+1}}) \cap A_{k_i} \neq \emptyset$, for $i = 1, \ldots, m-1$, and $c(A_{k_{i+1}}) \cap b(A_{k_i}) \neq \emptyset$ for at least one i. Let x_{k_i} be the element of $c(A_{k_{i+1}}) \cap A_{k_i}$ for all i and the element of $c(A_{k_{i+1}}) \cap b(A_{k_i})$ for at least one i. Let x_{k_1} be any element of $c(A_{k_1})$. Then $x_{k_i} \succeq^r x_{k_{i+1}}$ for all i, and $x_{k_i} \succ^r x_{k_{i+1}}$ for the distinguished i. We can similarly use $k'Ik$ to construct a sequence of revealed weak preferences from x_{k_m} back to x_{k_1}. But putting the two sequences of revealed weak preferences, one with a revealed strict preference in at least one step, would be a violation of SGARP.

Property 4 holds by construction. All four forms of "transitivity" called for in the property involve stringing together pairs of sequences used to define I and P and noting that: Two sequences that define I can be strung together to give another that defines I, and any two, as long as one has a strict revealed preference (i.e., is used for P) when strung together gives a sequence that defines P.

Hence we can apply the lemma and know that there is a function $V : K \to R$ such that kPk' implies $V(k) > V(k')$ and kIK' implies $V(k) = V(k')$. Let L be any number strictly less than $V(k)$ for all k. Define $U : X \to R$ by

$$U(x) = \begin{cases} V(k), & \text{if } x \in c(A_k) \text{ for some } k, \text{ and} \\ L, & \text{if } x \notin c(A_k) \text{ for all } k. \end{cases}$$

We must be sure that this is well-defined; i.e., if $x \in c(A_k) \cap c(A_{k'})$, then $V(k) = V(k')$. But if $x \in c(A_k) \cap c(A_{k'})$, then kIk', and $V(k) = V(k')$ follows.

Now define $\succeq$ as the weak preferences given by U. Obviously, $\succeq$ is complete and transitive. We are done if we show that for each k, $c(A_k) = c_{\succeq}(A_k)$.

To do this, fix A_k. The set $c_{\succeq}(A_k)$ contains all those elements of A_k that have the highest values according to U. By construction, all elements of $c(A_k)$ have the same value, namely $V(k)$. So we need only show that no $x \in b(A_k)$ has higher value than $V(k)$. But this is easy. If $x \notin c(A_{k'})$ for any other k', then $U(x) = L < V(k)$. Suppose $x \in c(A_{k'})$ Since $x \in b(A_k)$, we know that kPk'. Thus $U(x) = V(k') < V(k)$. Done. ∎

You are probably exhausted from all this work, but let me make a few remarks. First, the idea is not that hard: Because of SGARP, we are able to induce a revealed preference ordering among the $c(A_k)$. We select a utility function that reflects that ordering, giving everything that is never selected some utility less than anything that is ever selected (in the data). The hard part is in getting the right definition for the revealed preference ordering and showing that this is enough to produce a numerical representation. The structural properties needed to produce a numerical representation are properties 1 through 4, and the lemma and proposition establish the existence of the ordering. The definitions of I and P in this case, and the demonstration that SGARP implies property 1 for this definition, show that these structural properties hold.

Things are slightly easier if you assume that X is finite, as then you can work directly with X in the role of K. If you were able to do that much, you were doing quite well.

The adjective "simple" will be removed from GARP in Chapter 4, where we reconsider this result (and extend it) for the case of demand data.