CHAPTER 2

ELEMENTARY LOGIC

2–1 Introduction. Logic may be described roughly as the theory of systematic reasoning. Symbolic logic is the formal theory of logic. There are many kinds of logic, but we shall consider only the logic that is most commonly used in mathematics and other sciences.

Symbolic logic has important applications in science and industry. In the *New York Times* of November 25, 1956, a well-known producer of electric products advertised for "men with ideas" to work in the field of electronics. The advertisement called for students of mathematics who had done "creative and original work in all fields of mathematics," and who had "an interest in the theory of numbers, theory of groups, Boolean algebra and symbolic logic. . . ." In the following chapters the reader will get some inkling of why a great corporation is interested in such matters.

We introduce symbolic logic now for three reasons: (1) We can utilize its symbols and laws to simplify later work. (2) The axioms and proofs of elementary symbolic logic are simple and serve to illustrate the nature of a formal mathematical theory. (3) The laws and methods of logic will be useful to the reader in all his thinking in mathematics and other areas.

The purposes of this chapter are (1) to familiarize the student with the most important concepts and notations of symbolic logic, (2) to supply him with logical laws of wide applicability, (3) to develop his skill in reading formal mathematics, and (4) to apply logic to the algebra of real numbers.

2–2 Some simple logical formulas. Logical formulas are sentences whose variables stand for propositions. The purpose of this section is to familiarize the student with the following logical formulas.

	Logical formula	*Informal verbal synonym*
(1)	$\sim p$	It is false that p.
(2)	$p \wedge q$	p and q
(3)	$p \vee q$	p and/or q
(4)	$p \veebar q$	p or else q

We do not take the space constantly to suggest that the student carry out the operations listed in Section 1–17. Occasionally we ask questions, but the reader

should still refer to the check list and work over the material on his own. Item 3 on the list indicates that sentences should be substituted for p and q in (1) through (4). (a) Carry out (p: I shall buy a car, q: I shall sell my old car) in (1) through (4). (b) What is the difference between (3) and (4)?

Negation. We call $\sim p$ the *negation* of p. It is the sentence that denies p. We read $\sim p$ as "It is false that p," "p is false," or "not-p." The negation of a sentence is the sentence that is false when the original is true, and true when the original is false. We indicate this in table (5), in which 1 means truth and 0 means falsity.

(5)

p	1	0
$\sim p$	0	1

When a proposition is true we say that it has the *truth value* truth (represented here by 1), and when it is false we say that its truth value is falsity (represented here by 0). Then table (5) indicates in its first row the possible truth values of p and in the second row the corresponding truth values of $\sim p$. It shows that p and $\sim p$ always have opposite truth values.

To find the negation of a sentence, we must find a sentence that conforms to *both* columns of (5). For example, "He is a good hunter" and "He is not a good hunter" are each the negation of the other, because if one is true the other is false. However, "He is a bad hunter" is not the negation of "He is a good hunter," because both might be false (if he is not a hunter at all!).

(c) Why, in view of the meaning we attach to "proposition," does (5) represent all possibilities? (d) Why is "It is white" not the negation of "It is black"? (e) Express $\sim p$ in several ways if $p = (x \text{ is happy})$. (f) Why is "~ 3" nonsense? (g) How do we usually express $\sim(a = b)$?

Conjunction. We call $p \wedge q$ the *conjunction* of p and q. It is the sentence that asserts *both* p and q. We read it as "p and q," "Both p and q are true," or "p is true and q is true." The conjunction of p and q is true when both p and q have truth values 1, and it is false otherwise. This is indicated in table (6), which gives the possible combinations of truth values of p and q in the first two rows and the corresponding truth values of $p \wedge q$ in the last row.

(6)

p	1	1	0	0
q	1	0	1	0
$p \wedge q$	1	0	0	0

(h) True or false? $(2 + 2 = 4) \wedge$ ("3" is a name of 3). (i) What is the scope of $\wedge$ in (h)? (j) Why is $(2 = 4/2) \wedge (0 = 1)$ false? (k) Why is "$2 \wedge 5$" nonsense?

Disjunction, the inclusive "or." The word "or" in English has two distinct meanings. The exclusive "or" means "or else," as illustrated by "Either the Repubocrats will win or (else) I'll eat my hat!" The inclusive "or" means "and/or," as illustrated by "I intend to study French or (and/or) German." The inclusive "or" means one or both of the two possibilities; the exclusive "or" means one but not both. It turns out that the inclusive "or" is more frequently used in mathematics, and we adopt for it the symbol $\vee$.

We call $p \vee q$ the *disjunction* of p and q. It is a sentence that asserts that at least one of p and q is true. We read it as "p or q," "p and/or q," or "p or else q or else p and q," according to convenience. Table (7) shows how the truth value of $p \vee q$ depends on the truth values of p and of q.

(7)

p	1	1	0	0
q	1	0	1	0
$p \vee q$	1	1	1	0

(l) True or false? $(2 = 10/5) \vee (2 = 5)$. (m) True or false? (George Washington was our first president) $\vee$ (John Hancock signed The Declaration of Independence). (n) Why is "George $\vee$ Mary" nonsense?

Often we wish to state that one number, a, is less than or equal to another, b. We write $a \leq b$, defined as follows.

(8) Def. $$[a \leq b] = (a < b) \vee (a = b).$$

(o) Why is $2 \leq 3$? (p) Why is $2 \leq 2$? (q) Define $a \geq b$

Exclusive disjunction. The exclusive "or" is not used as much as $\vee$, but it is occasionally convenient, and for this reason we include it here. We call $p \veebar q$ the *exclusive disjunction* of p and q. It is the sentence claiming that one and only one of p and q is true. We read it "p or else q," "One and only one of the following is true: p or q," or "p or q but not both." Table (9) indicates the way in which the truth value of $p \veebar q$ depends on the truth values of p and of q.

(9)

p	1	1	0	0
q	1	0	1	0
$p \veebar q$	0	1	1	0

(r) Why is "up $\veebar$ down" nonsense? (s) Under what conditions is $p \vee q$ true but $p \veebar q$ false? (t) Can $p \veebar q$ be true and $p \vee q$ false for the same truth values of p and q? (u) Compare the meaning of "I will study mathematics $\vee$ I will study physics" with the same sentence with $(\vee : \veebar)$.

Definitions. The logical symbols introduced above are not independent. We can take some of them as undefined and use these to define the others. For example, we note from (7) and (6) that $p \vee q$ is false only when p and q are both false, that is, when and only when $\sim p \wedge \sim q$ is true. Hence $p \vee q$ is true when and only when $\sim p \wedge \sim q$ is false, that is, when $\sim(\sim p \wedge \sim q)$ is true. This suggests the definition

(10) Def. $$[p \vee q] = \sim(\sim p \wedge \sim q).$$

Similarly (9) and (7) suggest

(11) Def. $$[p \veebar q] = (p \vee q) \wedge \sim(p \wedge q).$$

(v) In (10) and (11) what symbols are taken as undefined? (w) Justify (11).

Problems

Translate 1 through 6 into words.

1. $\sim(2 > 3)$.
2. $(3 < 4) \wedge (4 < 5)$.
3. $(x = 3) \veebar (x = 4)$.
4. $(x = 3) \veebar (x < 0)$.
5. $(x < 0) \veebar (x = 0) \veebar (x > 0)$.
6. $\sim[x < 0 \wedge x > 0]$.

Translate 7 through 14 into symbols, using $\sim, \vee, \wedge, \veebar$.

7. 3 does not satisfy $x^2 = 10$.
8. 2 is less than 10, and 2 is a digit.
9. (1–13–16).
10. $2^2 = 2 \cdot 2$, but $3^2 \neq 2 \cdot 3$.
11. One and only one of the following holds: $a < b$, $a = b$, $a > b$.
12. $|5| = 5$, yet $|-5| \neq -5$.
13. $3 < 4$, however $-4 < -3$.
14. 0 is neither positive nor negative.

Suppose that p is true and q is false. Determine the truth value of 15 through 22.

15. $\sim q$.
16. $\sim p \vee q$.
17. $\sim(p \vee \sim q)$.
18. $p \wedge \sim q$.
19. $\sim p \vee \sim q$.
20. $\sim(p \vee q)$.
21. $p \veebar q$.
22. $\sim p \veebar q$.

23. Let $\bar{p}$ be the truth value of p. Then show from the truth tables that

(12) $$\overline{\sim p} = 1 - \bar{p},$$

(13) $$\overline{p \wedge q} = \bar{p} \cdot \bar{q},$$

(14) $$\overline{p \vee q} = \bar{p} + \bar{q} - \bar{p} \cdot \bar{q},$$

(15) $$\overline{p \veebar q} = |\bar{p} - \bar{q}|.$$

★24. Suppose we take $\sim$ and $\vee$ as undefined. Define $\wedge$.
★25. Define $\vee$ and $\wedge$ in terms of $\sim$ and $\veebar$.

Answers to Exercises

(a) (1) It is false that I shall buy a car. I shall not buy a car. (2) I shall buy a car and sell my old car. (3) I shall buy a car and/or sell my old car. (4) I shall buy a car or else I shall sell my old car. (b) The first means either or both, the second one but not both. (c) A proposition is either true or false but not both, by definition of "proposition." (d) Both might be false. (e) x is not happy; it is false that x is happy; not-(x is happy); (x is happy) is false. (f) Because "3" is not a sentence and hence not a significant substitute in (1). (g) $a \neq b$.
(h) 1. (i) "$2 + 2 = 4$" on left, "'3' is a name of 3" on right (j) See (6). (k) "2" and "5" are not sentences. (l) 1. (m) 1. (n) "George" and "Mary" are not sentences. (o) $2 < 3$. (p) $2 = 2$. (q) $[a \geq b] = \sim[a < b]$ or $[a \geq b] = (a > b) \vee (a = b)$. (r) "Up" and "down" are not sentences. (s) When p and q are both true. (t) No. (u) First means either or both, second either but not both. (v) $\sim$ and $\wedge$. (w) It says that $p \veebar q$ means that p is true or q is true but not both, which is what (9) indicates.

Answers to Problems

1. 2 is not greater than 3. 3. x is 3 or 4. 5. One and only one of the following holds: $x < 0$, $x = 0$, $x > 0$. 7. $\sim(3^2 = 10)$. 9. $[y = \sqrt{x}] = [y^2 = x \wedge y \geq 0]$. 12. $|5| = 5 \wedge |-5| \neq -5$. 15. 1. 17. 0. 19. 1. 21. 1. 23. From truth tables by considering all cases; or show (12) and (13), then use (10) and (11).

2–3 Implication. Sentences of the form "If p, then q" are very common in scientific discourse. The if-then idea is expressed in many ways, of which the following are synonymous examples.

(1) If p, then q.
(1′) q if p.

(2) p implies q.
(2′) q is implied by p.
(3) p only if q.
(4) Hyp: p, Con: q.
(5) p is a sufficient condition that q.
(6) q is a necessary condition that p.

In everyday discourse such expressions are used with various meanings and connotations. In scientific discourse these sentences are synonymous and have a precise technical meaning. In mathematics a special symbol is usually used, the most common being an arrow pointing from the hypothesis to the conclusion. We shall adopt this notation and write

(7) $$p \rightarrow q$$

as a synonym for sentences (1) through (6).

The meaning we attach to $p \rightarrow q$ is indicated by the formal definition (12) below. To prepare for the definition we indicate the significant substitutes in (7), the conditions under which it is true or false, and the nature of the information that it may convey.

(8) *The expression "$p \rightarrow q$" is a propositional formula in which significant substitutes for the variables are sentences and* only *sentences.*

(a) What does $p \rightarrow q$ become if for the variables we substitute statements? (b) numerals? (c) names of people? (d) Why is every value of $p \rightarrow q$ either true or false?

A statement of the form $p \rightarrow q$ is considered true under any one of the following three conditions:

(9)
p is true, and q is true.
p is false, and q is true.
p is false, and q is false.

It is considered false *only* in the following case:

(10) p is true, and q is false.

A statement of the form $p \rightarrow q$ makes the claim that one of the three possibilities listed in (9) is the case, but *it makes no other claim.* In particular, it says *nothing* as to whether p is true or false, as to whether q is true or false, as to the meanings of p and q, or as to the relation between these meanings.

We may summarize the above in terms of a table for $p \to q$.

(11)

p	1	1	0	0
q	1	0	1	0
$p \to q$	1	0	1	1

Which of the following are true? (e) $(2 = 2) \to (3 = 3)$, (f) $(2 = 3) \to (4 < 1)$, (g) $(2 = 3) \to (4 > 1)$, (h) $(2 = 2) \to (2 = 3)$.

We note from (11) that $p \to q$ is false just in the one case when p is true and q false. In other words, $p \to q$ is true just when $p \wedge \sim q$ is false, that is, when and only when $\sim(p \wedge \sim q)$ is true. This suggests

(12) **Def.** $$(p \to q) = \sim(p \wedge \sim q).$$

We call $p \to q$ the *conditional* of p and q. The special technical meaning assigned to $p \to q$ [and to the synonymous expressions (1) through (6)] by (12) may not seem entirely natural to the reader. He may think that we are not following the criteria of (1–13–9). It turns out that this idea of implication is entirely satisfactory for scientific purposes, is more in keeping with ordinary usage than first appears, and is more convenient than any alternative yet proposed. However, other kinds of implication are considered by logicians, and the one defined by (12) is called *material implication* to distinguish it from the rest.

For each of the following, first decide whether it is true or false on the basis of the everyday meaning, then decide the same question on the basis of (11): (i) If the ocean is mostly water, then it contains about twice as many hydrogen atoms as oxygen atoms. (*Note:* We assume that the formula for water is H_2O, i.e., each molecule of water contains 2 hydrogen atoms and one oxygen atom.) (j) If the ocean is entirely grade-A milk, then it contains about twice as many atoms of hydrogen as atoms of oxygen. (k) If the ocean is entirely grade-A milk, then ocean water is a nourishing beverage. (l) If the ocean is mostly water, then ocean water is a nourishing beverage.

The previous exercises serve as examples to indicate that unless we permit $p \to q$ to be true under any one of the three conditions of (9), we shall find a marked contradiction between our technical meaning and ordinary usage.

A still further illustration of the advantages of (9) is the way in which it facilitates the statement of laws. Consider, for example, the following law of elementary algebra (to be proved in Section 2–6):

(13) $$(a = b) \to (ca = cb).$$

We certainly are inclined to agree that this is a law, i.e., that if $a = b$, then $ca = bc$. Indeed, this is the law suggested by "If equals be multiplied by equals, the results are equal."

Now we recall that a law is a sentence that is true for *all* significant values of its variables. Since in (13) any number is a significant value of any of the variables, we are free to consider the following cases:

(14) $[2 = (6 - 4)] \to [3 \cdot 2 = 3 \cdot (6 - 4)]$ (13)$(a{:}2, b{:}6 - 4, c{:}3)$,

(15) $(2 = 3) \to (0 \cdot 2 = 0 \cdot 3)$ (13)$(a{:}2, b{:}3, c{:}0)$,

(16) $(2 = 3) \to (5 \cdot 2 = 5 \cdot 3)$ (13)$(a{:}2, b{:}3, c{:}5)$.

If (13) is a law, then (14) through (16) must each be true. In (14) both hypothesis and conclusion are true, and hence it is true by the first case in (9). In (15) the hypothesis is false and the conclusion true, and hence it is true by the second case. In (16) the hypothesis and conclusion are both false, and hence it is true by the third case in (9). We cannot find an example for which the hypothesis is true and the conclusion false, because (13) is a law. We see that unless we agree that $p \to q$ is true in all three cases in (9), we shall be unable to say that (13) is a law! The same is true of many other laws of the form $p \to q$.

(m) Discuss the law $(a = b) \to (a^2 = b^2)$ as we did (13).

In discussing and using the implication concept and the symbol $\to$, we shall make use of the different synonyms listed in (1) through (6). For this reason and because these synonyms appear very frequently in scientific discourse, it is important to be able to translate from any one form into any other, and particularly to and from the form $p \to q$. The essential thing is to think of the meaning and to recall that a hypothesis or sufficient condition is always at the heel of an arrow, whereas a conclusion or necessary condition is always at the point of an arrow, as indicated in Fig. 2–1. This reflects the fact that one argues from hypotheses (sufficient conditions) to conclusions (necessary conditions).

FIGURE 2–1

Translate into each of the forms (1) through (6): (n) (13). (o) The sentence of Exercise (i). (p) If two triangles are congruent, they are similar.

We call $q \to p$ the *converse* of $p \to q$. It is the sentence obtained by interchanging hypothesis and conclusion in $p \to q$. We call $p \leftrightarrow q$ the *biconditional* of p and q. It is the sentence that claims that $p \to q$ and $q \to p$; that is, $p \to q$ and its converse are both true.

(17) Def. $[p \leftrightarrow q] = [(p \to q) \wedge (q \to p)]$.

From this definition and (11) we easily get table (18):

(18)

p	1	1	0	0
q	1	0	1	0
$p \leftrightarrow q$	1	0	0	1

There are many ways to read $p \leftrightarrow q$. One is "$p \to q$ and conversely," and in this we may replace $p \to q$ by any one of the readings indicated in (1) through (6). Other forms are "p if and only if q," "q if and only if p," "p is a necessary and sufficient condition for q," "p has the same truth value as q," and "p is logically equivalent to q." The last form refers to the equality of the truth values of p and q when $p \leftrightarrow q$.

(q) True or false? $(2 = 4) \leftrightarrow (3^2 = 2 + 2)$, $(2 = 2) \leftrightarrow (2 = 3)$, $(24 = 2 \cdot 12) \leftrightarrow (2 = 2)$, $(1 = 2) \leftrightarrow (3 = 3)$. (r) Why is "$2 \leftrightarrow 2$" nonsense? (s) Give an example from geometry of a law whose converse is not a law. (t) What conclusion can be drawn if $(p \leftrightarrow q)$ is true and q is true? (u) and p is true? (v) and p is false? (w) and q is false?

Problems

In Problems 1 through 4 cite cases corresponding to the possibilities in (9).

1. If ABC is an equilateral triangle, then ABC is isosceles. (Draw figures for each case.)
2. $(x = y) \to (-x = -y)$.
3. $(AB \parallel A'B' \wedge AC \parallel A'C') \to (\angle BAC \cong \angle B'A'C')$.
4. $(x^2 = y^2) \to [(x = y) \vee (x = -y)]$.

In Problems 5 and 6 both the theorem and its converse are true. Cite cases to illustrate the possibilities. Why do you find only two?

5. $(ABC = 90°) \to (\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2)$ (The Pythagorean theorem).
6. $(a = b) \to (a + c = b + c)$.

★7. How are the following consistent with our definition of implication? "If to do were as easy as to know what were good to do, chapels had been churches and poor men's cottages princes' palaces." (*Merchant of Venice,* Act 1) "If all the year were playing holidays, to sport would be as tedious as to work." (*King Henry IV, Part 1,* Act 1)

Translate 8 through 23 into symbolic language.

8. There can be no great smoke arise, but there must be some fire.
9. There is no fire without smoke.

★10. "The existing economic order would inevitably be destroyed through lawless plunder if it were not secured by force." (*The Law as a Fact,* by K. Olivecrona, p. 137)

11. 3 is a root of $x^2 = 9$.

★12. In a good group you can't tell who the leader is.

★13. "In order for a country that imports capital goods to have a high rate of investment, it must have a large export industry." (*A Survey of Contemporary Economics,* Vol. II, p. 156)

14. "When an organism is conditioned to respond to one stimulus, it will respond in the same way to certain others." (*Principles of Psychology,* by F. S. Keller and W. N. Schoenfeld, p. 115)

★15. ". . . an adequate command of modern statistical methods is a necessary (but not sufficient) condition for preventing the modern economist from producing nonsense. . . ."

★16. "He is well paid that is well satisfied."

★17. I mean what I say.

★18. I say what I mean.

★19. "The surface of the leaf must be coated so as to prevent evaporation of the water that has been so laboriously gathered by the root system." (H. E. Stork, *Studies in Plant Life,* p. 36)

★20. You will make 6% provided the dividend is paid.

★21. He'll win as long as he's better.

22. "It has been shown . . . that the equality is necessary if the third and fourth marginal conditions are not to be violated."

23. It is known that the totally blind are able to detect objects at a distance. Although the blind often think they possess "facial vision" based on skin sensations, experiments have shown that such clues are neither necessary nor sufficient for the blind's perception.

★24. Restate each of Problems 8 through 23 in terms of necessary conditions.

★25. Restate 8 through 23 in terms of sufficient conditions.

★26. Explain the following: "The argument that no clear and present danger to American democracy now exists inside the country should not be taken to mean that no group would constitute a danger if it were powerful. For that would be a confusion of necessary and sufficient conditions." (R. G. Ross, "Democracy, Party, and Politics," *Ethics,* January 1954)

★27. One of the symptoms of tuberculosis is persistent coughing. Is coughing a necessary or a sufficient condition for tuberculosis? What can you say about medical symptoms generally?

★28. Read "What does 'if' mean?" in the *Mathematics Teacher* for January 1955.

★29. Does $p \to q$ mean that q follows p in time?

★30. Speaking of gamblers, Cardano (1501–1576) wrote, "If a man is victorious, he wastes the money won in gambling, whereas if he suffers defeat, then either he is reduced to poverty, when he is honest and without resources, or else to robbery, if he is powerful and dishonest, or again to the gallows, if he is poor and dishonest." Formalize and draw a conclusion. (Oystein Ore, *Cardano, The Gambling Scholar*, p. 187)

Answers to Exercises

(a) A statement. (b) Nonsense. (c) Same. (d) By (8), and see Section 1–7. (e) T. (f) T. (g) T. (h) F. (i) True, as is evident by arguing from the hypothesis to the conclusion; here p and q are true. (j) True, as above, since milk is mostly water; here p is false, and q is true. (k) True, since milk is nourishing; here p and q are false. (l) False. In each of the first three cases the conclusion follows from the hypothesis by everyday reasoning. It does not in the last, since many liquids containing mostly water are not nourishing. Here p is true and q is false. (m) $(a{:}2, b{:}2)$, $(a{:}2, b{:}-2)$, $(a{:}2, b{:}3)$.

(n) If $a = b$, then $ac = bc$. $a = b$ implies $ac = bc$. $a = b$ only if $ac = bc$. From $a = b$, it follows that $ac = bc$. Hyp: $a = b$, Con: $ac = bc$. $a = b$ is a sufficient condition that $ac = bc$. $ac = bc$ is a necessary condition that $a = b$. (o) The ocean is mostly water → it contains . . . . The ocean is mostly water implies it contains . . . . The ocean is mostly water only if it contains . . . . From the ocean is mostly water it follows that it contains . . . . Hyp: The ocean is mostly water. Con: It contains . . . . The ocean is mostly water is a sufficient condition that it contains . . . . (*Note:* The best procedure in performing translations of this kind is to first write the expression in the form $p \to q$, *being careful that p and q are sentences.* Then write in other forms and make whatever adjustments are required to conform to good English usage.) (p) A sufficient condition that two triangles be similar is that they be congruent. A necessary condition that two triangles be congruent is that they be similar. (q) T, F, T, F. (r) "2" is not a sentence. (t) p. (u) q. (v) $\sim q$. (w) $\sim p$.

Answers to Problems

1. Cite an equilateral triangle, one that is isosceles but not equilateral, and a scalene triangle. 3. Cite congruent angles, placed with their sides ‖, congruent angles placed otherwise, and noncongruent angles with sides not ‖.

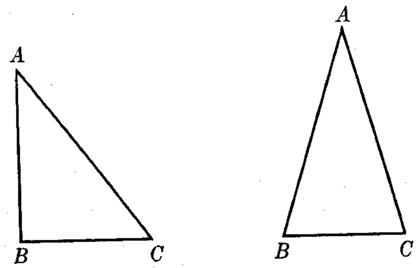

Figure 2–2

5. When the converse is also true we have only the possibilities of both hypothesis and conclusion true or both false by (18). Figure 2–2 shows two triangles ABC. In the first, the hypothesis and conclusion are both true; in the second they are both false. Hence, unless we wish to deny that the Pythagorean theorem is a law, we must agree that $p \to q$ is true when both p and q are false as well as when both p and q are true. This is an example of the convenience of the property of → embodied in the last column of (11). Of course, we do not care to apply a theorem when its hypothesis is false, but the meaning assigned to → and its various verbalizations makes the stating of laws much simpler.

9. There is fire → There is smoke. Fire is a sufficient condition for smoke. (Note here a violation of the agreement that only sentences be substituted for variables in $p \to q$. However, this is really an abbreviation for "A sufficient condition that there is smoke is that there is fire.") 11. $(x = 3) \to (x^2 = 9)$. 13. A country importing capital goods has a high investment rate → It has a large export industry. 27. Some are necessary but not sufficient (always present with the disease but also present at other times), some are sufficient but not necessary (their presence always indicates the disease but they may not always accompany it), some are both, and others are neither.

★2–4 Truth tables.

A *truth table* of a logical formula is a table that shows the truth value of the formula that corresponds to each combination of truth values of its variables. Truth tables are not essential to the development of logic, but they are a convenient tool for investigating logical formulas and provide a procedure for testing any logical formula to see whether it is a law. If we set up truth tables as our criteria for establishing laws in logic, we could use them as a device for proof. We prefer to work with axioms instead and to use tables as an informal device outside the theory.

For example, is $p \to (p \wedge q)$ a law? To answer the question we make table (1):

(1)

p	1	1	0	0
q	1	0	1	0
$p \wedge q$	1	0	0	0
$p \to (p \wedge q)$	1	0	1	1

The third row comes from (2–2–6). The fourth row is obtained from the first and third by reference to (2–3–11). We see that the formula is not a law, since it is false when p is true and q false.

To be able to compare the truth tables of different formulas it is essential to adopt a standard form of construction. To this end we always write the first rows in the following way. If one variable is present:

(2)

p	1	0

If two variables are present:

(3)

p	1	1	0	0
q	1	0	1	0

If there are three variables:

(4)

p	1	1	1	1	0	0	0	0
q	1	1	0	0	1	1	0	0
r	1	0	1	0	1	0	1	0

Note that each of these tables is formed from the previous one by rewriting each column of the previous one twice, the first time with a 1 below it and the second time with a 0 below it. For example, the first column of (3) reads 1, 1; and the first two columns of (4) read 1, 1, 1 and 1, 1, 0. The second column of (3) reads 1, 0; and the third and fourth columns of (4) read 1, 0, 1 and 1, 0, 0.

(a) Using the indicated procedure, write the first four rows in a truth table involving four variables.

In Section 2–5 we assert a number of identities, each of which claims that two logical formulas are synonymous. If two logical formulas are synonymous, certainly both should be true or both false in any given instance. In other words, their truth tables should have the same entries in the last row. In symbols, $(p = q) \rightarrow (p \leftrightarrow q)$. However, the converse is not true, since $p \leftrightarrow q$ whenever p and q have the same truth value, and this may happen without p and q being equal.

(b) Give an example of the observation in the preceding sentence.

From the above discussion, we see that we can test an alleged logical identity by making truth tables for its members. If their last rows are different, we know the equation is not an identity. If the last rows are the same, we have verification but not proof of the identity.

Verify the following by truth tables: (c) (2–5–2), (d) (2–5–4), (e) (2–5–7), (f) (2–5–8), (g) (2–5–23), (h) (2–5–28).

The truth table of a law of logic ought to contain only 1's in its last row. Consider, for example, the formula $p \vee \sim p$. Its table is (5).

(5)

p	1	0
$\sim p$	0	1
$p \vee \sim p$	1	1

Evidently this formula is a law. If we read it verbally, we have "p or not-p," "p is true or not-p is true," "p is true or false," or "any proposition is either true or false." We prove this in Section 2–7.

Make truth tables for the following laws: (i) (2–7–5), (j) (2–7–9), (k) (2–7–23), (l) (2–7–12).

In the following, change $=$ to $\leftrightarrow$ and make a truth table of the result: (m) (2–5–6), (n) (2–5–15), (o) (2–5–24).

Show that the following are not laws and indicate the truth values for which they fail: (p) $(p \rightarrow q) \rightarrow (q \rightarrow p)$, (q) $(p \rightarrow q) \rightarrow (\sim p \rightarrow \sim q)$, (r) $((p \rightarrow q) \wedge \sim p) \rightarrow \sim q$.

A logical formula that is a law, or a sentence in the form of such a formula, is called a *tautology*. The negation of a tautology is called a *contradiction.*

(s) Show that $p \wedge \sim p$ is a contradiction. (t) Show that $p \rightarrow \sim p$ is not a tautology. Show that it is not a contradiction! (u) Why is "I went or I did not go" a tautology? (v) Why is "I went and I did not go" a contradiction? (w) Show that the following is a tautology: It is snowing in Denver and it is not raining in Nashville, or it is not snowing in Denver and it is hailing in Kansas, or it is not hailing in Kansas, or it is raining in Nashville.

Whenever a statement is in the form of a tautology we know that it must be true without further consideration of the truth of its parts. Surprisingly often people try to prove a tautology by arguments about its terms, without realizing that this is not necessary. Consider, for example, the following quotation from a newspaper editorial. "There may be justification for a subsidy—defense needs, for example—but it should be clearly understood that it is never economically justifiable. So the reasons for a subsidy must be strong enough to override the drawbacks." This sounds as though the second sentence follows from the first. But the second sentence is a disguised tautology. It says that if a subsidy is a good thing its advantages must outweigh its disadvantages, i.e., subsidy is justified $\rightarrow$ reasons for it must override its drawbacks. But what do we mean by saying that anything is justified except that the reasons for it override its drawbacks? In other words, by definition, x is justified $=$ reasons for x override reasons against x. Hence the second sentence is logically equivalent to $p \rightarrow p$, with $p =$ the subsidy is justified, and is therefore true quite independently of the first sentence.

Problems

1. Make truth tables for other laws in Sections 2–5 and 2–7.

Which of 2 through 6 are laws?

2. $\sim(p \vee q) \leftrightarrow (\sim p \vee \sim q)$.
3. $[p \wedge (q \vee r)] \leftrightarrow [(p \wedge q) \vee r]$.
4. $(p \rightarrow q) \rightarrow [p \rightarrow (p \wedge q)]$.
5. $(p \rightarrow q) \rightarrow [s \rightarrow (p \rightarrow q)]$.
6. $(p \rightarrow q) \rightarrow [(p \wedge r) \rightarrow (q \wedge r)]$.

7. Make a truth table for (2–7–39). Show that its converse is not a law.
8. Experiment with other logical formulas.
9. Why is this a disguised contradiction? "Our speaker was born in Manhattan, the first of his family to venture north of the Mason-Dixon line."

★10. Is the following verse by John Donne a tautology?

I am unable, yonder beggar cries,
To stand, or move, if he say true, he lies.

11. Is an argument based on a tautology necessarily a poor argument?
12. Ogden Nash once wrote, "I regret that before people can be reformed they have to be sinners." Why is the clause beginning "before . . ." true?
13. Why is "Either an electron is excited to the fullest extent, or it is not excited at all" not a tautology? Why would it be a tautology if "to the fullest extent" were deleted?
14. How many columns are there in a truth table for a formula involving n variables?

Answers to Exercises

(b) (p:2 = 2, q:3^2 = 9).

p	1	1	0	0
q	1	0	1	0
$p \wedge q$	1	0	0	0
$\sim(p \wedge q)$	0	1	1	1

(e)

p	1	1	0	0
q	1	0	1	0
$\sim p$	0	0	1	1
$\sim q$	0	1	0	1
$\sim p \vee \sim q$	0	1	1	1

Note how we include an additional row for each formula that is a part of the final formula, then calculate each row in order.

(k)

p	1	1	0	0
q	1	0	1	0
$q \rightarrow p$	1	1	0	1
$p \rightarrow (q \rightarrow p)$	1	1	1	1

(p) Fails when q is true and p false. (q) Same. (r) Same. (w) If p = it is snowing in Denver, q = it is raining in Nashville, r = it is hailing in Kansas, then the sentence is $(p \wedge \sim q) \vee (\sim p \wedge r) \vee \sim r \vee q$, which is easily shown to be a tautology by a truth table.

Answers to Problems

4, 5, and 6 are laws. 7. The converse fails for q true, p and r false.

2–5 Logical identities. In this section we derive a number of useful identities from a few very plausible axioms. We let $\wedge$, $\sim$, and $=$ be undefined and define $\vee$, $\underline{\vee}$, $\rightarrow$, and $\leftrightarrow$ as in Sections 2–2 and 2–3. We specify that the following are sentences: $a = b$, $\sim p$, $p \wedge q$, $p \vee q$, $p \underline{\vee} q$, $p \rightarrow q$, $p \leftrightarrow q$, and expressions obtained by substituting for their variables other variables or sentences.

(1) **Ax.** $x = x$ (*Law of identity*),

(2) **Ax.** $p \wedge q = q \wedge p$,

(3) **Ax.** $p \wedge (q \wedge r) = (p \wedge q) \wedge r$,

(4) **Ax.** $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$,

(5) **Ax.** $p \wedge p = p$,

(6) **Ax.** $\sim\sim p = p$.

(a) Make the substitution (p:You can walk, q:You can swim, r:You can fly) in (1) through (6). (b) What is the scope of each $\sim$ in (6)? (c) Suggest names for (2), (3), and (4).

In building any mathematical theory, we can list laws in many different orders, provided, of course, that each proof uses only previously established laws. The order here has been chosen to make the proofs as short as possible and to bring out the similarity of the properties of $\vee$ and $\wedge$.

(7) $\sim(p \wedge q) = \sim p \vee \sim q$,

(8) $\sim(p \vee q) = \sim p \wedge \sim q$. (*De Morgan's laws*)

To prove Eq. (7) informally we note that (2–2–10)$(p{:}\sim p,\ q{:}\sim q)$ is $\sim p \vee \sim q = \sim(\sim\sim p \wedge \sim\sim q)$. But by (6) $\sim\sim p = p$ and $\sim\sim q = q$. Hence $\sim p \vee \sim q = \sim(p \wedge q)$. In more formal style,

(9) $\sim(p \wedge q) = \sim(\sim\sim p \wedge \sim\sim q)$ (6)$(p{:}q)$

(10) $= \sim p \vee \sim q$ (2–2–10)$(p{:}\sim p,\ q{:}\sim q)$.

Similarly, (8) is proved by

(11) $\sim(p \vee q) = \sim[\sim(\sim p \wedge \sim q)]$ (2–2–10)

(12) $= \sim p \wedge \sim q$ (6)$(p{:}\sim p \wedge \sim q)$.

(d) Rewrite this proof informally.

Laws (7) and (8), named after the English mathematician Augustus De Morgan (1806–1871), are useful in proving other laws and in stating the negations of compound sentences. The first law states that to deny a conjunction is to affirm the disjunction of the negations of its terms. The second states that to deny a disjunction is to assert the conjunction of the negations of its terms. To apply them, one must first state a compound sentence in the form $p \wedge q$ or $p \vee q$. For example, John and Jim are here = (John is here) $\wedge$ (Jim is here). Hence the negation is (John is not here) $\vee$ (Jim is not here), i.e., (John or Jim is not here).

Simplify: (e) $\sim$(We won and we are happy). (f) $\sim$(We won or we are happy). (g) $\sim$(John and Jim are 17).

State the negation of the following in two ways, first using logical symbols and then using familiar English: (h) John should honor his father and mother. (i) He is neither rich nor poor.

(13) $p \vee q = q \vee p,$

(14) $p \vee (q \vee r) = (p \vee q) \vee r,$

(15) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r),$

(16) $p \vee p = p.$

Remember Section 1–17! (j) Interchange $\vee$ and $\wedge$ in (13) through (16) and state your conclusion in words. (k) What must be the key to proving (13) through (16)? (l) Apply $(\vee{:}+,\ \wedge{:}\cdot,\ \sim{:}-)$ to (1) through (6) and (13) through (16) and state your conclusion in words.

To prove (13), we write

(17) $p \vee q = \sim(\sim p \wedge \sim q)$ (2–2–10)

(18) $= \sim(\sim q \wedge \sim p)$ (2)$(p{:}\sim p,\ q{:}\sim q)$

(19) $= q \vee p$ (2–2–10)$(p{:}q,\ q{:}p)$.

(m) Give reasons in the following proof of (16).

(20) $p \vee p = \sim(\sim p \wedge \sim p)$

(21) $= \sim(\sim p)$

(22) $= p.$

The others [(14) and (15)] are proved similarly.

From the above identities it is easy to derive many others. We list below those that are most frequently used or are required for later proofs in this book.

(23) $p \rightarrow q = \sim p \vee q,$

(24) $\sim(p \rightarrow q) = p \wedge \sim q,$

(25) $\sim(p \leftrightarrow q) = (\sim p \wedge q) \vee (p \wedge \sim q).$

Because of (3) and (14), parentheses are usually omitted when only $\vee$ or $\wedge$ are involved.

(26) $\sim(p \wedge q \wedge r) = \sim p \vee \sim q \vee \sim r,$

(27) $\sim(p \vee q \vee r) = \sim p \wedge \sim q \wedge \sim r.$

State the negation of: (n) If we lose, I'll eat my hat. (o) All three of our teams won today. (p) At least one of our three teams won today. (q) He will graduate if and only if he passes this test. (r) "There's neither honesty, manhood, nor good fellowship in thee." (*King Henry IV*)

(28) $p \rightarrow q = \sim q \rightarrow \sim p.$

Since $\sim q \rightarrow \sim p$ is called the *contrapositive* of $p \rightarrow q$, (28) asserts that an implication and its contrapositive are synonymous. To prove it, we note by (23) that $(p \rightarrow q) = (\sim p \vee q) = (q \vee \sim p) = (\sim\sim q \vee \sim p) = (\sim q \rightarrow \sim p)$, the last step being justified by (23)$(p{:}\sim q,\ q{:}\sim p)$.

(s) Write out the preceding proof with reasons. (t) Illustrate (23) for (p:He wins, q:I lose).

To prove an implication, we very often begin by assuming the negation of the conclusion and arguing to the negation of the hypothesis. This procedure is completely justified by (28). For example, in economics there is a theorem to the effect that when resources are allocated so as to yield maximum output, the marginal product (the additional output that would result from an additional unit of input) is the same for all enterprises. In symbols, Output is maximum $\rightarrow$ Marginal products are equal. The economist argues for this as follows. Suppose that marginal products were not equal. Then an additional unit of input in one enterprise would result in greater output than would be lost by removing this unit from another location. Since output can be increased by shifting resources, it is not maximum. What the economist has proved by this argument is that $\sim$ (Marginal products are equal) $\rightarrow \sim$ (Output is maximum).

(u) If no tracks are observed in snow we usually conclude that no one has passed by. Why is this legitimate?

The following are listed for future reference:

$$p \leftrightarrow q = q \leftrightarrow p, \tag{29}$$

$$[p \rightarrow (q \rightarrow r)] = [(p \wedge q) \rightarrow r], \tag{30}$$

$$[p \rightarrow (q \rightarrow r)] = [q \rightarrow (p \rightarrow r)], \tag{31}$$

$$[p \rightarrow (q \rightarrow r)] = [\sim r \rightarrow (p \rightarrow \sim q)], \tag{32}$$

$$[(p \wedge q) \rightarrow p] = [q \rightarrow (p \vee \sim p)], \tag{33}$$

$$[p \rightarrow (q \rightarrow [p \wedge q])] = [\sim(p \wedge q) \vee (p \wedge q)], \tag{34}$$

$$[(p \wedge q) \rightarrow r] = [(p \wedge \sim r) \rightarrow \sim q], \tag{35}$$

$$[(p \rightarrow q) \wedge (p \rightarrow r)] = [p \rightarrow (q \wedge r)]. \tag{36}$$

Rewrite in as simple form as possible: (v) $\sim p \rightarrow q$, (w) $(p \rightarrow q) \vee (\sim p \rightarrow \sim q)$, (x) $\sim(p \wedge \sim q)$, (y) $\sim[p \wedge (q \vee r)]$, (z) $\sim(\sim p \vee \sim(\sim p \vee \sim(\sim a \vee \sim b)))$.

Problems

Derive the identities in Problems 1 through 6.

1. $\sim p = \sim\sim\sim p$.
2. $\sim p \wedge \sim p = \sim p$.
3. $\sim(\sim p \wedge q) = p \vee \sim q$.
4. $\sim(p \vee \sim q) = \sim p \wedge q$.
5. (23).
6. $[p \rightarrow \sim q] = [q \rightarrow \sim p]$.

In Problems 7 through 17 write the negation in good English after first writing it in logical symbols.

7. I am rich and happy.
8. John or Jim is mistaken.
9. Mathematics and physics are sciences.
10. Neither of you is right.
11. If it rains today, I'll stay home.
12. One and only one of you may go.
13. 7 and 14 are primes.
14. John, Jack, and Jim are 17.
15. New York is a big city if and only if Chicago is not a big city.
16. If he wins, I can't lose.
17. Both countries are at fault.

18. Assuming that if one cannot do mathematics he cannot be an engineer, show that if one is able to be an engineer he must be able to do mathematics.

19. Argue that if an enterprise is engaging in various activities in such a way as to maximize its profits, the additional revenue from increasing expenditure by one unit is the same in all activities.

20. Prove $a \neq 0 \rightarrow -a \neq 0$.

★21. Investigate the properties of $\underline{\vee}$, looking for laws involving it. Is it associative? Distributive over $\vee$? What is the negation of $p \,\underline{\vee}\, q$? Show that $p \,\underline{\vee}\, q = p \leftrightarrow \sim q$.

★22. Start with $\rightarrow$ and $\sim$ as undefined and define the other logical symbols in terms of them.

★23. Suppose we take p/q as our only basic term, where we take it to mean that it is false that both p and q are true, that is, $p/q = \sim(p \wedge q)$. Make a table for p/q as we did for other formulas in Section 2–2. Then define all other formulas in terms of "/". (*Note:* This is called the *stroke* and is of interest because it shows that all logical formulas can be defined in terms of just one.) Work out some laws involving the stroke.

★24. What does "p unless q" mean? Is it synonymous with $\sim q \rightarrow p$ or with $\sim q \leftrightarrow p$?

★25. Every logical identity involving only $\vee$, $\wedge$, and $\sim$ remains an identity if $\vee$ and $\wedge$ are interchanged. Verify this in particular cases. Prove that it is true for all identities derivable from our axioms.

Answers to Exercises

(a) There are various possibilities; for example, one rendering of (2) is: To say that you can walk and you can swim is the same as to say that you can swim and you can walk. (b) $\sim p$ and p. (c) Commutative, associative, and distributive laws. (e) We did not win and/or we are not happy. (f) We did not win and we are not happy. (g) John and/or Jim is not 17. (h) John does not have to honor both his parents, or (John need not honor his father) $\vee$ (John need not honor his mother). (i) He is either rich or poor. (j) The results are all laws of the theory. (k) The relation between $\wedge$ and $\vee$ embodied in (2–2–10), (7), and (8). (l) All but (5), (15), and (16) are laws of elementary algebra. Evidently the algebra of logic has *some* similarities to the algebra of numbers, but it *is* different. (m) (2–2–10)$(q{:}p)$, (5)$(p{:}\sim p)$, (6). (n) We'll lose and I won't eat my hat. (o) At least one of them lost. (p) None of them won. (q) He will graduate and not pass, or he will not graduate and will pass. (r) Thou hast some honor, manhood, or good fellowship in thee, and possibly more than one of these virtues. (s)

(37) $p \rightarrow q = \sim p \vee q$ (23)

(38) $= q \vee \sim p$ (13)$(p{:}\sim p)$

(39) $= \sim\sim q \vee \sim p$ (6)$(p{:}q)$

(40) $= \sim q \rightarrow \sim p$ (23)$(p{:}\sim q,\ q{:}\sim p)$.

(u) (28). (v) $p \vee q$. (w) $p \vee \sim p \vee q \vee \sim q$. (x) $p \rightarrow q$. (y) $\sim p \vee (\sim q \wedge \sim r)$. (z) $p \wedge (\sim p \vee (a \wedge b))$.

Answers to Problems

1. (6)$(p{:}\sim p)$. 3. (2–2–10)$(q{:}\sim q)$. 5. Apply (2–5–7) to (2–3–12). 7. I am either not rich or not happy. 9. Not both mathematics and physics are sciences. Either one or the other or both are not sciences. 11. It rains today and I don't stay home. 13. Either 7 or 14 is not prime. 15. New York and Chicago are big cities or they are both not big cities.

2–6 Rules of proof. Everyone is familiar with the fact that what seems reasonable, even obvious, to one person may seem quite unreasonable and obscure to another. An argument that convinces some may not appear at all convincing to others. Nevertheless, in science, and particularly in mathematics, we wish to prove laws in such a way as to obtain universal agreement. How can this be done in view of the diversity of opinions and experiences of men?

To see the answer to this question, let us imagine that we wish to convince someone of the correctness of a certain statement. We might try to convince him by citing some authority in which he believes (sometimes called "proof by intimidation"), by getting him to "see" ("proof by intuition"), by appealing to his emotions ("proof by waving the red flag"), by confusing and tricking him with words and faulty logic ("elastic inference"), or by other methods familiar to everyone. These methods have two serious disadvantages: they can be used to establish false statements just as easily as to establish true ones, and they do not lead to universal agreement.

If we want a more satisfying procedure we must begin by assuming some axioms, as indicated in Section 1–16. But what if our listener does not agree that our axioms are truly laws? Is there no way out if he will not accept the axioms we propose? There *is* a way out; namely, we may ask

our listener to agree merely that *if* our axioms are indeed laws, *then* our theorems are also laws. If we take this tack, we eliminate argument about the truth of both the axioms and the theorems. We ask our listeners merely to agree that the theorems really do follow from the axioms! This is precisely the procedure we follow in mathematics. The mathematician claims only that the theorems follow from the axioms, and he has no objection if someone prefers to adopt different axioms.

But suppose that our listener objects to the manner in which we derive the theorems from the axioms? Then we shall have to come to some agreement with him as to what procedures are legitimate. And if we cannot agree with him on method of proof? Then we take the same way out as before. We say to him, "Let us merely agree that if these axioms are accepted and if these methods of proof are used, then these theorems can be obtained." In constructing a formal mathematical theory we state in advance the acceptable methods of deriving theorems from axioms. Then all we claim is that the theory is derived from the axioms according to these rules. If the axioms and the rules are acceptable to anyone, the theorems should be also. If the axioms are applicable to any particular situation (i.e., are true in a particular case), the theorems may be applied to that situation.

Our first two rules of proof are the Rule of Replacement and the Rule of Substitution introduced in Section 1–11 and used frequently since. For convenience of reference we recapitulate them here.

(1) Rule of Replacement: *If a term in an expression is replaced by a synonym, the resulting expression is synonymous with the original. If the original is a law, so is the result.*

To use the first part of this rule we need to cite the law that asserts the synonymity of the term and its replacement. Thus we write

(2) $a + b = -(-a) + b$ Rep, (1–9–33).

The reason indicates that a replacement, justified by (1–9–33), has been made in the left member to get the right member. To use the second part of the rule we must have two previously established laws, the law in which we intend to make the replacement and the law asserting that the replacement is a synonym of the term it replaces. We use the abbreviation "Rep" followed by a reference to the law in which the replacement is made and to the law asserting the synonymity. For example,

(3) $a + b = b + (a + 0)$ Rep, (1–9–7), (1–9–22).

The reason on the right indicates that (3) was obtained by replacement in (1–9–7) of a by $a + 0$, which is a synonym according to (1–9–22).

(4) Rule of Substitution: *If a sentence is obtained by making significant substitutions for the variables in a law, the result is a law.*

When applying this rule we use the abbreviation "Sub" followed by reference to the law in which the substitution is made and an indication of the substitution. For example,

(5) $(b + c)a = a(b + c)$ Sub, (1–9–14)$(a{:}b + c, b{:}a)$.

Recall that substitutions are permitted *only for variables* and that *they must be made throughout*, whereas replacements may be made for any term and need not be made throughout.

(a) Review Section 1–11. (b) Review previous uses of these rules.

Our next rule is based on the meaning of implication indicated in Section 2–3. As indicated in (2–3–11), when $p \to q$ and p are true, q must be true. Schematically,

(6)
$$\begin{array}{c} p \to q \\ \underline{\quad p \quad} \\ \therefore q \end{array}.$$

We embody this idea in the following rule of proof.

(7) Rule of Inference: *If an implication is a law and its hypothesis is a law, then its conclusion is a law.*

It is this rule that enables us to detach a conclusion and state it separately when we know that a hypothesis from which it follows is true. When using it we write "Inf" followed by reference to the implication and the hypothesis. For example, from

(8) $2 = \sqrt{4} \to 3 \cdot 2 = 3\sqrt{4}$ Sub, (2–3–13)$(a{:}2, b{:}\sqrt{4}, c{:}3)$,

(9) $2 = \sqrt{4}$ (assumed here),

(10) $3 \cdot 2 = 3\sqrt{4}$ Inf, (8), (9).

(c) Would we get a satisfying rule of proof by interchanging "hypothesis" and "conclusion" in (7)? Explain.

If every law were stated as a theorem, axiom, or definition, each proof would consist of a single statement justified by one of the above rules. However, many laws are not worth displaying as theorems even though they are needed to prove other more interesting laws. Accordingly, a proof usually consists of several steps, each one justified by a rule of proof.

A *complete formal proof* is a sequence of steps such that:

(11)
I. *Each step is a sentence.*
II. *Each step is an established law (axiom, definition, or previously proved theorem) or is the result of applying a rule of proof to a previous step or to an established law.*
III. *The last step is the theorem to be proved.*

A *formal proof* is a complete formal proof or an abbreviation of one obtained by omitting or consolidating steps and reasons. An *informal proof* is an expression of the steps of a formal proof in paragraph style by using everyday language. In this book we use the word "proof" to apply only to formal or informal proofs. "Show" or "argue for" are used for other discussions tending to convince. We call such arguments *heuristic discussions* or *plausibility arguments*. (See Section 2–12.)

(d) Cite examples of different kinds of proofs given in this book so far.

If we relied only on the three rules of proof given above, proofs would be unreasonably long and cumbersome. It is therefore customary to cut proofs short by omitting steps, abbreviating reasons, and adopting further rules of proof. Such measures are considered perfectly legitimate provided they really do amount merely to ways of abbreviating proofs that *could* be carried out by the use of Rep, Sub, and Inf.

One very useful rule of proof, known to the reader from high-school plane geometry, applies to proving laws of the form $p \to q$. This rule permits inserting p as a step "by hypothesis." Then if q can be derived from this assumption, we consider that $p \to q$ is proved. For example, consider the theorem

(12) *If three sides of one triangle are equal in length respectively to three sides of another, the triangles are congruent.*

When we use letters for variables, this becomes

(12′) $[\overline{AB} = \overline{XY},\ \overline{BC} = \overline{YZ},\ \text{and}\ \overline{CA} = \overline{ZX}] \to [\Delta ABC \cong \Delta XYZ]$.

Here $\overline{AB}$ means the length of the segment AB. In proving the theorem, we assume the equations in the hypothesis and then reason step by step to the conclusion. When the conclusion is reached, the proof of (12) is considered complete, and we write "Q.E.D.," an abbreviation for a Latin phrase meaning "that which was to have been demonstrated."

In using this method of proof, we treat the variables in the hypothesis as though they were constants. For example, the above theorem is supposed to hold for all values of A, B, C, X, Y, Z (that is, for all points), but when we assume $\overline{AB} = \overline{XY}$ by hypothesis in the proof, we do not assume that $\overline{AB} = \overline{XY}$ for all A, B, X, Y. To assume this would be to assume that all segments are equal! What we do, rather, is the following. We imagine A, B, C, X, Y, Z to be definite but unspecified points for which the hypothesis holds; i.e., we treat the symbols as constants. Then we show from this assumption that the conclusion holds for these points. However, since we have made no special assumption about these points, the same reasoning would enable us to reason from hypothesis to conclusion in every case. Hence the theorem is proved, since we are assured that its conclusion is true whenever its hypothesis is true, which is all that is required by (2-3-11).

(13) RULE OF HYPOTHESIS: *In order to prove an implication, its hypothesis may be introduced as a step and treated as a law, except that the Rule of Substitution may not be applied to its variables. If the conclusion appears as a step, the implication is proved.*

When using this rule, we write "Hyp" after an assumed hypothesis. When the conclusion is obtained, it is justified by whatever rule is appropriate. Then the implication is justified by writing Q.E.D. and referring to the steps asserting the hypothesis and conclusion. Or we may omit the statement of the implication and just write Q.E.D. when the conclusion is justified. Note that *any* sentence may be assumed by Hyp as a step. Note, also, that a step assumed by Hyp and steps derived from it are not necessarily laws. For this reason, substitution is not permitted in such steps; however, Rep and Inf can be applied to them.

To illustrate the use of Hyp, we prove several laws of elementary algebra in continuation of Section 1-16.

(14) $(a = b) \rightarrow (b = a)$.

(15) Proof of (14):

(a)	$a = a$	Sub, (2-5-1)$(x{:}a)$,
(b)	$a = b$	Hyp,
(c)	$b = a$	Rep, (a), (b),
(d)	$(a = b) \rightarrow (b = a)$	Q.E.D., (b), (c).

Note that (c) was obtained by replacing the first a in (a) by b, which is a synonym according to the assumption (b). We could have omitted step (d) and simply written Q.E.D. after step (c).

(e) Similarly prove one of the following.

(16) $(a = b) \rightarrow (a + c = b + c)$,

(17) $(a = b) \rightarrow (ac = bc)$,

(18) $(a = b) \rightarrow (-a = -b)$,

(19) $(a = b) \rightarrow (a - c = b - c)$,

(20) $(a = b) \rightarrow (1/a = 1/b)$,

(21) $(a = b) \rightarrow (a/c = b/c)$,

(22) $(a = b) \rightarrow (a^2 = b^2)$,

(23) $(a + c = b + c) \rightarrow (a = b)$,

(24) $(c \neq 0 \wedge ac = bc) \rightarrow (a = b)$,

(25) $(a = b \wedge c = d) \rightarrow (ac = bd \wedge a + c = b + d)$.

(f) Does (20) hold for all numbers a and b? (g) Does (21) hold for any number c? (h) In (15), which steps are laws? (i) Prove that $2 = 4 \rightarrow 10 = 20$. (j) Could we use (i) and Inf to prove that $10 = 20$?

An argument (or proof) is called *valid* when it proceeds by applying laws of logic and rules of proof, or could be justified by reference to such laws and rules. The reader should note that any law of logic may be the basis of a valid argument. He should also notice that the validity of an argument does not depend on the truth or falsity of its premises or conclusions. One cannot argue validly from true premises to false conclusions, but one can argue validly from false premises to either true or false conclusions. On the other hand, because a man's conclusions are correct it does not follow that his argument is valid.

(k) Give examples to illustrate the comments in the preceding paragraph. Show that the following two arguments are valid: (l) If the price of butter increases, demand decreases. The price has increased. Hence the demand has decreased. (m) Free competition leads to price cutting and maximum output. Price cutting is rare in our economy and output is usually below capacity. Hence free competition is not universal. (n) Kant held that all proofs of the existence of God were fallacious. Does this show that he was an atheist?

An argument that is not valid is called *invalid*. A line of reasoning that is contrary to logical laws is called a *fallacy*. Many fallacies are based on misapplication of laws of logic or application of a logical formula that is not actually a law. Fallacies are often difficult to detect, but it is usually helpful to restate (or attempt to restate) a suspect argument in symbolic form. It should be kept in mind that the word "fallacy" refers to reasoning and not to the premises or conclusion taken alone.

(o) Collect and/or construct examples of invalid arguments and explain the fallacies involved.

When we construct a theory on the basis of explicitly stated axioms and rules of proof, we have not really eliminated all possibility of controversy. For one thing, there may be differences of opinion as to whether we have correctly applied the rules of proof. But this is just a question of whether we have made a mistake. Mistakes may be hard to find, but such differences of opinion can be solved by sufficiently careful examination of the theory. Unsolvable disagreement is still possible, however, on whether the axioms and rules of proof should have been adopted at all, whether the theory that results from them is a good theory, and so on. Such questions are not answered by the theory itself, but they are placed outside the theory when we agree to argue on the basis of the axioms and methods of proof. Hence we may expect universal agreement *within* the theory, but no universal agreement *about* it. To settle arguments about a theory we should have to construct a second theory about it. A theory about a theory is called a *metatheory*. Of course, there would remain areas of possible disagreement about the assumptions of any metatheory. Evidently we cannot eliminate disagreement or controversy, but we can construct a theory in such a way that disagreement is possible only about certain parts of it, namely the axioms and methods of proof. This is a great advantage because it leads to universal agreement over a considerable area, avoids arguing about matters that can be agreed upon, and identifies the really controversial issues.

Problems

1. Suppose we have proved the Pythagorean theorem: $\angle ABC$ is a right angle $\rightarrow \overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$. Now suppose that we have a particular triangle XYZ in which $\angle XYZ$ is a right angle, $\overline{XY} = 7$ and $\overline{YZ} = 5$. Give an informal proof that $\overline{XZ}^2 = 5^2 + 7^2$. Indicate the rules of proof used.

2. Prove one of (16) through (22) with a complete formal proof. Rewrite it in informal style.

3. Prove others in the list (16) through (22).

4. Prove that if the ocean is lemonade it contains citric acid, indicating your rules of proof.

5. Accepting as a fact that the ocean contains salt, prove it contains sodium, indicating rules of proof.

6. What rules of proof were used in Section 2–5? Insert reasons in the following complete proof of (2–5–7).

(a) $\sim(p \wedge q) = \sim(p \wedge q)$,
(b) $\sim\sim p = p$,
(c) $\sim\sim q = q$,
(d) $\sim(p \wedge q) = \sim(\sim\sim p \wedge \sim\sim q)$,
(e) $\sim p \vee \sim q = \sim(\sim\sim p \wedge \sim\sim q)$,
(f) $\sim(p \wedge q) = \sim p \vee \sim q$.

7. Similarly prove (2–5–8).

8. Give a complete proof of (2–5–28).

9. Explain why the following schema is valid, and state a possible rule of proof based on it.

(26) $$\frac{\begin{array}{c} p \rightarrow q \\ \sim q \end{array}}{\therefore \sim p}. \quad \text{(valid)}$$

10. Explain why the following schemata are invalid.

(27) $$\frac{\begin{array}{c} p \rightarrow q \\ \sim p \end{array}}{\therefore \sim q}. \quad \text{(fallacious!)} \qquad \frac{\begin{array}{c} p \rightarrow q \\ q \end{array}}{\therefore p}. \quad \text{(fallacious!)}$$

11. From $1.5 = 3/2$ and the law in (22), by what pattern can you conclude that $(1.5)^2 = (3/2)^2$?

12. If two lines are parallel, they do not meet. If two particular lines do meet, what conclusion can you draw and why?

13. Suppose you know that the squares of two numbers are equal. Can you draw from (22) the conclusion that the numbers are equal? Explain.

14. Suppose two numbers are not equal. Can you draw the conclusion that their squares are not equal by relying on (22)? Explain.

15. Prove (23) by using $(13)(a{:}a + c,\ b{:}b + c,\ c{:}-c)$.

16. Similarly prove (24).

17. Show that $(ac = bc) \rightarrow (a = b)$ is *not* a law. Detect the error in the following *fallacious* proof. By $(17)(a{:}ac,\ b{:}bc,\ c{:}1/c)$ we have $ac = bc \rightarrow ac(1/c) = bc(1/c)$ or $ac = bc \rightarrow a = b$.

18. Show that the converse of (22) is not a law.

★19. Prove

(28) $$(p = q) \rightarrow (\sim p = \sim q),$$

(29) $$(p = q) \rightarrow (p \wedge r = q \wedge r).$$

20. What conclusions can you draw from $a^2 \neq b^2$?

21. If we know that an argument is valid and that it leads to a false conclusion, what can we conclude?

22. If we know that an argument leads from correct assumptions to false conclusions, what can we conclude?

Answers to Exercises

(c) No; from $p \to q$ and q, p does *not* follow. (d) In Section 1–16 we gave informal and formal, but not complete formal proofs. (e) (16) $a + c = a + c$; $a = b$; $a + c = b + c$; $a = b \to a + c = b + c$. (f) Not for $a = b = 0$; but (20) is a law as it stands, since a law is a sentence all of whose values are true, and (20) is not a sentence for $(a{:}0, b{:}0)$. (g) Not for $c = 0$. (h) (a) and (d) only. (i) (17)$(a{:}2, b{:}4, c{:}5)$. (j) No, since we do not know that $2 = 4$.

Answers to Problems

1. We have given as true that $\angle XYZ$ is a right angle, $\overline{XY} = 7$, $\overline{YZ} = 5$. Now $\angle XYZ$ is a right angle $\to \overline{XY}^2 + \overline{YZ}^2 = \overline{XZ}^2$ by Sub, (Pythagorean theorem)$(A{:}X, B{:}Y, C{:}Z)$. Also, XYZ is a right triangle by Hyp. Hence $\overline{XY}^2 + \overline{YZ}^2 = \overline{XZ}^2$ by Inf and the two previous steps. Hence $7^2 + 5^2 = \overline{XZ}^2$ by Rep. The equation to be proved then follows by Sub, (14)$(a{:}7^2 + 5^2, b{:}\overline{XZ}^2)$ and Inf. 2. (16) By the law of identity, $a + c = a + c$. Since $a = b$ by hypothesis, we may replace a by b in the right member to get $a + c = b + c$, which is the desired conclusion. 4. Use Hyp and others. 5. Use Inf and others. 9. (2–5–28) and Inf. 17. The proof is informal and incomplete, but it would still be in error if omitted steps were inserted. The point is that "$1/c$" is undefined for $c = 0$. Hence the substitution is not significant unless $c \neq 0$. Since $ac = bc \to a = b$ is still a sentence when $c = 0$, we must exclude this possibility by inserting $c \neq 0$ in the hypothesis, as we have done in (24).

2–7 Laws of implication. We now use our additional rules of proof to prove theorems that enable us to reason from a sentence to another that is not synonymous with it.

(1) $p \to p$ (Law of tautology).

This law is so obvious that it would seem out of the question to prove it. However, it can be proved by use of the Rule of Hypothesis.

(2) Proof of (1):

(a) p Hyp,

(b) $p \to p$ Q.E.D., (a).

Note that (b) is justified, since we have assumed p and so, of course, obtained p as a step, in conformity with (2–6–13)!

(3) $(p = q) \to (p \to q)$.

This is proved by assuming $p = q$ by hypothesis and then replacing p by q in the right member of (1).

(a) Does it follow from (3) that $4 \to 2^2$ since $4 = 2^2$? (b) Prove from (3) that $\sim\sim p \to p$. (c) Derive (4) and (5) from (1) by using (2–5–23) and (2–5–7).

(4) $p \vee \sim p$ (Law of excluded middle),

(5) $\sim(p \wedge \sim p)$ (Law of contradiction),

(6) $(p \wedge q) \to p$.

To prove (6) it is convenient to first prove $q \to (p \vee \sim p)$. Since this law is not important except for the purpose of proving (6), we do not list it as a theorem. Instead we call it a *lemma*.

(7) Lemma: $q \to (p \vee \sim p)$.

Proof:

(a) q Hyp,

(b) $p \vee \sim p$ (4),

(c) $q \to (p \vee \sim p)$ Q.E.D., (a), (b).

(8) Proof of (6):

(a) $[(p \wedge q) \to p] = [q \to (p \vee \sim p)]$ (2–5–33),

(b) $(p \wedge q) \to p$ Rep, (7), (a).

(9) $p \to (p \vee q)$.

★(d) Prove (9) by showing that it is synonymous with (6)$(q{:}\sim q)$. ★(e) Show by a truth table that (6) and (9) are not laws if we change $\to$ to $\leftrightarrow$. (f) Show that the converse of (9) is not a law. (g) Prove that $(p \wedge q) \to q$ is a law by showing that it is synonymous with (6)$(p{:}q, q{:}p)$.

Although we cannot include the proofs of all theorems, we list those that are most interesting and not too complicated, and in an order such that each can be proved conveniently by relying on those already listed.

(10) $p \to [q \to (p \land q)]$.

Theorem (10) follows from (2–5–34) and (4). It permits us to infer that $p \land q$ is true if we know that two propositions p and q are each true, for by (10), from p and Inf we can conclude that $q \to (p \land q)$. Then from q and Inf we can conclude $(p \land q)$.

(11) $(p \to q) \to [(p \lor r) \to (q \lor r)]$.

Theorem (11) enables us to "add" the same proposition to both members of an implication.

(12) $[(p \to q) \land (q \to r)] \to (p \to r)$ (Law of syllogism).

Theorem (12) is the principle of logic that permits us to construct chains of reasoning. Thus, if we know $a \to b, b \to c, c \to d, \ldots, x \to y$, and $y \to z$, then we may conclude by using (12) repeatedly that $a \to c$, $a \to d$, $a \to x$, $a \to y$, and finally that $a \to z$. Arguments based on it are called "syllogistic."

★(13) Proof of (12):

(a) $(\sim q \to \sim p) \to [(\sim q \lor r) \to (\sim p \lor r)]$	Sub, (11)$(p{:}\sim q, q{:}\sim p)$,
(b) $(p \to q) \to [(\sim q \lor r) \to (\sim p \lor r)]$	Rep, (a), (2–5–28),
(c) $(p \to q) \to [(q \to r) \to (p \to r)]$	Rep, (b), (2–5–23),
(d) $[(p \to q) \land (q \to r)] \to (p \to r)$	Rep, (c), (2–5–30).

(14) $[p \land (p \to q)] \to q$,

(15) $[\sim q \land (p \to q)] \to \sim p$,

(h) How are laws (14) and (15) related to the discussion in Section 2–6?

(16) $p \land (q \lor r) \to (p \land q) \lor r$,

(17) $p \lor (q \land r) \to (p \lor q)$.

Laws (16) and (17) illustrate how implication and identity are quite different. With = in place of →, they would certainly not be laws.

(18) $p \leftrightarrow p$,

(19) $(p \leftrightarrow q) \to (q \leftrightarrow p)$,

(20) $[(p \leftrightarrow q) \land (q \leftrightarrow r)] \to (p \leftrightarrow r)$.

(i) Which of (18) through (20) hold when ↔ is changed to →? (j) to =?

(21) $(p = q) \to (p \leftrightarrow q)$.

(k) From (21) prove that $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is a law.

Note that (21) permits us to write a theorem on equivalence corresponding to each identity in logic.

(22) $[(x = y) \land (y = z)] \to (x = z)$.

★(22) Proof of (22):

(a) $x = y \land y = z$	Hyp,
(b) $x = y$	Inf, (6)$(p{:}x = y, q{:}y = z)$,
(c) $y = z$	same, (2–5–2),
(d) $x = z$	Rep, (c), (b), Q.E.D.

The following laws are stated here as examples from the infinite list of theorems provable from the axioms of logic. They were selected for their own interest or because we shall use them later.

(23) $p \to (q \to p)$,

(24) $\sim p \to (p \to q)$,

(25) $(p \to q) \to [(p \land r) \to (q \land r)]$,

(26) $[(p \to q) \land (r \to s)] \to [(p \land r) \to (q \land s)]$,

(27) $(p \to q) \to [(p \land r) \to q]$,

(28) $(p \to q) \to [p \to (q \lor r)]$.

The first asserts that a true statement is implied by any statement. The last two say that a true implication remains true if we conjoin a statement to its hypothesis or disjoin one to its conclusion.

(29) $[(p \to r) \land (q \to r)] \to [(p \lor q) \to r]$,

(30) $q \to [(p \land q) \leftrightarrow p]$,

(31) $\sim q \rightarrow [(p \lor q) \leftrightarrow p]$,

(32) $q \rightarrow [(p \lor q) \leftrightarrow q]$,

(33) $\sim q \rightarrow [(p \land q) \leftrightarrow q]$,

(34) $(p \rightarrow q) \leftrightarrow [(p \land q) \leftrightarrow p]$,

(35) $(p \rightarrow q) \leftrightarrow [(p \lor q) \leftrightarrow q]$,

(36) $(p \leftrightarrow q) \rightarrow [(p \land r) \leftrightarrow (q \land r)]$,

(37) $[(p \leftrightarrow q) \land (r \leftrightarrow s)] \rightarrow [(p \lor r) \leftrightarrow (q \lor s)]$,

(38) $[(p \rightarrow r) \land (p \rightarrow s)] \rightarrow [p \rightarrow (r \land s)]$,

(39) $[[\sim p \rightarrow (q \leftrightarrow r)] \land (p \rightarrow r)] \rightarrow [r \leftrightarrow (p \lor q)]$.

Any argument based on assuming a negation is called *indirect*. The basis of such arguments is (2–5–28), but they take many forms, of which the following are typical.

(40) $(\sim q \rightarrow q) \rightarrow q$,

(41) $\{p \land [(p \land \sim q) \rightarrow q]\} \rightarrow q$,

(42) $\{p \land [(p \land \sim q) \rightarrow \sim p]\} \rightarrow q$,

(43) $[\sim q \rightarrow (p \land \sim p)] \rightarrow q$,

(44) $[(\sim q \rightarrow p) \land \sim p] \rightarrow q$,

(45) $[(p \land \sim q) \rightarrow (r \land \sim r)] \rightarrow (p \rightarrow q)$,

(46) $[(p \land \sim q) \rightarrow \sim p] = (p \rightarrow q)$,

(47) $[(p \land \sim q) \rightarrow q] = (p \rightarrow q)$.

An example of the use of (44) is the proof that two different lines parallel to the same line are parallel to each other. Let the theorem be written: $(a \parallel c \land b \parallel c \land a \neq b) \rightarrow (a \parallel b)$. Suppose $\sim(a \parallel b)$. Then a meets b, say in P. Then $(a \parallel c) \land (b \parallel c) \land$ (a and b pass through P). But this is false since only one line can be drawn through a point and parallel to a given line. Letting $q = a \parallel b$, $p = (a \parallel c) \land (b \parallel c) \land$ (a and b pass through P), we have $\sim q \rightarrow p$ and $\sim p$. Hence by (44) and Inf we may conclude q, that is, $a \parallel b$. Then the Rule of Hypothesis establishes the desired implication.

(l) Use a similar argument to show that if two lines are perpendicular to the same line, then they are parallel. (m) Argue that if a man invests his money so as to maximize his return, then his rate of profit must be the same on all his investments.

Proofs by *elimination* are based on showing that the law to be proved is one term of a disjunction that is a law and each of whose other terms is false. It is based on such laws as the following.

(48) $[(p \lor q) \land \sim q] \rightarrow p$,

(49) $[(p \lor q \lor r) \land \sim q \land \sim r] \rightarrow p$.

(n) Argue for the plausibility of (48) and (49). (o) If one angle of a triangle is greater than another, the side opposite the larger angle is larger. To prove this, we note that the first side must be greater than, equal to, or less than the second. The side opposite the greater angle cannot be equal to the other, since if it were the two angles would be equal. It cannot be less, for then the opposite angle would be less instead of greater. (We assume it is known that if one side of a triangle is larger than another, the angle opposite the first is greater than the angle opposite the second.) Hence it must be greater. Explain how (49) applies. (p) What other logical ideas are involved in this argument?

It would be possible to formulate additional rules of proof based on the theorems of logic. For example, because of (6), we could say that if $p \land q$ is a law, then p is a law. Instead of formulating such rules, we refer directly to the theorems when they are needed. (q) Formulate a rule of proof corresponding to (2–5–28). (r) Do the same for (15).

Problems

1. Show that the converse of (3) is not a law.
2. Is $[(p \lor q) \leftrightarrow r] \leftrightarrow [\sim p \rightarrow (q \leftrightarrow r)]$ a law? Is it a law if the central $\leftrightarrow$ is changed to $\rightarrow$? Justify your answers.
3. Rewrite (7) informally.

★4. Explain (13) by giving a more complete proof.

★5. Rewrite (13) informally.

★6. Discover and prove some laws involving $\underline{\lor}$.

7. Prove that $(p \leftrightarrow q) \leftrightarrow (\sim p \leftrightarrow \sim q)$.
8. Formulate a rule of proof corresponding to (3).
9. Formulate rules of proof corresponding to other laws of implication.
10. Prove $(p \leftrightarrow q) \rightarrow (q \leftrightarrow p)$.
11. Assuming that there is one and only one line passing through two different points, prove that two lines intersect in at most one point.
12. A prime number is by definition a natural number greater than 1 with no factors other than itself and 1. Euclid proved that there is no greatest prime number in the following way. Suppose that there is a greatest prime. Let us call it N. Now consider all the prime numbers, $2, 3, 5, 7, \ldots, N$. Let us construct a number M equal to the product of all of these plus 1, that is, $M = (2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \ldots N) + 1$. Now M is not divisible by any of these primes, since there is always a remainder of one after such a division. Hence M is a

prime, since if it had a factor other than itself or 1, this factor would have to be a prime or be divisible by a prime, and in either case M would be divisible by one of the primes. But obviously $M > N$, and hence N is not the largest prime! Explain the use of indirect argument in this proof.

13. Show that the converse of (21) is not a law.

Answers to Exercises

(a) No; *only* sentences are significant substitutes for the variables in (3). (b) $\sim\sim p = p$. Hence by (3) and *Inf* $\sim\sim p \rightarrow p$. (c) $(p \cdot\rightarrow p) = (\sim p \vee p) = p \vee \sim p = \sim(\sim p \wedge \sim\sim p) = \sim(\sim p \wedge p) = \sim(p \wedge \sim p)$. (d) $(6)(q{:}\sim q) = [(p \wedge \sim q) \rightarrow p] = \sim(p \wedge \sim q) \vee p$ [by (2–5–23)] $= \sim p \vee \sim\sim q \vee p = \sim p \vee (q \vee p) = \sim p \vee (p \vee q) = p \rightarrow (p \vee q)$. (f) When p is false and q true, it fails. (g) The substitution yields $(q \wedge p) \rightarrow q$, which is the same as $(p \wedge q) \rightarrow q$. (h) They embody the schemata (2–6–6) and (2–6–26). (i) First and third only. (j) All. (k) (2–5–28).

(l) $\sim(a \parallel b) \rightarrow$ [a meets b in some point]. Call this point P. Then through P there are two lines $\perp c$. But this is false, since only one line can be drawn from a given point perpendicular to a given line. (m) If his rates of profit are different, he can increase his return by transferring some resources to the more profitable investment, and hence his profit is not maximum. The conclusion follows by (2–5–28). (o) (49)(p:first greater than second, q:first equals second, r:first less than second). (p) Each possibility is shown to be false by showing that it implies a false statement, i.e., by applying $(44)(q{:}\sim q)$. (q) To prove an implication it is sufficient to prove its contrapositive. (r) To prove a statement false, it is sufficient to prove that it implies a false statement.

Answers to Problems

1. $(2 = 5) \rightarrow (2 = 2)$. 3. We may assume q by hypothesis; also, $p \vee \sim p$ is a law by (4). Hence, in accordance with the Rule of Hypothesis, the desired conclusion appears as a step in a proof in which the hypothesis is assumed, and we have proved the implication. 7. $(p \leftrightarrow q) = (p \rightarrow q) \wedge (q \rightarrow p) = (\sim q \rightarrow \sim p) \wedge (\sim p \rightarrow \sim q) = (\sim p \leftrightarrow \sim q)$. Then use (21). 12. The main argument is justified by (43)(q:There is no largest prime, p:N is the largest prime). To show M is prime we use (44)(q:M is prime, p:M is divisible by a prime).

2–8 Manipulating algebraic equations. Much scientific discourse is formulated in terms of equations. The following are some of the problems that arise.

I. *To prove that a given equation is an identity.* Numerous examples in logic and the algebra of numbers have already been given.

II. *To prove that a given equation is not an identity.* This is often most easily done by citing a *counterexample* (a single example for which it is false).

III. *To prove that a given value of the variable in an equation is a solution.* The proof is accomplished by substituting in the equation and proving that the result is an identity. Hence this problem reduces to I.

IV. *To find some solutions of an equation.* This is quite a different matter from III. Indeed, before we can try to prove that a number is a solution, we must find it! The difference between III and IV is very much like the difference between the jobs of a prosecuting attorney and a detective. A trial cannot be held without an accused! The prosecuting attorney (III) wishes to *prove* that the accused is guilty, and he is bound by very strict rules (rules of evidence, court procedure, etc.) He wishes to *prove* that the accused is a solution of the sentence "x committed the crime." On the other hand, the detective (IV) wishes merely to *find* the guilty party. In this search he may guess, use his intuition, listen to gossip, and do all sorts of things that would not be considered proof by any court of law. Similarly, we may use any convenient method to find solutions of an equation, but we must not imagine that this is the same thing as proving that what we have found is indeed a solution.

V. *To find all solutions of an equation,* i.e., to solve it. This involves more than III or IV.

VI. *To derive some equations from others.* This problem is more general than I, since here we do not require that the equations be identities. It arises very frequently in science when the scientist assumes that certain relations hold among values of his variables and desires to discover the consequences of this assumption.

To solve such problems we make use of manipulations of several kinds:

A. *Manipulations yielding synonymous sentences.* If we replace an expression in an equation by a synonym, the result is an equation synonymous with the original. Moreover, by (2–7–21), the original and resulting equations are logically equivalent. Because of the equivalence, any value of the variable that is a solution of one is a solution of the other. If one is an identity, so is the other. If one is not, neither is the other. If we find all solutions of one, we have found all solutions of the other.

(a) Why does an equation imply any equation found by a manipulation of Type A above?

For example, to prove that 2 is a solution of $x^2 + x - 6 = 0$, we may write

(1) $$2^2 + 2 - 6 = 0 \leftrightarrow 4 + 2 - 6 = 0 \leftrightarrow 0 = 0.$$

Here each change involves a replacement. Since the last equation is an identity, so is the first.

(b) Prove that -3 is a solution of $x^2 + x - 6 = 0$.

B. *Manipulations yielding logically equivalent sentences.* These include not only those of Type A but also other manipulations that yield sentences that are equivalent without being synonymous. For example,

(2) $$2x - 1 = 0 \leftrightarrow 2x = 1,$$

because $2x - 1 = 0 \rightarrow 2x = 1$ by (2–6–16)(a:$2x - 1$, b:0, c:1), and $2x = 1 \rightarrow 2x - 1 = 0$ by (2–6–16)(a:$2x$, b:1, c:-1). Manipulations of this kind have the same advantages arising out of the equivalence as do those of Type A.

The most useful laws that justify manipulations of this type are the following.

(3) $$(a = b) \leftrightarrow (a + c = b + c),$$

(4) $$(c \neq 0) \rightarrow (a = b \leftrightarrow ac = bc),$$

(5) $$(ab = 0) \leftrightarrow (a = 0 \vee b = 0).$$

The first law follows from (2–6–16), (2–6–23), and (2–3–17). To prove the second, we note that $a = b \rightarrow ac = bc$. Hence by (2–7–23)($p$:$a = b \rightarrow ac = bc$, q:$c \neq 0$) and Inf, we may assert $c \neq 0 \rightarrow (a = b \rightarrow ac = bc)$. From this and (2–6–24) we have (4) by using (2–7–38) and Inf.

We can argue informally for (5) as follows. First to prove $ab = 0 \rightarrow a = 0 \vee b = 0$, we assume $ab = 0$ by hypothesis. If $a = 0$, we have the conclusion. If not, (4) justifies multiplying both members of $ab = 0$ by $1/a$ to get $b = 0$. Second, $(a = 0 \vee b = 0) \rightarrow (ab = 0)$ follows immediately from (1–9–30) and (1–9–31). Hence, by the definition of logical equivalence we have (5).

To prove (5) more rigorously we use (2–7–39)(p:$c = 0$, q:$a = b$, r:$ac = bc$) to get $[[c \neq 0 \rightarrow (a = b \leftrightarrow ac = bc)] \wedge (c = 0 \rightarrow ac = bc)] \rightarrow [ac = bc \leftrightarrow (c = 0 \vee a = b)]$. The hypothesis of this implication is a law, since its first term is just (4) and its second term follows immediately from (1–9–30). Hence by Inf, its conclusion is a law; that is,

(6) $$(ac = bc) \leftrightarrow (c = 0 \vee a = b).$$

This law is sometimes useful as it stands. We can get (5) from it immediately by the substitution (b:0).

We illustrate the use of these laws by the following manipulations.

(7) $2x^2 + 2x = 12 \leftrightarrow x^2 + x = 6$ (4)

(8) $\leftrightarrow x^2 + x - 6 = 0$ (3)

(9) $\leftrightarrow (x + 3)(x - 2) = 0$ (1–12–7)

(10) $\leftrightarrow x + 3 = 0 \vee x - 2 = 0$ (5)

(11) $\leftrightarrow x = -3 \vee x = 2$ (3).

Each sentence is logically equivalent to the preceding one for the reason given. Hence we have simultaneously solved problems III, IV, V for this equation. We know that -3 and 2 are roots and all the roots.

(c) Which steps in (7) through (11) involve replacements? (d) Which sentences are synonymous? (e) Solve $3 - 2x = 3x + 1$ by such manipulations, giving reasons. (f) Do the same for $x^2 - x - 6 = 0$ and for (g) $2x^2 + 8x + 8 = 0$.

An equation of the form $ax + b = 0$ (where a and b do not involve x, and $a \neq 0$) is called a *linear equation in one variable*. Equations that can be reduced to this form by replacements and manipulations like those in (3) and (4) are also called linear. The solution of such an equation is trivial.

(12) $$ax + b = 0 \leftrightarrow ax = -b$$

(13) $$\leftrightarrow x = -b/a.$$

(h) Solve, giving reasons: $x/2 + 16 - 5x = -2(x + 3)$.

An equation of the form $ax^2 + bx + c = 0$ (where $a \neq 0$, and a, b, and c are independent of x) and equations that can be reduced to this form are called *quadratic*. Manipulations similar to (7) through (11) may be used to solve any quadratic equation.

$ax^2 + bx + c = 0$

(14) $\leftrightarrow x^2 + (b/a)x + (c/a) = 0$

(15) $\leftrightarrow x^2 + (b/a)x + (b/2a)^2 - (b/2a)^2 + (c/a) = 0$

(16) $\leftrightarrow (x + b/2a)^2 - (b^2/4a^2 - c/a) = 0$

(17) $\leftrightarrow (x + b/2a)^2 - (b^2 - 4ac)/4a^2 = 0$

(18) $\leftrightarrow (x + b/2a + \sqrt{b^2 - 4ac}/2a)(x + b/2a - \sqrt{b^2 - 4ac}/2a) = 0$

(19) $\leftrightarrow x + \dfrac{b + \sqrt{b^2 - 4ac}}{2a} = 0 \vee x + \dfrac{b - \sqrt{b^2 - 4ac}}{2a} = 0$

(20) $\leftrightarrow x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a} \vee x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}.$

(i) Give the reasons for steps (14) through (17), (19), and (20).

Step (18) is obtained by factoring (17) as a difference of two squares, that is, by using

$$[a^2 - b^2 = (a + b)(a - b)](a{:}x + b/2a, b{:}\sqrt{b^2 - 4ac}/2a).$$

However, if $b^2 - 4ac < 0$, $\sqrt{b^2 - 4ac}$ is not a real number and does not even exist in the number system we have used so far. (Why? For a proof, see Section 3–5.) Hence, for the present when $b^2 - 4ac < 0$ we say that the equation has no real roots. In Chapter 6 we extend the number system so that any number has a square root. Then (14) through (20) proves that any quadratic equation with $b^2 - 4ac \neq 0$ has two roots given by (20). When $b^2 - 4ac = 0$ there is only one root, but it is customary to say that the two roots are equal. We call $b^2 - 4ac$ the *discriminant* of the quadratic. Usually (20) is written

(21) $$ax^2 + bx + c = 0 \leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

and the expression giving the two values of x is called the *quadratic formula.*

(j) Carry through (14) through (20) with $x^2 + 2x - 4 = 0$. Use the quadratic formula to find the roots of: (k) $x^2 - 2x - 4 = 0$, and (l) $x^2 + 6x + 9 = 0$. Solve: (m) $3x^2 + 5x - 1 = 0$, and (n) $3x^2 - 5x + 1 = 0$.

C. *Manipulations yielding a necessary or sufficient, but not a logically equivalent, sentence.* For example, to solve $x^2 = x$ we may divide both members by x to get $x = 1$. According to (2–6–17)$(a{:}x, b{:}1, c{:}x)$, $x = 1 \rightarrow x^2 = x$. Hence 1 is a root of $x^2 = x$. But the converse is false; that is, $x^2 = x \rightarrow x = 1$ is not a law. Indeed, its hypothesis is true and its conclusion false for $x = 0$. This manipulation yielded a sufficient but not necessary condition for the original equation, and one root was lost. Of course, (6) yields

(22) $$(x^2 = x) \leftrightarrow (x = 0 \lor x = 1).$$

Or we could solve the equation by rewriting it as $x^2 - x = 0$ and then factoring or using the quadratic formula.

As a second example, consider the equation $1/x + 3 = 1/x$. Multiplying both members by x, we have $1 + 3x = 1$, $3x = 0$, $x = 0$. But 0 is not a root of the original, since "1/0" is undefined. The difficulty arose because $1/x + 3 = 1/x \rightarrow 1 + 3x = 1$, but the converse is false. Hence we gained a root by this manipulation. Here we found a necessary but not sufficient condition. Again the difficulty could be avoided by sticking to manipulations of Type B. By (4) we have

(23) $$x \neq 0 \rightarrow (1/x + 3 = 1/x \leftrightarrow 1 + 3x = 1).$$

And since 0 is the only solution of the right member of the equivalence, there is no root.

(o) Show directly that $1/x + 3 = 1/x$ has no solution by showing it logically equivalent to $3 = 0$.

The above examples make clear the risks in manipulations of this type when solving equations. As long as we stick to manipulations of Type B we neither gain nor lose roots, but those of Type C may do either. However, manipulations of Type C are convenient in some situations, provided they are used carefully.

For example, to solve the equation $\sqrt{x} + 1 = \sqrt{4x}$, we may use (2–6–22) to write

(24) $$\sqrt{x} + 1 = \sqrt{4x} \rightarrow x + 2\sqrt{x} + 1 = 4x$$

(25) $$\rightarrow 2\sqrt{x} = 3x - 1$$

(26) $$\rightarrow 4x = 9x^2 - 6x + 1$$

(27) $$\rightarrow 9x^2 - 10x + 1 = 0$$

(28) $$\rightarrow (9x - 1)(x - 1) = 0$$

(29) $$\rightarrow x = 1/9 \lor x = 1.$$

Since the converse of (2–6–22) is not a law, we cannot use equivalence signs here. The implication in (29) means that if x satisfies the original equation, it must equal one of the values given by (29). But it does not follow that if x equals one of the values given by (29), then it must satisfy (24). Indeed, substitution in the original equation shows that 1 is a root but 1/9 is not. Here this procedure caused no error, since our manipulations did not lose any roots and we tested the alleged roots it yielded.

(p) Verify the statements about 1 and 1/9. (q) Justify (24) through (29).

As an example of the usefulness of manipulations of Type C in deriving some equations from others in science, we consider the following equations taken from an article in the Autumn, 1956, issue of *Educational and Psychological Measurements.*

(30) $$\text{A.L.} = \frac{(A_5 - A_4) + (A_5 - A_3) + (A_5 - A_2) + (A_5 - A_1)}{4},$$

(31) $$\text{Acc} = A_1 + A_2 + A_3 + A_4 + A_5,$$

(32) $$\text{A.L.} = (5A_5 - \text{Acc})/4.$$

The authors say that (30) and (31) imply (32). Now by replacements we find that (30) is equivalent to

(33) $$\text{A.L.} = (4A_5 - (A_1 + A_2 + A_3 + A_4))/4.$$

Also, (31) is equivalent to

(34) $$\text{Acc} - A_5 = A_1 + A_2 + A_3 + A_4.$$

Now we make in (33) a replacement of the right member of (34) by its left member. After slight simplification, the result is (32) as desired. We see that by assuming (30) and (31) we can derive (32). Hence $[(30) \wedge (31)] \rightarrow (32)$ by the Rule of Hypothesis. But the converse certainly does not hold.

(r) Carry through all the details required to verify the above argument. Show that the converse fails by finding numerical values for which (32) holds but at least one of (30) and (31) is false. (s) The following equations are taken from an article in the Spring, 1955, issue of *Educational and Psychological Measurements.*

(35) $$R_i = (1 - c)T_i + cn,$$

(36) $$R_i + W_i = n,$$

(37) $$T_i = R_i - \frac{c}{1-c} W_i.$$

Derive (37) from (35) and (36).

The discussion of this section refers to equations. However, the general comments under I through VI and A through C apply more generally to any sentences. Examples will be given in later sections.

Problems

1. Prove that $(x_1^2 + y_1^2)(x_2^2 + y_2^2) = (x_1x_2 - y_1y_2)^2 + (x_1y_2 + x_2y_1)^2$ is an identity.
2. Prove that $a^2 = b \rightarrow a = \sqrt{b}$ is not a law.
3. Prove that $1/a$ is a solution of $x^2 + 1/x^2 = a^2 + 1/a^2$.
4. Find three other solutions of the equation in Problem 3.
5. Find all the solutions of $x(x^2 - 1)(x + 2)(x^2 - 3x - 4) = 0$.
6. Show that if $p = \dfrac{3P - 1}{2}$, then $P = p + (1/3)(1 - p)$.
7. Prove

(38) $$(a^2 = b^2) \leftrightarrow (a = b \vee a = -b).$$

8. Prove $(a \neq 0 \wedge b \neq 0) \rightarrow (ab \neq 0)$.
9. Prove $(a \neq 0 \wedge ab = 0) \rightarrow (b = 0)$.

10. Sometimes people try to prove an identity by showing that it implies a true statement. Thus, in place of (1) they write $2^2 + 2 - 6 = 0 \rightarrow 4 + 2 - 6 = 0 \rightarrow 0 = 0$. What is the fallacy in this argument?
11. Find the fallacy in the following: Let $2 = y$. Then $4 = 2y$ or $4 - 2y = 0$. Factoring, we find $2(2 - y) = 0$. Dividing by $2 - y$, we find $2 = 0$(!).
12. Solve $x/2 - x/3 = 4$.
13. Solve $x/3 - (1 - 2x) = 7x + 9$.
14. Solve $ax + b = cx + d$.
15. Solve $x^2 + x - 1 = 0$.
16. Solve $x^2 + a_1x + a_0 = 0$.
17. Solve $5x^2 - x - 1 = 0$.

★18. Prove by substitution that the quadratic formula gives roots of the quadratic.

★19. Find (21) by the change of variable $x = y - b/2a$ in $ax^2 + bx + c = 0$, solving the result for y, and substituting back for x.

★20. Show that if r_1 and r_2 are the roots of $x^2 + a_1x + a_0 = 0$, then $r_1 + r_2 = -a_1$ and $r_1r_2 = a_0$.

21. Show $[h = (N - x_1^2)/2x_1 \wedge x_2 = x_1 + h] \rightarrow [x_2 = (N + x_1^2)/2x_1]$.
22. From the conclusion of Problem 21 and $x_3 = (N + x_2^2)/2x_2$, find a formula for x_3 in terms of x_1 and N.
23. (This and the next problem are taken from articles in *Educational and Psychological Measurements*, Summer, 1956.) Show that

$$\frac{U(n - L) - L(n - U)}{n^2} = \frac{U - L}{n}.$$

24. Solve for r_{12}:

$$1 - (r_{12}^2 + r_{13}^2 + r_{23}^2) + 2r_{12}r_{13}r_{23} = 0.$$

25. (This and Problem 26 are taken from *The Geometry of International Trade*, by J. E. Meade.) $(a + b = c) \rightarrow [a/b = 1/(c/a - 1)]$.
26. $[e' = E/Q \wedge i = I/e(Q + E)] \rightarrow [i = (1/e)(I/Q)/(1 + e') \wedge (1/e)(I/Q) = i(1 + e')]$.
27. Solve for x:

$$\frac{2x - a}{b} = \frac{bc - cx}{a} \quad \text{for } a = 2,\ b = -1,\ c = 3.$$

28. Solve for x:

$$x - a = \frac{bc}{d} + \frac{c^2x}{de} \quad \text{for } a = -3,\ b = 0,\ c = -2,\ d = -2,\ e = 4.$$

29. Solve for x:

$$\frac{x + a}{x - a} - \frac{x - a}{x + a} = \frac{1}{x - a} - \frac{1}{x^2 - a^2} + \frac{1}{x + a} \quad \text{for } 6a + 7 = 0.$$

30. Solve for x:

$$\frac{2b - x - 2a}{bx} = \frac{x - 4a}{ab - b^2} - \frac{4b - 7a}{ax - bx}.$$

31. Show that if $b \neq 0$, $[(-a)(-b) = -ab] \leftrightarrow (-a = a)$.

32. Show that if $[Pb^{++}] = [HSO_4^-] + [SO_4^=]$, $[Pb^{++}][SO_4^=] = K_{sp}$, and $[H^+][SO_4^=] = K_2[HSO_4^-]$, then $[Pb^{++}]^2 = \frac{K_{sp}[H^+]}{K_2} + K_{sp}$. (*Note:* The brackets stand for concentrations of the chemical named within.)

33. (This and Problems 34 and 35 are taken from *Mathematical Introduction to Economics*, by G. C. Evans.) Solve for u: $2u/a - b/a - 2Au - B = 0$.

34. Solve for p: $\frac{b + Ba}{2 - 2Aa} = ap + b$.

35. Solve for p: $\frac{p - B}{2A} = ap + b$.

36. (This and the next problem are taken from *Mathematical Biology of Social Behavior*, by N. Rashevsky, pp. 44 and 139.) Solve for $x_1 + x_2$:

$$a_1 + a_2 - 2b_1(x_1 + x_2) = 2b_2x_1 + 2b_2x_2.$$

37. Find x if $1/(1 + x) - B = 0$.

38. Show that $-4(x - 1/2)(y - 1/4) + 5/2 = -4xy + x + 2y + 2$. (*Introduction to the Theory of Games*, by J. C. C. McKinsey, p. 22)

39. Show that if $\Delta y = y_1 - y_0$ and $\Delta x = x_1 - x_0$, then

$$y_0 + \frac{(x - x_0)\,\Delta y}{\Delta x} = \frac{1}{x_1 - x_0}\,[(y_1 - y_0)x + y_0x_1 - x_0y_1].$$

40. (From *An Introduction to the History of Mathematics*, by H. Eves, p. 184) Solve for x:

$$\frac{\sqrt{[(2/3)(7/4)(3x)]^2 - 52} + 8}{10} = 2.$$

41. (From *Statistical Methods*, by G. W. Snedecor) Solve for x:

$$\frac{(a - x)(d - x)}{(b + x)(c + x)} = 1.$$

42. Show that

$$P = \frac{1}{1 + a} = 1 - Q \rightarrow \frac{P}{Q} = \frac{1}{a}.$$

43. Show that

$$p = q - 1 \wedge P = Q - 1$$
$$\rightarrow \frac{(pN - PN)^2}{PN} + \frac{(qN - QN)^2}{QN} = \frac{N}{PQ}\,(p - P)^2(P + Q).$$

44. Show that

$$jd - j(A - d) = \log\frac{E}{E'} \rightarrow d = \frac{1}{2}\left[\frac{\log(E/E')}{j} + A\right].$$

(*Note:* You do *not* need to know the meaning of $\log(E/E')$ to solve this problem.)

Answers to Exercises

(a) $(p = q) \rightarrow (p \rightarrow q)$. (b) $(-3)^2 + (-3) - 6 = 0 \leftrightarrow 9 - 3 - 6 = 0 \leftrightarrow 0 = 0$. (c) (9) only. (d) (8) = (9). (e) $3 - 2x = 3x + 1 \leftrightarrow -2x = 3x - 2 \leftrightarrow -5x = -2 \leftrightarrow x = 2/5$. By adding -3 to both members, adding $-3x$ to both members, dividing both members by -5, and simplifying. See Sections 1–9 and 1–14. (f) $x^2 - x - 6 = 0 \leftrightarrow (x - 3)(x + 2) = 0 \leftrightarrow x - 3 = 0 \vee x + 2 = 0 \leftrightarrow x = 3 \vee x = -2$. (g) $x^2 + 4x + 4 = 0 \leftrightarrow (x + 2)^2 = 0 \leftrightarrow x = -2$. (h) Check solution by substitution. See Sections 1–9 and 1–14.

(i) (14):(4), (15):(1–9–20) and (1–9–22), (16):(1–12–9), (17):(1–14–2) and (1–14–3), (19):(5), (20):(3). (j) Instead of substituting $(a{:}1, b{:}2, c{:}-4)$ throughout, carry through each step. (k) $x = (2 + \sqrt{20})/2 = 1 \pm \sqrt{5}$. Note that by (1–13–22), $\sqrt{20} = \sqrt{4 \cdot 5} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$. (l) Double root, -3. (m) $(-5 \pm \sqrt{37})/6$. (n) $(5 \pm \sqrt{13})/6$. (o) Add $-1/x$ to both sides. (p) By substitution. (q) (25) and (27) by (2–6–19). (s) Replace n in (35) from (36) and solve for T_i.

Answers to Problems

1. Expand each member. 2. $(a{:}-1)$. 4. $a, -a, -1/a$. 5. $0, \pm 1, -2, 4, -1$. 6. Solve for P and rearrange. 7. $a^2 = b^2 \leftrightarrow a^2 - b^2 = 0 \leftrightarrow (a + b)(a - b) = 0$. 10. (2–6–27). 11. Division by $2 - y$ is division by 0. 12. Check alleged solutions by substitution. 14. $(d - b)/(a - c)$. 15. $(-1 \pm \sqrt{5})/2$. 17. $(1 \pm \sqrt{21})/10$.

2–9 The functional notation. It is often convenient to have an abbreviation for a formula that appears several times in a discussion. For example, we might wish to talk about $x^2 - x + 1$ and its values for various values of x. To avoid having to write out the formula each time we wish to mention it, we could adopt a single letter as an abbreviation. Thus we might let $s = x^2 - x + 1$. However, if we wish to indicate what happens when substitutions are made in a formula, a single letter is unsatisfactory because it does not involve the variable explicitly. The difficulty is overcome by adopting an abbreviation in which x appears. We choose some letter, such as f, and follow it by the variable in parentheses, as in $f(x)$ (read "f of x"). Then we adopt a temporary definition, such as

(1) $$\text{Let } f(x) = x^2 - x + 1.$$

Now (1) says that $f(x) = x^2 - x + 1$ is an identity by definition. Hence we get identities from it by any significant substitution for x. We must, of course, substitute for x throughout. Thus

(2) $$f(2) = 2^2 - 2 + 1,$$

(3) $f(a) = a^2 - a + 1,$

(4) $f(y) = y^2 - y + 1,$

(5) $f(x + y) = (x + y)^2 - (x + y) + 1,$

(6) $f(x^2) = (x^2)^2 - x^2 + 1,$

(7) $f(x^3) = (x^3)^2 - x^3 + 1,$

(8) $f(a^2 + 1) = (a^2 + 1)^2 - (a^2 + 1) + 1.$

In this way we can use the notation to indicate any substitution in the original formula. Thus $f(2)$ means the value of $f(x)$ when $x = 2$; that is, $f(2) = [f(x)](x{:}2)$. Indeed, for any value of a,

(9) $f(a) = [f(x)](x{:}a).$

We read $f(x)$ as "f of x." It does *not* mean "f times x." Multiplication is not involved in any way. It means the value of a certain formula corresponding to the value of x. To find $f(a)$, where a is any significant substitute whatever for x, we simply substitute a, however complicated it may be, into the formula for which $f(x)$ has been adopted as an abbreviation. Thus if we let $f(x) = [x = 3]$, then $f(2) = [2 = 3]$, $f(3) = [3 = 3]$, $f(a^2 + 2) = [a^2 + 2 = 3]$, and so on.

(a) Let $f(x) = x - 1$. Find $f(1)$, $f(2)$, $f(-3)$, $f(b)$, $f(x - 1)$, $f(x + 2)$, $f(x + 4)$, $f(a + b + c)$. (b) Let $N(x) = \sim x$. Find $N(p)$, $N(\sim p)$, $N(N(p))$, $N(p \vee q)$, $N(p \wedge q)$. Comment on "$N(7)$."

How can the reader know whether an expression such as $s(x)$ means "s times x" or "s of x"? He has to tell by the context. However, he is aided by the following conventions: (1) Ordinarily letters near the center of the alphabet are used in the way explained in this section, especially f, g, h, F, G, H. (2) When multiplication is meant, and confusion might occur, we write $s \cdot x$, $(s)(x)$, or sx. Hence, in the absence of indications to the contrary, $s(x)$ stands for "s of x."

Find $f(-1)$ for: (c) $f(x) = (x - 1)^2$, (d) $f(x) = -x$, (e) $f(x) = x^2$, (f) $f(x) = x + y$, (g) $f(x) = x^3$. (h) Argue that if $n(x) = -x$, then $n[n(x)] = x$.

If $F(x)$ = the father of x, $F[F(x)]$ = the father of the father of x = the paternal grandfather of x. Let $M(x)$ = the mother of x. Find: (i) $M[M(x)]$, (j) $M[F(x)]$, (k) $F(F[F(x)])$.

Reading "f-of-x" for $f(x)$ may seem strange, but it is a natural outcome of ordinary ways of speaking. Thus, suppose we let $f(x) = x^2$ = the

square of x. Here f takes the place of "the square" and the parentheses indicate "of." Similarly, in the last exercise F stands for "the father." We shall discuss the meaning of f in $f(x)$ more fully in Chapter 5. For the present the reader may think of it as standing for the relation between the values of x and the corresponding values of the formula.

The functional notation is very widely used in mathematics, and for this reason it is worth while to develop some skill with it. It is very simple to use if we remember simply that whatever appears within the parentheses after f in $f(\ )$ is to be substituted for x in the formula for which $f(x)$ is an abbreviation. Then the result is to be interpreted as the value of the formula corresponding to this value of x. If a is a constant and the formula contains no other variables, $f(a)$ is a constant synonymous with the constant resulting from substituting a for x in the formula. For example, if $f(x) = x^2$ and $a = 2$, $f(a) = f(2) = 4$. If a is a variable or the formula involves other variables, $f(a)$ is a formula synonymous with the new formula obtained by substituting a for x. For example, if $f(x) = 2y + x$, $f(2) = 2y + 2$ and $f(x^2) = 2y + x^2$.

Let $f(x) = 3x + 2$ and $g(x) = 1 - x$. Find: (l) $f(4)$, (m) $g(4)$, (n) $f(x + 2)$, (o) $g(1 - x)$, (p) $f(f(x))$, (q) $g(g(x))$, (r) $f(g(x))$, (s) $g(f(x))$.

Often the functional notation is used to stand for an unspecified formula. When $f(x)$ is used in this way, both f and x are variables, whereas when $f(x)$ has been temporarily defined, only x is a variable. Indeed we have been using the functional notation in this way, for we intended (9) to hold no matter what formula had been abbreviated by $f(x)$.

(t) How would you interpret $p(x) \rightarrow q(x)$? What sort of formulas would be significant substitutes here for $p(x)$ and $q(x)$? (u) Suggest a substitute for $p(x)$ and $q(x)$ for which the implication is a law.

Later, when the functional notation is used to stand for any one of many formulas, we shall be able to make substitutions for the letter in front of the parentheses. For the present we treat "$f(x)$" as though it were a variable, and we consider any term to be a significant substitute for "$f(x)$" and similar expressions. But we have as yet no constants that are significant substitutes for "f." The reason we say that any term may substitute for "$f(x)$" is that we may wish to substitute a constant or an expression not involving "x." For example, suppose we let $f(x) = x - x$. Since the right member is 0, we have $f(x) = 0$ for all values of "x." This may seem strange, but it is often convenient to let "$f(x)$" stand for a constant.

(v) Let $f(x) = (x - 1)^2 - x^2 + 2x$. Find $f(3)$, $f(7)$, $f(b)$, $f(-3)$. Show that $f(x) = 1$ for all values of x.

The functional notation can easily be extended to deal with formulas in which more than one variable appears. We simply adopt some letter as before and place in parentheses after it a list of the variables in which we are interested. Thus we may let $f(x, y) = x - y$. Then we may as before make any significant substitutions throughout. For example, $f(y, x) = y - x$, $f(2, 4) = 2 - 4$, $f(0, 1) = 0 - 1$, $f(1, 0) = 1 - 0$. It is important, of course, to notice the order of the variables in the abbreviation and place the substitutes in the same order.

Let $h(x, y) = x + 2y$. Find: (w) $h(2, 3)$, (x) $h(a, b)$, (y) $h(-1, -2)$, (z) $h(0, 0)$.

PROBLEMS

In Problems 1 through 3 let $f(x) = 1 - x^2$, $g(x) = 2x$.

1. Find $f(x) + 1$, $f(x + 1)$.
2. Find $g(x) - 3$, $g(x - 3)$.
3. Find $f(x) + f(y)$, $f(x + y)$.

4. State a conclusion from Problems 1 through 3 in terms of the notion of distributivity.

Find $f(f(x))$ for $f(x)$ defined as in 5 through 10.

5. x^2. 6. $-x$. 7. $\sim x$.
8. $1/x$. 9. $2x$. 10. $3 - x$.

Find $f(g(x))$ and $g(f(x))$ where $f(x)$ and $g(x)$ are defined as in 11 through 15.

11. x^2, $\sqrt{x}$ with $x \geq 0$. 12. $2x$, $x/2$. 13. $-x$, $-x$.
14. $2x - 1$, $(1 + x)/2$. 15. $1 + 1/x$, $1/(x - 1)$.

In Problems 16 through 26 let $T(x)$ = the truth value of x.

★16. What are the significant values of x?
★17. What are the values of $T(x)$?
★18. Show that $T(p \wedge q) = T(p) \cdot T(q)$.
★19. $T(\sim p) = 1 - T(p)$.
★20. $T(p \vee q)$ = the larger of $T(p)$ and $T(q)$ or either if they are equal.
★21. $T(p \wedge q)$ = the smaller of $T(p)$ and $T(q)$ or either if they are equal.
★22. $T(p \vee q)$ = the smaller of $T(p) + T(q)$ and 1.
★23. $T(p \rightarrow q) = T[T(p) \leq T(q)]$.
★24. $T(p \leftrightarrow q) = T[T(p) = T(q)]$.
★25. $T(p \vee q) = T(p) + T(q) - T(p) \cdot T(q)$.
★26. $T(p \underline{\vee} q) = |T(p) - T(q)|$.

27. If $f(x, y) = x - y$, prove $f(y, x) = -f(x, y)$.
28. Find some definitions of $f(x, y)$ so that $f(x, y) = f(y, x)$.
29. $f(x, y) = x^2 + y - xy$. Find $f(2, y)$, $f(2, x)$, $f(y, x)$, $f(x, f(x, y))$.

30. Repeat Problem 29 for $f(x, y) = (y = 2x)$.
31. Repeat Problem 29 for $f(x, y) = y/x$.

In Problems 32 through 35 let $J(p, q) = p \wedge q$, $D(p, q) = p \vee q$, $N(p) = \sim p$, $C(p, q) = p \rightarrow q$.

★32. Argue that $D(p, q) = D(q, p)$.
★33. Show that $N[D(p, q)] = J[N(p), N(q)]$.
★34. Define D and C in terms of J and N.
★35. State (2-7-23) and (2-7-38) using these symbols.

36. If h = heredity, i = environment, and p = the personality, what would it signify to write $p = f(h, i)$?

37. If t = the tax rate of x, I = the income of x subject to tax, and T = the tax paid by x, what would we mean by $T = g(t, I, x)$? Would it be adequate to write $T = g(t, I)$?

38. As the reader may have observed, we are almost always omitting the quotation marks called for by the discussion in Section 1-3. He may find it interesting to decide where quotation marks could properly be inserted, especially in the present section.

ANSWERS TO EXERCISES

(a) $f(1) = 1 - 1$, $f(2) = 2 - 1$, $f(-3) = (-3) - 1$, $f(b) = b - 1$, $f(x - 1) = (x - 1) - 1$, $f(x + 2) = (x + 2) - 1$, $f(x + 4) = (x + 4) - 1$, $f(a + b + c) = (a + b + c) - 1$. (b) $\sim p, \sim\sim p, \sim\sim p, \sim(p \vee q), \sim(p \wedge q)$. "$N(7)$" is nonsense. (c) $(-1 - 1)^2$. (d) $-(-1)$. (e) $(-1)^2$. (f) $-1 + y$. (g) $(-1)^3$. (h) $n[n(x)] = n(-x) = -(-x) = x$. (i) The maternal grandmother of x. (j) The paternal grandmother of x. (k) The paternal great grandfather on the father's side of x.

(l) 14. (m) -3. (n) $3(x + 2) + 2$. (o) $1 - (1 - x)$. (p) $3(3x + 2) + 2$. (q) $1 - (1 - x)$. (r) $3(1 - x) + 2$. (s) $1 - (3x + 2)$. (t) Some propositional formula implies some propositional formula; sentences. (u) Let $p(x) = (x = 1)$, $q(x) = (2x = 2)$. (v) All substitutes yield 1, since $f(x) = x^2 - 2x + 1 - x^2 + 2x = 1$. (w) 8. (x) $a + 2b$. (y) -5. (z) 0.

ANSWERS TO PROBLEMS

1. $2 - x^2$, $1 - (x + 1)^2$. 2. $2x - 3$, $2(x - 3)$. 3. $1 - x^2 + 1 - y^2$, $1 - (x + y)^2$. 5. x^4. 7. x. 9. $4x$. 11. x. 13. x. 15. x. 16. Propositions. 17. 0 and 1. 27. $f(y, x) = y - x = -(x - y) = -f(x, y)$. 29. $4 + y - 2y$, $4 + x - 2x$, $y^2 + x - xy$, $x^2 + (x^2 + y - xy) - x(x^2 + y - xy)$. 31. $y/2$, $x/2$, x/y, y/x^2. 36. That h and i determine p, that is, that heredity and environment determine the personality. 37. That the tax is determined by the tax rate, the total income, and the individual. Not if we wished to indicate explicitly all the factors determining a person's tax. 38. This has been done in the paragraph preceding Exercise (v).

2–10 Quantifiers. As we have seen, a sentence may be true for all, some, or none of the significant values of its variables. If we wish to claim that a sentence is true for all values of its variables, we can simply display it as a theorem according to the convention explained in Section 1–8. However, it is convenient to have special symbols to indicate how many of the values of a sentence are true propositions. We call such symbols *quantifiers*.

We adopt the symbol $\forall x$ to mean "for all significant values of x." Thus, $\forall x\, p =$ (for all significant values of x, p). Usually we say "For all x, p" or "p is a law." If $f(x)$ is a sentence involving no variables other than x, then to claim that $\forall x\, f(x)$ is the same as to claim that $f(x)$ is a law. For example, $\forall x\, x^2 = x \cdot x$ is true because $x^2 = x \cdot x$ is a law. Similarly, $\forall p\, (p \to p)$ is true. We call $\forall$ the *universal quantifier*.

State in words: (a) $\forall x\, f(x)$, (b) $\forall x \sim f(x)$, (c) $\sim\forall x\, f(x)$, (d) $\sim\forall x \sim f(x)$.
True or false? (e) $\forall x\, x - x = 0$, (f) $\forall x\, (x + 1)^2 = x^2 + 1$.

We adopt the symbol $\exists x$ to mean "for some values of x" where "some" is understood to mean "one or more." *This meaning of "some" is universal in mathematics. In mathematics, some = at least one.* We also read $\exists x$ as "there exists an x such that." For example, $(\exists x\, x^2 = 0) =$ (for some value of x, $x^2 = 0$) $=$ (there exists an x such that $x^2 = 0$). We call $\exists$ the *existential quantifier*.

State in words: (g) $\exists x\, f(x)$, (h) $\exists x \sim f(x)$, (i) $\sim \exists x\, f(x)$, (j) $\sim \exists x \sim f(x)$.
True or false? (k) $\exists x\, x = x + 1$, (l) $\exists x\, x^2 \neq 0$.
Express, using quantifiers: (m) $\sim\sim p = p$ is a law, (n) $p \to \sim p$ is sometimes true, (o) $p \wedge \sim p$ is never true.

We take $\forall x\, f(x)$ as undefined and define $\exists x\, f(x)$. It is to be understood that significant substitutes for $f(x)$ in $\forall x\, f(x)$ are only sentences or formulas standing for sentences.

(1) **Def.** $$[\exists x\, f(x)] = \sim\forall x \sim f(x).$$

This definition merely reflects the idea that to claim that $f(x)$ is true for *some* x is the same as to say that it is *not false for all x!* Several examples have appeared in the exercises above.

Still further light is shed on the definition and the meaning of quantifiers by considering the special case when the quantified variable has only two significant values, say a and b. Then $\forall x\, f(x) =$ [$f(a)$ and $f(b)$ are true] $=$ $[f(a) \wedge f(b)]$, and $\exists x\, f(x) =$ [at least one of $f(a)$ and $f(b)$ is true] $= [f(a) \vee f(b)]$. Then $\sim\forall x \sim f(x) = \sim[\sim f(a) \wedge \sim f(b)]$. But the right member, by De Morgan's law, is synonymous with $f(a) \vee f(b)$, that is, with $\exists x\, f(x)$. Hence (1) is verified in this case.

The following theorems are easily derived.

(2) $$\sim[\forall x\, f(x)] = \exists x \sim f(x),$$

(3) $$\sim[\exists x\, f(x)] = \forall x \sim f(x).$$

(p) Illustrate (1) through (3) for $f(x)$: $x^2 \geq 0$, and for (q) $f(x)$: $x \wedge \sim x$.
(r) Give reasons in the following proofs of (2): $\exists x \sim f(x) = \sim \forall x \sim (\sim f(x)) = \sim \forall x\, f(x)$, and (3): $\sim \exists x\, f(x) = \sim(\sim\forall x \sim f(x)) = \forall x \sim f(x)$.

We now state an axiom that embodies the essential property of the universal quantifier.

(4) **Ax.** $$[\forall x\, f(x)] \to f(y).$$

In words, if a sentence is true for all values of x, it is true for any chosen value. In (4) we may substitute for $f(x)$ any sentence involving x, and for y any constant that is a significant substitute for x in $f(x)$. For example, if we make the substitution $[f(x){:}x/x = 1, y{:}2]$, we have $[\forall x\, x/x = 1] \to 2/2 = 1$.

The axiom (4) embodies formally the Rule of Substitution, because it says that if a sentence is a law, then any sentence obtained by a significant substitution is also a law. It permits us to "drop" the universal quantifier. Our convention about displaying laws, discussed in Section 1–8, simply means that we write laws in a form ready for substitution. If we did not have such a convention, we should have quantifiers in front of all laws and be obliged to apply (4) whenever we wished to substitute. If we did not have the convention of Section 1–8, we should write (4) as $\forall y\, [[\forall x\, f(x)] \to f(y)]$.

(5) $$f(y) \to [\exists x\, f(x)].$$

Theorem (5) says that if a sentence is true for a particular value of its variable, it is true for some value. With all quantifiers shown, it is $\forall y\, [f(y) \to [\exists x f(x)]]$. It is proved by the substitution $(f(x){:}\sim f(x))$ in (4) and then the use of $(p \to q) = (\sim q \to \sim p)$.

(s) Write out the proof just outlined. (t) Treat (2) through (5) in the case of only two significant values of x.

A theorem of the form $\exists x\, p$ is called an *existence theorem*. By (5), to prove $\exists x\, f(x)$, it is sufficient to exhibit a significant value of x, say y, such that $f(y)$ is true. Proofs of this kind are called *constructive*. For example,

(6) $\exists x\, a + x = a$.

Proof:

(a) $a + 0 = a$ (1–9–22),

(b) $a + 0 = a \rightarrow \exists x\, a + x = a$ Sub, (5)($f(x)$: $a + x = a$, y: 0),

(c) $\exists x\, a + x = a$ Inf, (b), (a).

(u) Similarly prove that $\exists x\, a \cdot x = a$.

Another way in which existence theorems are sometimes proved is to assume the contrary and derive a contradiction. Because of (1), to prove $\exists x\, p$, one may show that $\sim(\forall x \sim p)$, that is, that $\forall x \sim p$ is false. Such indirect existence proofs are called *nonconstructive*.

(v) Prove (6) using indirect proof to establish $\sim \forall x\, a + x \neq a$.

(7) $$\sim f(y) \rightarrow [\sim \forall x\, f(x)],$$

(8) $$[\forall x\, f(x)] \rightarrow [\exists x\, f(x)].$$

(w) Prove (7) and (8).

If we know a law of the form $\exists x\, f(x)$, we know that $f(x)$ for some value of x, but we may not know any particular solution. However, it would seem legitimate to adopt some name for the unknown value (or for one of the unknown values) of x, provided we do not assume anything about this value except that it is a solution of $f(x)$. For example, if we know that [$\exists x\, x$ has run the mile race in less than four minutes], we may draw some conclusions about whoever has done so without knowing who it is. In carrying on such a discussion it would be convenient to say: Let A be a person such that A has run the mile race in less than four minutes. Then we might say some things about A, such as that he must have been in good physical condition at the time. These considerations lead us to adopt the following additional rule of proof.

(9) RULE OF CHOICE: *If a law asserts the existence of a solution of a sentence "f(x)," then "f(c)" may appear as a step in a proof, provided "c" is treated as a constant and no other assumption is made about it.*

Like the Rule of Hypothesis, this rule could be omitted at the cost of lengthening and complicating proofs considerably. It is merely a convenience, and any proof using it could be replaced by one relying only on the previous rules. We shall see later how it is used.

PROBLEMS

Prove or disprove 1 through 6.

1. $\forall x\, x/x = 1$.
2. $\exists x\, x/x = 1$.
3. $\forall x \sim (x^2 = 2x)$.
4. $\exists x\, x = 2x$.
5. $\sim \exists x\, x^2 \neq x \cdot x$.
6. $\exists x\, x^2 = x \cdot x$.

Express 7 through 12 in ordinary language.

7. $\exists x\, x = -x$.
8. $\sim \forall x\, x^2 \neq 2x$.
9. $\sim \forall x\, |x| = x$.
10. $\exists x\, |x| = 0$.
11. $\forall x\, \forall y\, x^2 = y^2 \leftrightarrow x = y \lor x = -y$.
12. $\forall x\, \exists y\, x + y = 2$.

Express 13 through 18 in symbols two ways, one using $\exists$ and the other $\forall$.

13. $x = x + 1$ is false for all x.
14. $x^2 = x$ is not always false.
15. $x^2 = x$ is sometimes true.
16. "All plant and animal tissues are made up of units known as cells.... All cells contain a living substance known as protoplasm." (Harvey E. Stork, *Studies in Plant Life*, 1945, p. 19)
17. $|x| = x$ is sometimes false.
18. There is an x such that for all y $y/y = x$.

19. Complete the following verbalizations of (2) and (3): (2): To say that it is false that $f(x)$ is true for all x is to say that there is some x for which $f(x)$ is ________. (3): To say that it is ________ that ________ is true for some x is to say that $f(x)$ is ________ for ________ x.
20. Write out (1) through (5), (7) and (8) for three values of x: a, b, and c.
21. What is the scope of the first $\sim$ in (1)? of $\forall x$?
22. Would (1) mean anything different if we substituted y for x in only the left member? in both members?
23. What theorem in this section expresses the idea that to prove that a sentence is not a law it is sufficient to cite a counterexample?
24. Which theorem expresses the idea that in order to prove that a sentence is sometimes true it is sufficient to cite a case in which it is true?
25. Prove that it is false that all natural sciences are based primarily on controlled experiment.
26. Prove $\sim \forall x\, (x < 1{,}000{,}000)$.
27. Prove $\forall a\, 1/a \neq 0$.

Quantifiers are helpful in clarifying the meaning of statements involving "all," "some," and "none." For example,

(10) All squares are positive $= \forall x\, x^2 > 0 = \sim (10')$.

(10′) Not all squares are positive $= \sim \forall x\, x^2 > 0 = \sim (10)$

$= \exists x \sim (x^2 > 0)$

= Some squares are not positive.

(11) Some squares are positive $= \exists x\, x^2 > 0 = \sim (11')$.

(11′) No squares are positive $= \sim \exists x\, x^2 > 0 = \sim (11)$

$= \forall x \sim (x^2 > 0)$

= All squares are not positive.

Often the last form in (11′) is used carelessly when (10′) is meant. Such usage is incorrect (see H. W. Fowler, *A Dictionary of Modern English Usage*, under) "not"). To avoid confusion it is best not to use the last form in (11′).

28. Justify the identities (10) through (11′).
29. Which of the sentences within the identities (10) through (11′) are true?
30. Repeat (10) through (11′) with the sentence "All men are mortal" ($\forall x$ x is a man $\rightarrow$ x is mortal).

Taking the negation of a quantified statement is easy if one can express it with quantifiers as above and then make use of laws of quantification. It should be noted that because of (1) through (3) and the flexibility of the spoken language, there are many ways of stating each idea. The solution to any confusion may be found by translating into mathematical terms, re-expressing, and translating back, being sure at each translation that the meaning is unchanged. Express in normal English in two ways the negation of each of 31 through 34.

31. All Negroes favor an end to segregation.
32. Some Negroes favor an end to segregation.
33. No Negroes favor an end to segregation.
34. Not all Negroes favor an end to segregation.

Express 35 through 43 symbolically. Then express the negation in normal English in two ways.

35. Nothing is good enough for him.
36. Not all white people favor segregation.
37. He is always happy.
38. He is not always happy.
39. He is always not happy.
40. He is sometimes happy.
41. He is sometimes not happy.
42. "All that glisters is not gold."
43. There's nothing new under the sun.

The following laws express formally the meaning of equality.

(12) $[x = y] \leftrightarrow \forall f\, [f(x) = f(y)]$,

(13) $[x = y] \leftrightarrow \forall f\, [f(x) \leftrightarrow f(y)]$.

44. What rule of proof corresponds to (12)?
★45. Prove that the left member of (12) implies the right.
★46. Prove the converse by letting $f(x) = x$.
★47. Why do these two results prove (12)?
★48. Prove (13).
49. Write out the laws of quantification for the case of four values of x: a, b, c, and d.
★50. Justify the following:

(14) Ax. $[\forall x(f(x) \wedge g(x))] = [\forall x\, f(x)] \wedge [\forall x\, g(x)]$,

(15) $[\exists x(f(x) \vee g(x))] = [\exists x\, f(x)] \vee [\exists x\, g(x)]$,

(16) Ax. $[\forall x(f(x) \rightarrow g(x))] \rightarrow [(\forall x\, f(x)) \rightarrow (\forall x\, g(x))]$,

(17) $[\forall x(f(x) \rightarrow g(x))] \rightarrow \{[\exists x\, f(x)] \rightarrow [\exists x\, g(x)]\}$.

51. Show that (14) and (15) are false if $\vee$ and $\wedge$ are interchanged.

Answers to Exercises

(a) [For all x, $f(x)$] = [Every significant value of x is a solution of $f(x)$] = [$f(x)$ is always true] = [$f(x)$ is a law]. (b) [For all x, not-$f(x)$] = [$f(x)$ is always false] = [No value of x is a solution of $f(x)$] = [$f(x)$ is never true] = [For no x, $f(x)$]. (c) [For not all x, $f(x)$] = [Not every significant value of x is a solution of $f(x)$] = [$f(x)$ is not always true] = [$f(x)$ is not a law]. (d) [For not all x, $\sim f(x)$] = [Not every significant value of x fails to be a solution of $f(x)$] = [$f(x)$ is not always false] = [the negation of $f(x)$ is not a law] = [For some x, $f(x)$]. (e) 1. (f) 0, since the sentence is false for $x = 1$.

(g) [For some x, $f(x)$] = [At least one significant value of x is a solution of $f(x)$] = [$f(x)$ is sometimes true] = [There exists an x such that $f(x)$] = [$\sim f(x)$ is not a law] = Exercise (d). (h) [For some x, not-$f(x)$] = [Some significant value of x is not a solution of $f(x)$] = [$f(x)$ is sometimes false] = [There exists an x such that $\sim f(x)$] = Exercise (c). (i) [For no x, $f(x)$] = [No significant value of x is a solution of $f(x)$] = [There exists no x such that $f(x)$] = Exercise (b). (j) [For no x, not-$f(x)$] = [No significant value of x fails to be a solution of $f(x)$] = [There exists no x such that $f(x)$ is false] = Exercise (a). (k) 0. (l) 1.

(m) $\forall p \sim \sim p = p$. (n) $\exists p\, p \rightarrow \sim p$. (o) $\sim \exists p\, p \wedge \sim p$ or $\forall p \sim (p \wedge \sim p)$. (p) (For some x, $x^2 \geq 0$) = (It is false that for all x, $x^2 < 0$). (It is false that for all x, $x^2 \geq 0$) = (For some x, $x^2 < 0$). (There is no x such that $x^2 \geq 0$) = (For all x, $x^2 < 0$). (r) (1) and $\sim\sim p = p$. (s) $f(y) \rightarrow \sim[\sim f(y)] \rightarrow \sim[\forall x \sim f(x)] \rightarrow \exists x\, f(x)$ by (2–5–6), (2–7–3), (4), (2–5–28), (1). (t) (2), (3), (4), (5) become (2–5–7), (2–5–8), (2–7–6), and (2–7–9).

The results of Exercise (t) should not be surprising. The universal quantifier asserts the conjunction of all the values of the formula that follows it, since $\forall x\, f(x)$ means that every value of $f(x)$ is true. Hence $\forall x\, f(x) = f(a) \wedge f(b) \wedge f(c) \wedge \ldots$, where the values of x are $a, b, c, \ldots$. Similarly $\exists x\, f(x) = f(a) \vee f(b) \vee f(c) \vee \ldots$.

(u) (1–9–25). (v) $(\forall x\; a + x \neq a) \to (\forall x\; x \neq 0) \to (0 \neq 0)$; by (2–5–28), $(0 = 0) \to (\sim \forall x\; a + x \neq a)$. (w) (7) = (4) by (2–5–28); (8) follows from (4), (5), and (2–7–12).

Answers to Problems

1. True. Zero is not a value of x here. 3. False. $2^2 = 2 \cdot 2$. 5. True, since $\forall x \sim (x^2 \neq x \cdot x)$. 7. There is a number equal to its own negative. 9. Not all numbers are equal to their absolute values. 11. If the squares of two numbers are equal, the numbers are equal or one is the negative of the other. 13. $\forall x\; x \neq x + 1; \sim \exists x\; x = x + 1$. 15. $\exists x\; x^2 = x, \sim \forall x\; x^2 \neq x$. 17. $\exists x\; |x| \neq x; \sim \forall x\; |x| = x$. 19. False; false; $f(x)$; false; all. 21. $[\forall x \sim f(x)]; \sim f(x)$.

28. Use the theorems with $(f(x){:}x^2 > 0)$. 29. $x^2 = 0$ for $x = 0$; otherwise $x^2 > 0$. 30. All men are mortal = ~ (Not all men are mortal) = ~ (Some men are not mortal). Some men are mortal = ~ (No men are mortal). 31. Not all Negroes favor an end to segregation = Some Negroes do not favor an end to segregation. 32. No Negroes favor an end to segregation = All Negroes are against an end to segregation. 33. Some Negroes favor an end to segregation = Not all Negroes do not favor an end to segregation. 34. All Negroes favor an end to segregation = No Negroes do not favor an end to segregation. 35. $\sim \exists x\; x$ is good enough. 37. $\forall t$ He is happy at time t. 42. $\sim \forall x\; x$ glisters $\to x$ is gold. ($\neq \forall x\; x$ glisters $\to x$ is not gold). This is an example of improper usage. 44. Replacement.

★2–11 Multiple quantification. The most useful applications of quantifiers occur when more than one variable is quantified. We have the following possibilities for two variables.

(1) $\forall y\, \forall x \ldots$ = For every y, for every $x \ldots$.

(2) $\forall x\, \forall y \ldots$ = For every x, for every $y \ldots$.

(3) $\forall x\, \exists y \ldots$ = For every x there exists a y such that ....

(4) $\forall y\, \exists x \ldots$ = For every y there exists an x such that ....

(5) $\exists y\, \forall x \ldots$ = There exists a y such that for every $x \ldots$.

(6) $\exists x\, \forall y \ldots$ = There exists an x, such that for every $y \ldots$.

(7) $\exists x\, \exists y \ldots$ = There exists an x such that there exists a y such that ....

(8) $\exists y\, \exists x \ldots$ = There exists a y such that there exists an x such that ....

Read (1) through (8) with each of the following in place of the dots, and decide on truth values: (a) $x + y = 2x$, (b) $xy = x$.

Exercises (a) and (b) suggest that (1) = (2) and (7) = (8), so that the order of quantification is immaterial when only like quantifiers are involved. But clearly no two of (3) through (6) are synonymous, so that unlike quantifiers are *not commutative.*

(9) **Ax.** $[\forall x\, \forall y\, f(x, y)] = [\forall y\, \forall x\, f(x, y)]$,

(10) $[\exists x\, \exists y\, f(x, y)] = [\exists y\, \exists x\, f(x, y)]$.

To avoid unnecessary parentheses, we follow the convention that the scope of a quantifier is the entire following expression or extends to the end of the smallest parentheses within which it lies. This convention is illustrated in (1) through (8) where, for example, $\forall y\, \forall x \ldots = \forall y[\forall x \ldots]$, in (2–10–1) where the right member means $\sim[\forall x[\sim f(x)]]$, and in (2–10–8) where the parentheses are essential. Because (1) = (2) and (7) = (8) we usually read $\forall x\, \forall y$ as "for all x and y" and $\exists x\, \exists y$ as "for some x and y." We also adopt the following abbreviations.

(11) **Def.** $\forall x{,}y\; f(x, y) = \forall x\, \forall y\, f(x, y)$,

(12) **Def.** $\exists x{,}y\; f(x, y) = \exists x\, \exists y\, f(x, y)$.

It is easy to extend the laws on negation to two quantifiers by simply applying (2–10–2) and (2–10–3) carefully.

(13) $[\sim \forall x\, \forall y\, f(x, y)] = [\exists x\, \exists y \sim f(x, y)]$,

(14) $[\sim \forall x\, \exists y\, f(x, y)] = [\exists x\, \forall y \sim f(x, y)]$,

(15) $[\sim \exists x\, \forall y\, f(x, y)] = [\forall x\, \exists y \sim f(x, y)]$,

(16) $[\sim \exists x\, \exists y\, f(x, y)] = [\forall x\, \forall y \sim f(x, y)]$.

(c) Prove (13) through (16). (d) Negate each statement made in Exercise (a) and compare truth values.

Numerous other quantifiers could be defined, but only one is in general use. It is $\exists! x$, which is read "for one and only one value of x" or "for just one x."

(17) **Def.** $[\exists! x\, f(x)] = [\exists x\, f(x)] \wedge \forall x{,}y\, [(f(x) \wedge f(y)) \to x = y]$,

(18) $[\exists! x\, f(x)] \to [\exists x\, f(x)]$,

(19) $[\sim \exists! x\, f(x)] = \{[\forall x \sim f(x)] \vee \exists x{,}y\, [f(x) \wedge f(y) \wedge y \neq x]\}$.

(e) Verbalize (17) through (19) and argue for them informally. (f) Discuss $\exists! x\, f(x)$ for the case where x has only two significant values and the case where it has only three. (g) Prove (18) and (19). (h) Prove $\forall y\, \exists! x\; y + x = y$. (i) Does $\exists!$ commute with $\forall$ or $\exists$?

A variable that is within the scope of a quantifier is said to be *quantified*. From the discussion in this and the previous section the reader is already aware that some care is necessary when dealing with quantified variables. Consider, for example,

(20) $$\exists x\, \forall y\; x \text{ dislikes } y,$$

which says that there exists an x such that for all y, x dislikes y; that is, there exists a person who dislikes everybody. Now if we substitute x for y we have

(21) $$\exists x\, \forall x\; x \text{ dislikes } x,$$

which says that there exists an x such that for all x, x dislikes x. If it makes sense at all, it certainly means something different from (20). On the other hand, if we substitute z for y in (20), we get

(22) $$\exists x\, \forall z\; x \text{ dislikes } z.$$

This evidently has the same meaning as (20). We see that for a quantified variable we may safely substitute another variable only if the latter does not already appear within the scope of the quantifier.

Since $\forall x\, f(x)$ means that the formula $f(x)$ is true for all values of the variable x, the substitution of a constant for x would result in nonsense. Similar remarks apply to $\exists$ and $\exists!$. We see then that we cannot substitute constants for quantified variables. Of course, if $\forall x\, f(x)$ is a law, we can substitute a constant for x in $f(x)$ standing alone, but not in the whole expression $\forall x\, f(x)$. For similar reasons it is not convenient to substitute formulas for quantified variables.

If we recall the definition of "variable" in Section 1–3, we see that in the context $\forall x\, f(x)$, x is not a variable (since constants are not to be substituted for it), even though it is a variable in the formula $f(x)$ considered alone. A symbol that is a variable in part of its context considered alone but is not a variable in the whole context is said to be a *dummy* in the whole context. For example, x is a variable in $x - x = 0$, but it is not a variable in $\forall x\; x - x = 0$, so we call it a dummy in the second context. We limit substitutions for dummies to variables not appearing elsewhere in the same context.

From the preceding discussion it is evident that a quantified variable is a dummy. In (20) both x and y are dummies. Hence (20) actually involves no variables, which fits with the fact that when we express (20) in words it is seen to be a statement and not a propositional formula. However, "$\forall y\; x$ dislikes y" involves the variable x, and it is a propositional formula.

(j) When we use the Rule of Hypothesis, any variable in the hypothesis becomes a dummy. Why? (k) Why in using the Rule of Hypothesis is it legitimate to substitute a variable not already appearing elsewhere in the context?

Problems

Letting S = x is the father of y, where x and y are men, state 1 through 18 in words and decide on truth values.

1. $\forall x\, \forall y\, S$.	2. $\forall y\, \forall x\, S$.	3. $\forall x\, \exists y\, S$.
4. $\forall y\, \exists x\, S$.	5. $\exists x\, \forall y\, S$.	6. $\exists y\, \forall x\, S$.
7. $\exists x\, \exists y\, S$.	8. $\exists y\, \exists x\, S$.	9. $\forall x\, \exists! y\, S$.
10. $\forall y\, \exists! x\, S$.	11. $\exists! x\, \forall y\, S$.	12. $\exists! y\, \forall x\, S$.
13. $\exists x\, \exists! y\, S$.	14. $\exists y\, \exists! x\, S$.	15. $\exists! x\, \exists y\, S$.
16. $\exists! y\, \exists x\, S$.	17. $\exists! x\, \exists! y\, S$.	18. $\exists! y\, \exists! x\, S$.

19. Repeat 1 through 18 for S = the season x immediately follows the season y.

State 20 through 26 symbolically.

20. Every man has one and only one father.
21. Some men are brothers.
22. All men are brothers.
23. There are two numbers whose product is 16.
24. Every man has ancestors.
25. Any even number greater than 4 is the sum of 2 odd primes (Goldbach's conjecture).
26. Ideas have consequences.

27. Negate each of Problems 1 through 18.
28. Do the same for Problem 19.
29. Prove $\exists! x\, \forall y\; x + y = y + x = y$.
30. Prove $\exists! x\, \forall y\; x \cdot y = y \cdot x = y$.
31. Prove $\forall x\, \exists! y\; x + y = y + x = 0$.
32. Prove $\forall x\; x \neq 0 \rightarrow \exists! y\; xy = 1$.
33. Is $\exists! x\, \exists! y\, f(x, y) = \exists! y\, \exists! x\, f(x, y)$ a law?

Answers to Exercises

(a) F, F, T, T, F, F, T, T. (b) F, F, T, T, T, T, T, T. (c) Apply (2–10–2) and (2–10–3). (d) Apply (13) through (16); for example, $[\sim \exists y\, \forall x\; x + y = 2x] = [\forall y\, \exists x\; x + y \neq 2x]$ = [For every y there exists an x such that $x + y$ does not equal $2x$]. (e) The right member of (17) reads "There exists an x

such that $f(x)$, and if both x and y are solutions of $f(x)$ then x and y are identical." The right member of (17) $\rightarrow$ its first term, by (2–7–6); hence (18) holds. To get (19) we negate both members of (17). The right member of the result, by De Morgan's law, becomes $\sim \exists x\, f(x) \vee \sim \forall x,y\, [f(x) \wedge f(y) \rightarrow x = y]$. The first term in this conjunction is $\forall x \sim f(x)$ by (2–10–3). The second simplifies as follows: Let $p = f(x) \wedge f(y)$, and let $q = (x = y)$. Then we have $[\sim \forall x\, \forall y\, p \rightarrow q] = [\exists x\, \exists y \sim (p \rightarrow q)] = [\exists x\, \exists y\, p \wedge \sim q]$ by using (16) and (2–5–24).

(f) For two significant values, $[\exists! x\, f(x)] = [f(a) \veebar f(b)]$. (g) Along lines indicated in Exercise (e). (h) $\forall y\, y + 0 = y$. Hence $\forall y\, \exists x\, y + x = y$. (i) No. (j) We $[y + x = y \wedge y + x' = y] \rightarrow [y + x = y + x'] \rightarrow [x = x']$. (k) It makes excluded substitutions in a hypothesis introduced by this rule. no difference what symbols we use so long as they are not otherwise involved.

Answers to Problems

1. Every man is the father of all men; 0. 2. Same as Problem 1. 3. Every man is the father of some man; 0. 4. Every man is a son of some man; 1. (This truth value and others below may be considered controversial by some, but this book is not concerned with theology.) 5. Some man is the father of all men; 0. 6. Some man is a son of all men; 0. 7. Some man is the father of some man; 1. 8. Same as Problem 7. 9. Every man has just one son; 0. 10. Every man has just one father; 1. 11. Just one man is the father of all men; 0. 12. Just one man is a son of all men; 0. 13. Some man is the father of just one son; 1. 14. There is a man who is a son of just one father; 1. 15. Just one man is the father of a son; 0. 16. Just one man is a son of some man; 0. 17. Just one man is the father of just one son; 0. 18. Same as 17.

20. $\forall x\, [x$ is a man $\rightarrow \exists! y\; y$ is the father of $x]$. 21. $\exists x\, \exists y\; x$ and y are brothers. 22. $\forall x\, \forall y\; x$ and y are brothers. 23. $\exists x\, \exists y\; xy = 16$. 24. $\forall x\, [x$ is a man $\rightarrow (\exists y\; y$ is an ancestor of $x)]$. 25. $\forall x\, [x$ is even $\wedge\; x > 4 \rightarrow \exists y\, \exists z\; x = y + z \wedge y$ and z are odd primes$]$. 26. $\forall x\, [x$ is an idea $\rightarrow \exists y\; y$ is a consequence of $x]$.

2–12 Heuristic. Logic is concerned with deductions that are certain. As explained in Section 2–6, in a deductive theory all the theorems follow with certainty if the axioms, definitions, and rules of proof are accepted. Although strict logical thinking is very important in science and in daily living, it is by no means the only kind of worth-while intellectual activity. Indeed, without observation, experiment, discovery, and inspiration there would be nothing to be logical about!

Before a theorem can be proved, it must be conceived and considered plausible. Before axioms can be used, they must be formulated. Before a theory can be built about certain concepts, basic terms must be created and defined. Before a logical theory can be constructed, its subject matter must be understood to some extent. Before an answer can be justified, it must be found. Such activities are exploratory and uncertain, as are all activities of creation and discovery.

The word "heuristic" is used as an adjective to describe activities serving to discover or reveal, including arguments that are persuasive and plausible without being logically rigorous. It is used as a noun to refer to the science and art of heuristic activity.

Heuristic is obviously very important, and the high rewards that sometimes go to its most successful devotees are well deserved. Why, then, is not more attention devoted to studying and perfecting this art? The most obvious answer is that, by definition, it includes just those things that have not yet been systematized and made scientific. As soon as we discover a formula for solving a particular kind of problem, we no longer need heuristic for dealing with it. It is possible, also, that most heuristic experts prefer to practice the art rather than to talk about it!

(a) Look up "heuristic" in an unabridged dictionary. (b) Could heuristic be entirely formalized?

We do not attempt here to discuss heuristic in any detail. Instead we make some brief observations and refer the reader to the very few books on the subject. Among these, the best one for a beginning is *How To Solve It*, by G. Polya (Princeton University Press, 1945, and also in a paperback edition issued by Doubleday, 1957.) In it we read: "The first rule of discovery is to have brains and luck. The second rule of discovery is to sit tight and wait till you get a bright idea." A third rule might be: while waiting for brains, luck, and a bright idea, spend some time experimenting, some time thinking, and some time resting. If there is any rule that applies to all heuristic situations, it is: observe, experiment, think, and rest. The greatest difficulty comes in combining thought and action. Some persons act without thinking. Others are afraid to act until they know the answer. In Shakespeare's classic lines,

Our doubts are traitors,
And make us lose the good we oft might win
By fearing to attempt. (Measure for Measure)

Polya gives a very useful outline of steps that may be helpful in problem solving. The main steps are: (1) understand the problem, (2) devise a plan, (3) carry out the plan, (4) check and reconsider the results. Although the bare outline may not seem to be very helpful, Polya gives many practical suggestions under each step. The methods are applicable to all kinds of heuristic activity, both to problems where a simple answer is sought and problems consisting of discovering a proof. Perhaps a fifth step might be added: (5) clean up the problem. This suggestion would involve reformulating the problem and its solution in a logical and clear way after eliminating all the experiments and confusion of the heuristic process. Step (5) has usually been performed on the mathematics that students see in books, which is the reason there is often no clue to how the results were discovered.

It is possible to formulate heuristic rules that serve as guides in creative work. They differ from rules of proof or theorems of mathematics since they suggest possible lines of activity and do not claim to be laws. For example,

Heuristic Rule: An identity may sometimes be proved by writing an identity whose left member is the left member of the identity to be proved and then modifying the right member by replacements until it is the same as the right member of the identity to be proved.

(c) Give an example of our use of the procedure described in the preceding rule. (d) Suggest an obvious modification of the rule. (e) Formulate a heuristic rule to cover the cases in which we work with both sides of an identity. (f) Formulate other heuristic rules for doing proofs. Keep a list and add to it. (g) Read Polya's *How to Solve It.*

Among the very few books on heuristic, the student will find the following most helpful:

How to Study—How to Solve, by H. M. Dadourian, Addison-Wesley, 1951. Emphasizes problems in elementary mathematics.

The Psychology of Invention in the Mathematical Field, by J. Hadamard, Princeton University Press, 1949. A fascinating little book by a great mathematician.

Also interesting are:

Mathematics and Plausible Reasoning (two volumes), by G. Polya, Princeton University Press, 1954.

The Creative Process, by B. Ghiselin, University of California Press, 1952.

Problem-Solving Processes of College Students, by B. S. Bloom and L. J. Broder, University of Chicago Press, 1950.

Productive Thinking, by M. Wertheimer, Harper & Brothers, 1945.

Problems

Problems 1 through 12 are to be solved in the light of Polya's discussion of heuristic and his outline. The answer should include an analysis of how the problem was solved as well as a justification of the solution. Of course, all problems in this book are exercises in heuristic.

1. How much must a merchant mark up an article costing him $12 in order that he may later mark it down 25% from its sale price and still make a profit of 20% on it?

2. A man agrees to build a garage to be owned jointly by himself and his neighbor, each to share equally in the cost. The man does all the labor himself, working 200 hours, and the neighbor contributes $500 worth of material. If the man's labor is worth $1.50 an hour, how should settlement be made?

3. According to a recent news item, a mail-order house installed a $500,000 electronic computer with which ten girls could do the work formerly done by 150. Is this a money-saving move?

4. Which would you rather have, four annual salary increases of $100 each or two salary increases of $200 each at two-year intervals?

5. Did Greek authors of the pre-Christian era use footnotes?

6. A lady asks how long she should make a valance to go all around a circular table of diameter 3 feet $6\frac{1}{2}$ inches. What do you tell her?

7. Fill in the blank in the following quotation from the *Wall Street Journal.* "Alden's, Inc., voted a quarterly of 30 cents on the increased common, payable July 1, to stock of record June 14. A 50% common stock dividend was paid May 7. Before that 37 1/2 cents was paid quarterly on the old common. In announcing the increase, Robert W. Jackson, president, said the 'previous common quarterly dividend of 37 1/2 cents is equivalent to 25 cents based on outstanding shares reflecting a 50% distribution made May 7, 1954. The current rate of 30 cents, therefore, represents a ____% increase in the common stock dividend.'"

8. A reed growing in the middle of a pond 8 feet in diameter just touches the edge when it is pulled over. How deep is the pool if one foot of the reed is above the water? (An old Chinese problem.)

9. Find a formula for the long diagonal of a rectangular box of sides a, b, and c.

10. Show that in chess it is impossible to reach a position in which all pawns of one color are in the same file.

★11. In chess, a "won position" for White (Black) is one from which White (Black) can force a win no matter how Black (White) plays. A "draw position" is one in which either player can force at least a draw. It has been proved that every position in chess is a won position for Black, a won position for White, or else a draw. There is no known method of deciding which of these possibilities is the actual one in any case, however. If we modify the rules to permit a player not to make a move when it is his turn, it follows that the initial position is either a draw or a won position for White. Prove this last statement.

12. A chess board can easily be completely covered by dominoes of a size that cover two adjacent squares. Show that this cannot be done if two diagonally opposite corner squares are removed from the board.

Let S, C, T, s, c, t be undefined terms satisfying the following axioms and subject to the laws of algebra of real numbers.

$$S^2 + C^2 = 1, \tag{1}$$

$$T = S/C, \tag{2}$$

$$t = C/S, \tag{3}$$

$$s = 1/S, \tag{4}$$

$$c = 1/C. \tag{5}$$

Prove 13 through 20.

13. $T^2 + 1 = c^2$.
14. $1 + t^2 = s^2$.
15. $CT = S$.
16. $Ss = Cc = Tt = 1$.
17. $(1 - S)(c + T) = C$.
18. $c/(T + t) = S$.
19. $T^2 = S^2T^2 + S^2$.
20. $(c^2 + s^2)/(sc) = T + t$.

21. An auditorium with 800 seats was not full one evening. Exactly $340 was taken in. Adults paid 50¢, children 20¢. Prove that more than three times as many adults as children were present.

If a sentence is true in a particular case we say that it is *verified* in that case. Verification is important. In empirical sciences it is the way in which scientific hypotheses are checked, but it should not be confused with proof. The physicist checks his hypotheses by attempting to verify them experimentally. If he finds a counterexample, he has to reject an alleged law. If he does not, he accepts the law tentatively. Usually he tries to prove the alleged law by mathematical methods from other laws thought to be correct. In short, he tries to prove his hypotheses by constructing a deductive theory, but this process is quite different from the process of checking by experiment and observation.

22. Write a sentence that is true for 100,000,000 values of its variable but is not a law.
23. Write one that is true for an infinite number of values of its variable, and also false for an infinite number!
24. Cite an example of an accepted scientific law and explain why it could not be verified for every case.
25. Could verification by numerous cases ever prove a law?

According to Webster, evidence is "that which furnishes, or tends to furnish proof, . . ." If $p \rightarrow q$, we call p *conclusive evidence* of q, but evidence for q includes also any proposition that verifies, makes plausible, or otherwise supports belief in q. For example, F. H. Elwell in *Elementary Accounting* (p. 83) writes: "The fact that a trial balance 'proves' does not necessarily mean that the ledger accounts accurately record all transactions of the period. . . . In general, however, the fact that the bookkeeper obtains a trial balance is usually accepted as evidence that the ledger accounts are correct."

26. Letting p = ledger accounts are correct, and q = books balance, which is true, $p \rightarrow q$ or $q \rightarrow p$?
27. In what sense is q evidence for p?

Since evidence includes so much, it is impossible to indicate all types. Certainly, however, when $p \rightarrow q$, q is used as evidence for p, and this is justified on the ground that the truth of q at least establishes conclusively that p *may* be true.

28. Justify the last remark by showing the implications of q being false in such a situation.
29. If $p \rightarrow q$, is $\sim q$ conclusive evidence for $\sim p$?

In legal practice evidence includes only such matter as conforms to the established rules of evidence. *Direct evidence* is evidence that "immediately establishes the fact to be proved by it," while *indirect evidence* is that which "establishes immediately collateral facts from which the main fact may be inferred."

30. If q is the fact to be proved, p expresses the direct evidence, and r the indirect evidence, indicate symbolically the relations between p, q, and r.
31. X is accused of stabbing Y. A testifies that he saw X stabbing Y. Direct or indirect? Conclusive?
32. Z testifies that he saw X running from the scene with a bloody knife, that Y gasped out X's name, and that Z saw no one else in the area. Direct or indirect? Convincing? Conclusive?

The last question indicates the frequency, and indeed the necessity, of making decisions on the basis of evidence that is, strictly speaking, not conclusive. The law requires merely certainty "beyond a reasonable doubt," which is much less certain than strict deductive proof. For example, in Problem 32 there may be doubts about Z's honesty, his memory, his accuracy of observation, the meaning of Y's last words, and the possibility of someone else having been present and committed the crime without Z's knowing it. Nevertheless, the evidence is convincing because we feel it very unlikely that these things be accompanied by Z's testimony. We ignore here the aspects of evidence that involve probability judgments, and consider merely examples in which (the evidence) $\rightarrow$ (the fact) and those in which it does not. Evidence of the latter type is usually called *circumstantial evidence.*

33. Would you call Z's testimony circumstantial?

Let a = (X committed the crime), and b = (X was present at the time of the crime). An *alibi* is a claim on the part of X to have been elsewhere, i.e., a claim that $\sim b$.

34. Which is correct, $a \rightarrow b$ or $b \rightarrow a$?
35. Why is a valid alibi conclusive proof of innocence?

Let G = the defendant is guilty, $I = \sim G$, C = evidence for G such that $G \rightarrow C$, but $\sim(C \rightarrow G)$, and E = the evidence such that $E \rightarrow G$.

36. Write various relations between G, I, and E.
37. What is the basis in logic for the requirement that the prosecution must prove G ("assumed innocent until proven guilty") rather than the alternative of requiring the defense to prove I ("guilt by accusation")? In this connection consider the possibility of both truth and falsity for E and C, and also the possibility that all evidence is inconclusive.