CHAPTER 3

ELEMENTARY THEORY OF SETS

3–1 The set concept. When we think about several things, we may consider them individually or we may view them as a single whole, class, collection, or aggregate. Mathematicians call such a whole a *set*, and they refer to the individual objects of which it is composed as *members* that *belong* to the set.

Words that designate sets are called collective nouns by grammarians. In ordinary discourse, collective nouns are used sometimes to refer to a set and sometimes to its members. For example, we may say "That bunch of grapes is heavy" or "That bunch of grapes have thick skins." The first sentence refers to the set, the second to the individual members. In mathematics, the name of a set refers only to the set as a whole.

An example of a set is the United States Senate. Its members are senators. Other examples of sets are the Los Angeles Dodgers, the planets, the living ex-Presidents, the even numbers, the possible opening moves in a game of chess, and the possible outcomes of an experiment.

(a) Describe the members of each of the sets named in the last sentence. (b) Among words used as variables to stand for sets are "collection," "flock," "herd," "bunch," and "crowd." Indicate the kinds of sets described by these words, and give other examples of collective common nouns.

The set concept plays a very important role in mathematics, and indeed in all clear thinking. It is one of the basic concepts in terms of which other mathematical ideas are explained. The purposes of this chapter are (1) to familiarize the student with the concepts and notations for dealing with sets, (2) to show the relations between sets and sentences, and (3) to apply set theory to further development of elementary algebra and to linear analytic geometry.

Answers to Exercises

(a) Baseball players; planets; Hoover, Truman; . . . -6, -4, -2, 0, 2, 4, 6, . . . ; any one of 20 different moves; say, different readings on a dial.

3–2 Designating sets. Since we frequently wish to say that an object is a member of some set, we adopt the special notation $a \in C$ to mean that a belongs to C. This is an informal definition. The symbol $a \in C$ will be introduced as a basic term of the theory. We write $a \notin C$ to mean that a is not a member of C.

(a) Define $a \notin C$ formally in terms of $a \in C$ and logical symbolism.

Suppose we wish to designate the set whose members are just the numbers 1, 2, and 3. We wish to distinguish between the set considered as a whole and the individual members. Hence it would be unsatisfactory to write "the set 1, 2, and 3." We could, of course, always write "the set whose members are 1, 2, 3," but it is convenient to have an abbreviation. We adopt the device of writing a list of the members within braces. For example, $\{1, 2, 3\}$ = the set whose only members are 1, 2, and 3. Note that the braces are read "the set whose members are . . . ."

Describe in words: (b) $\{1, 2\}$, (c) $\{2\}$, (d) $\{a, b, c\}$, (e) $\{0, 1\}$, (f) $\{0\}$.

We call the above way of designating sets the *roster notation*. We refer to the list of members that appear within the braces as the *roster* of the set. Note that the roster is an expression made up of names of the members; it is not a name of the set.

Use the roster notation to designate the following sets. (g) The positive integers less than 5. (h) The solutions of $2x = 5$. (i) The three major religious faiths in the U. S. (j) The digits greater than 2 and less than 7.

The roster notation is very convenient for sets with only a few members. However, there are sets with many members (e.g., the population of the world) and even with an infinite number of members (e.g., the points on a line). For such sets the roster notation is awkward or impossible. To think of some things as a set we do not need to know or list every member. All we need to know is some sentence that tells us which things are members and which are not. That is, we need to know of some sentence like "A thing is a member of the set if and only if the thing . . . ," for example, "A person is a citizen of the U.S. if and only if the person is born in the U.S. or is naturalized." This sentence completely determines the set called "the citizens of the U.S." even though we would hardly care to list all the members. Replacing "a thing" by "x," a rule for designating a set takes the form "x is a member of the set if and only if x . . . ," where the dots indicate the rest of some sentence of which x is the subject. In our example, we have "x is a citizen if and only if x is born in the U.S. or x is naturalized." Letting $f(x)$ = (x is born in the U.S. or is naturalized), we may say that (the citizens of the U.S.) = (the set of x's such that $f(x)$).

Complete the following: (k) The student body of this college = the set of x's such that x ________. (l) The living ex-Presidents = the set of x's such that x ________ and x ________.

We now adopt an abbreviation for the formula "the set of x's such that $f(x)$", namely, $\{x \mid f(x)\}$. In this notation, the braces are read "the set" and

In Problems 14 through 20 use the defining-sentence notation to name the indicated set.

14. The set whose numbers are 1 and 2.
15. The positive integers between $\frac{1}{2}$ and $\frac{3}{2}$.
16. The positive integers that are greater than 1 and also less than 2.
17. The States of the U. S. touching an ocean.
18. The integers that are multiples of 3.
19. The integers that leave remainders of 1 when divided by 7.
20. Integers that are perfect squares.

Answers to Exercises

(a) $a \notin c = \sim(a \in c)$. (b) The set whose members are just 1 and 2. (c) The set whose one member is 2. (d) The set whose members are just a, b, and c. (e) The set whose members are just 1 and 0. (f) The set whose one member is 0. (g) $\{1, 2, 3, 4\}$. (h) $\{2.5\}$; note that we speak of "the solutions" even though the set turns out to have only one member. (i) {Catholicism, Protestantism, Judaism}. (j) $\{3, 4, 5, 6\}$. (k) is registered here. (l) is living; was but is no longer President. (m) $\{1\}$. (n) $\{3, 4\}$. (o) All orphans. (p) All felons. (q) The sentence is a law. (r) All numbers except 1. (s) Set of all men. (t) $-b/a$ is unique solution. (u) through (x) See Section 2–8. (y) sentences; set. (z) sets; sentence.

Answers to Problems

1. The digits. 3. $\{-1 + \sqrt{2}, -1 - \sqrt{2}\}$. 5. The even integers. 7. The odd integers. 9. The prime numbers. 13. Since $f(x) = f(x)$ for just the values of x that have meaning in the context. 15. $\{x \mid x \in J^+ \wedge 1/2 < x < 3/2\}$. 17. $\{x \mid x$ is a State $\wedge$ x touches an ocean$\}$. 19. $\{x \mid x \in J \wedge \exists y\, y \in J \wedge x = 7y + 1\}$.

3–3 Special sets. In any discussion, we talk about some set of objects. The set of all things under discussion in any context is called the *universe of discourse* or *universal set*, and we shall designate it by U. For example, in elementary algebra the universe of discourse is usually the real numbers. From the members of U we can construct sets, which we call *subsets* of U. Then $a \in S$ if a is a member of S and S is a subset of U. Also $\{x|f(x)\}$ is a subset of U if the values of x in $f(x)$ are members of U, that is, if the domain of $f(x)$ is a subset of U.

Suppose, now, that the domain of $f(x)$ is a subset of U but that $f(x)$ is false for all values of x. We still consider $\{x \mid f(x)\}$ to be a set and a subset of U even though it has no members. We call a set with no members *empty* or *vacuous*. We define $\emptyset$ as follows.

(1) **Def.** $$\emptyset = \{x \mid x \neq x\}.$$

Clearly $\emptyset$ is vacuous; that is, we have the law

(2) $$\sim(x \in \emptyset).$$

(a) Why?

We call $\emptyset$ *the null set*. The word "the" is used because any vacuous set is equal to $\emptyset$. [See (8) below.]

To prove (2) or any other theorem about sets, we need some axioms containing our undefined terms. The first axiom below embodies the essential idea in our notion of $\{x \mid f(x)\}$, namely that it is just the set of objects satisfying $f(x)$.

(3) **Ax.** $$[a \in \{x \mid f(x)\}] = f(a).$$

(b) In (3) what are the variables, and what terms, if any, are dummies?

Our next axiom indicates that we regard a set as just the collection of its members. To say that sets A and B are identical is to say that they have the same members.

(4) **Ax.** $$(A = B) = \forall x\,(x \in A \leftrightarrow x \in B).$$

Now we can prove

(5) $$A = \{x \mid x \in A\},$$

(6) $$[\{x \mid f(x)\} = \{x \mid g(x)\}] = [\forall x(f(x) \leftrightarrow g(x))].$$

This theorem tells us that two logically equivalent sentences define the same set, and conversely. Evidently every set A has many defining sentences, among which, by (5), $x \in A$ is one.

The following axiom indicates the nature of the universal set.

(7) **Ax.** $$U = \{x \mid x = x\}.$$

(c) Could the second $=$ in (6) be changed to $\leftrightarrow$? (d) Give an example of three sentences defining the same set. (e) Give a sentence that defines the set of all statements.

We can now prove that any vacuous set is identical with the null set $\emptyset$; that is,

(8) $$[\forall x\, x \notin A] \leftrightarrow [A = \emptyset].$$

Indeed, by (6), $A = \{x \mid x \in A\} = \{x \mid x \neq x\} = \emptyset$ if and only if $\forall x[x \neq x \leftrightarrow x \in A]$. But by hypothesis, the right member of this equivalence is always false, and we know that $x \neq x$ is always false. Hence $x \neq x$ and $x \in A$ are logically equivalent for all x, as required.

(f) Show that $\exists! A\ \forall x\ x \notin A$. (g) We have seen that the null set is unique. Is this true of the universe of discourse U; that is, does (7) determine a unique U? (h) What is $\{x \mid 1/x = 0\}$?

Now we introduce the roster notation by definition.

(9) **Def.** $\{a\} = \{x \mid x = a\}$,

(10) **Ax.** $a \neq \{a\}$,

(11) $a \in \{a\}$,

(12) $\exists! x\ x \in \{a\}$,

(13) $(\{a\} = \{b\}) \leftrightarrow (a = b)$.

We call a set with just one number a *singleton*. Note how the "oneness" of a singleton is expressed by (12) without using numbers.

(i) Why did we include (10) as a law? (j) What is the variable in (12) and what is the dummy? (k) Write each of (9) through (13) with quantifiers as you would if we did not have the convention that we omit quantifiers that stand before a law.

(14) **Def.** $\{a, b\} = \{x \mid x = a \lor x = b\}$,

(15) **Def.** $\{a, b, c\} = \{x \mid x \in \{a, b\} \lor x = c\}$.

When $a \neq b$, we call $\{a, b\}$ a *pair*. Similarly, when $a \neq b$, $b \neq c$, and $a \neq c$, we call $\{a, b, c\}$ a *triple*. When designating sets by their rosters, it is customary to name each member only once in the roster. Hence, ordinarily $\{a, b\}$ is a pair and $\{a, b, c\}$ is a triple.

(l) Define $\{a, b, c, d\}$ and $\{a, b, c, d, e\}$. (m) Distinguish between a, $\{a\}$, "a," $\{$"a"$\}$, and "$\{a\}$."

(16) $\{a, a\} = \{a\}$,

(17) $\{a, b, c\} = \{x \mid x = a \lor x = b \lor x = c\}$,

(18) $\{a, b\} = \{b, a\}$,

(19) $[\{a, b\} = \{c, d\}] \leftrightarrow [(a = c \land b = d) \lor (a = d \land b = c)]$,

(20) $[\{a\} = \{b, c\}] \leftrightarrow (a = b = c)$.

★*Special legislation.* Constitutions often prohibit special legislation, that is, laws that apply to a single community, individual, or group. One way of getting around this is to pass a law applying to a certain set of communities, but to word the defining sentence of the set so that it actually defines a singleton! Professor Ralph Fjelstad, after a study of the Minnesota legislature, reported that in the 1953 session there were hundreds of "special bills couched in general terms." For example, "Any village having a population of more than 1,300 but less than 1,500 persons according to the last applicable state or federal census and having an assessed valuation of real and personal property in excess of \$1,000,000 may . . . ." It "so happened" that only one village, Aurora, met these qualifications. The formula for a law of this kind is "Any x such that $f(x)$ may . . . ," where one takes care that $\exists! x\, f(x)$ and $f(a)$, where a is the individual community, person, or group for which one wishes to pass special legislation "couched in general terms."

(n) Express the defining sentence in the previous example, using x as the variable. (o) Using the expression in Exercise (n), write symbolically that there is only one such village. (p) Use the symbols of this section to express the fact that this one solution is the village of Aurora. (q) Suggest other kinds of sentences that might define a singleton whose one member is a town. (r) Give, if you can, examples of similar methods of evading prohibition of legislation directed against special religious, racial, national, or political groups.

Problems

In Problems 1 through 6 identify the sets.

1. $\{x \mid x^2 = 16\}$.
2. $\{x \mid 2x + 1 = 0\}$.
3. $\{x \mid 1/x = x\}$.
4. $\{x \mid (x - 2)(x + 1)(x + 1000) = 0\}$.
5. $\{x \mid 1/x = 1/x\}$.
6. $\{x \mid x = \sqrt{9}\}$.
7. Does $\emptyset = 0$?
8. Show that $\{x \mid x > x\} = \{x \mid x \cdot 0 = 2\}$.
9. Prove (2).
10. Prove (5) and (6).
11. Prove some of (9) through (13) and (16) through (20).
12. Show that $[A \neq B] = \exists x\ (x \in A \land x \notin B) \lor (x \in B \land x \notin A)$.
13. The following laws indicate the relation between quantifiers and set membership. Justify them.

(21) $[\forall x\, f(x)] = [\{x \mid f(x)\} = \{x \mid f(x) = f(x)\}]$,

(22) $[\exists x\, f(x)] = [\{x \mid f(x)\} \neq \emptyset]$.

14. Prove the following:

(23) $$[x = y] \leftrightarrow \forall S\,(x \in S \leftrightarrow y \in S),$$

(24) $$[x = y] \leftrightarrow [\{S \mid x \in S\} = \{S \mid y \in S\}].$$

15. Verbalize (23) and (24).
16. Prove (25).

(25) $$x \in U.$$

17. Define $A \neq B$ (for sets) without using $=$ in the defining term.

Axiom (10) formalizes our distinction between a set and its members. Failure to make this distinction results in confusion. The fallacy of taking a statement about the members of a set to be about the whole set is called the *fallacy of composition.* The opposite fallacy of mistaking a statement about a set to be a statement about its members is called the *fallacy of division.*

★18. Let A = the barrel of apples is ripe. Does A = the barrel $\in$ ripe things?
★19. Let B = the barrel of apples is heavy. Does B = the barrel $\in$ heavy things?
★20. What is the fallacy in the following? The even numbers, E, are divisible by 2; hence, E is divisible by 2, and so E is an even number.
★21. Explain the fallacy in the following: The Russians claim to have liquidated the kulaks, but this is obviously false since some of them escaped.
★22. What two sets of Americans were eliminated by the Civil War? Were their members eliminated?
★23. A corporation may be sued for debt. Mr. Jones is a member of the corporation, hence he may be sued. Valid?
★24. In the eyes of the law a corporation is a person. A person may be sued, hence a corporation may be sued. Valid?

Answers to Exercises

(a) $\forall x\, x = x$, hence $\forall x \sim (x \neq x)$ and every x fails to satisfy the defining sentence. (b) "a" and "f" are variables, "x" is a dummy. (c) Yes, since $(p = q) \rightarrow (p \leftrightarrow q)$. (d) $x - 3 = 0$, $(x - 3)^2 = 0$, $2x = 6$. (e) $x \leftrightarrow x$. (f) Recall (2–11–17). $\exists A\ \forall x\, x \notin A$ by (2). Suppose $\forall x\, x \notin A$ and $\forall x\, x \notin B$. Then by (8) $A = \emptyset$ and $B = \emptyset$. Hence the second term in the right member of (2–11–17) with $(a{:}A, y{:}B)$ is satisfied. (g) No. U depends on the context. (h) Null set. (i) To be sure that "$a \in \{a\}$" is a sentence. (j) "a" is a variable, "x" a dummy. (k) For example, (9) becomes $\forall a\, \{a\} = \{x \mid x = a\}$.

(l) $\{a, b, c, d\} = \{x \mid x \in \{a, b, c\} \vee x = d\}$. (m) The object a, the singleton whose one member is a, the letter "a," the singleton whose member is the first letter of the alphabet, and a name of the singleton whose one member is a. (n) x is a village having a population . . . and having . . . $1,000,000.

(o) $\exists! x\ x$ is a . . . . (p) {Aurora} = $\{x \mid x$ is a village . . .$\}$. (q) There are many, but the trick is to find one that does not appear to name a single village! (r) Literacy tests, country-of-origin tests, etc., in certain circumstances.

Answers to Problems

1. $\{4, -4\}$. 3. $\{1, -1\}$. 5. All nonzero numbers. 7. No! 0 is a number. It is the number of members in $\emptyset$, but it is not $\emptyset$ itself! 14. To prove that the right member of (23) implies the left member, assume the right member by Hyp, use (2–10–4) with $(y{:}\{x\})$ to get $x \in \{x\} \leftrightarrow y \in \{x\}$. Since $x \in \{x\}$, $y \in \{x\}$, from which $y = x$. 17. See Problem 12.

3–4 Subsets. We call the set A a *subset* of the set B if every member of A is a member of B. We use the abbreviation $A \subseteq B$. An alternative verbalization is that "*B includes A.*"

(1) Def. $$A \subseteq B = \forall x\,[x \in A \rightarrow x \in B].$$

(a) Read definition (1). (b) Show that:

(2) $$[\{x \mid f(x)\} \subseteq \{x \mid g(x)\}] = \forall x\,[f(x) \rightarrow g(x)].$$

We may illustrate conveniently by using small sets. Let $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$, $B = \{2, 3\}$, $C = \{1, 4, 5\}$. Then $A \subseteq U$, $B \subseteq U$, $B \subseteq A$, but $\sim(A \subseteq B)$ and $\sim(A \subseteq C)$.

An alternative illustration arises from letting U = the set of points in the interior of the square in Fig. 3–1 and letting A and B equal the sets of points within the labeled circles. In the figure, $A \subseteq B$, $A \subseteq U$, and $B \subseteq U$.

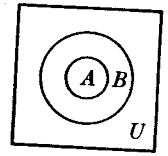


Figure 3–1

True or false? (Give reasons.) (c) $\{1\} \subseteq \{1, 2\}$; (d) $\{4, 5\} \subseteq \{1, 2, 3, 4, 5\}$; (e) $\{2\} \subseteq \{3, 4\}$; (f) $\{2\} \in \{2, 3\}$; (g) $\{2\} \subseteq \{2, 3\}$.

It is important to distinguish carefully between membership and inclusion; as suggested in Exercises (f) and (g), they are quite different. It is possible to construct examples in which a member of a set is also one of its subsets. Thus $\{2\} \in \{\{2\}, 2, 3\}$ and $\{2\} \subseteq \{\{2\}, 2, 3\}$. The first is true because "{2}" appears in the roster. The second is true because "2" appears in the roster, and hence {2} is a set made up of members of the given set. It should be kept in mind that $\in$ is used to indicate a relation between a member and a set, whereas $\subseteq$ indicates a relation between two sets. Of course, a set may be a member of another set, as indicated in the example.

Given any set, we may form various subsets. For example, $\{1, 2, 3\}$ has the subsets $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$. The set of all subsets of this set is then $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Note that the null set and the set itself are members of this set. We call the set of all subsets of A the *power set* of A. Let A be a women's club. Then the power set of A is the set of all possible committees, including the committee of the whole (the club itself) and the committee to which no one belongs. We designate the power set by $\mathcal{P}(A)$.

Find the power set of: (h) $\{1, 2\}$, (i) $\{2, 3\}$, (j) $\{1\}$, (k) $\emptyset$, (l) $\{1, 2, 3, 4\}$.

The examples given suggest that the number of members in the power set of A is $2^{\mathfrak{N}(A)}$, where $\mathfrak{N}(A)$ is the number of members in A. Such is indeed the case. This is the origin of the name "power set."

Linear graphs. Let U be the set of all points on an axis. Because of the one-to-one correspondence between the points of U and the real numbers, we may think of U also as the set of all real numbers. Let $f(x)$ be a sentence whose domain (range of x) is a subset of U. Then the truth set of $f(x)$ is a set of points on the axis, a subset of U. We call this set of points the *linear graph* of $f(x)$. Thus the graph of $x^2 = 1$ is the set of points (real numbers) $\{1, -1\}$. The graph of a sentence is then just a geometrical picture of its truth set. Here the graph of $x^2 = 1$ is a geometrical picture of $\{x \mid x^2 = 1\}$. The graph may be a finite set or it may contain an infinite number of points. For example, the graph of $x < 2$ contains all points to the left of 2, as suggested in Fig. 3-2. We often use the defining sentence as a name for its graph.

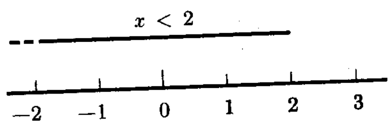


FIGURE 3-2

Sketch the linear graph of each of the following sentences: (m) $x^2 = 9$, (n) $x > -1$, (o) $x < 3 \vee x > 4$, (p) $x < 4 \wedge x > 3$, (q) $(x - 1)(x + 5) > 0$, (r) $x^3 \leq 0 \wedge x^3 \geq 8$. (s) Describe $\mathcal{P}(U)$ where U = the axis of real numbers.

Intervals of real numbers. We say that x lies *between* a and b, where $a < b$, if $a < x \wedge x < b$. It is natural to adopt the following abbreviation.

(3) Def. $(a < x < b) = (a < x \wedge x < b)$.

We call $\{x \mid a < x < b\}$ the *open interval* from a to b and designate it by (a_b). The word "open" refers to the fact that it does not contain its endpoints.

(4) Def. $(a\_b) = \{x \mid a < x < b\}$.

It is convenient to think of the set of all points to the left of a given point (all numbers less than a given one) as an open interval stretching "to infinity" toward the left. We designate it by $(-\infty\_b)$, but the student should keep in mind that ∞ is not a numeral. Similarly, (a_∞) is the set of all points (numbers) to the right of a.

(5) Def. $(-\infty\_b) = \{x \mid x < b\}$,

(6) Def. $(a\_\infty) = \{x \mid x > a\}$.

Describe in set notation, in words, and sketch: (t) (2_4), (u) (-1_1), (v) $(a - \epsilon\_a + \epsilon)$.

When the endpoints belong to an interval, it is called *closed* and is designated by $(\underline{a\quad b})$.

(7) Def. $(a \leq x \leq b) = (a \leq x \wedge x \leq b)$,

(8) Def. $(\underline{a\quad b}) = \{x \mid a \leq x \leq b\}$.

When only one of the endpoints belongs, the interval is said to be closed on the right or on the left. Thus $(\underline{a\ \ }b)$ is *closed on the left* and *open on the right*, $(a\underline{\ \ b})$ is *open on the left* and *closed on the right*. An infinite interval cannot be closed, since it does not have a first or last point in the direction in which it "stretches to infinity."

Give definitions of: (w) $a < x \leq b$, (x) $a \leq x < b$, (y) $a > c > b$, (z) $(a\underline{\ \ b})$, (aa) $(\underline{a\ \ }b)$, (bb) $(\underline{a\ \ }\infty)$, (cc) $(-\infty\underline{\ \ b})$. Sketch and describe in words: (dd) (3_4), (ee) $(\underline{-1\quad 1})$, (ff) $(\underline{0\ \ }\infty)$, (gg) $(-\infty\underline{\ \ 0})$, (hh) $(\underline{-100\quad 100})$, (ii) $(\underline{a - e\quad a + e})$.

PROBLEMS

1. Find $\mathcal{P}\{0\}$.
2. Find $\mathcal{P}\{1, 2, 3, 4, 5\}$.

Graph Problems 3 through 10.

3. $|x| \leq 1$.
4. $x^2 + 2x - 3 = 0$.
5. $x^2 + 4x + 4 = 0$.
6. $3x - 1 = 0$.
7. $3x - 1 \leq 0$.
8. $x \notin (-1\_1)$.
9. $x^2 < 0$.
10. $x \in (-\infty\_\infty)$.

In Problems 11 through 19 write inequalities synonymous to each expression.

11. $x \in (2\_\_3)$. 12. $x \in (\underline{2\_\_}3)$. 13. $x \in (2\underline{\_\_3})$.
14. $x \in (\underline{-1\_\_}\infty)$. 15. $x \in (-\infty\_\_3)$. 16. $x \in (\underline{-1\_\_5})$.
17. $x \notin (\underline{2\_\_8})$. 18. $x \notin (-\infty\underline{\_\_-1})$. 19. $x \notin (-1\_\_\infty)$.

Tell whether each of Problems 20 through 23 is true or false.

20. $(2\_\_3) \subseteq (\underline{2\_\_3})$. 21. $(\underline{-1\_\_}0) \subseteq (-1\_\_1)$.
22. $(3\_\_5) \subseteq (4\_\_5)$. 23. $(\underline{1\_\_5}) \subseteq (2\_\_6)$.

24. Identify $(\underline{3\_\_3})$.
25. Comment on $(\underline{3\_\_2})$, $(2\_\_2)$, and $(\underline{2\_\_}-\infty)$.
26. Is $2 \subseteq (\underline{2\_\_}3)$?
★27. Name several subsets of the U. S. Senate.

Answers to Exercises

(a) "A is included in B" means that any member of A is also a member of B. (b) (1) and (3–3–3). (c) T. (d) T. (e) F. (f) F. (g) T. (h) $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. (i) $\{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$. (j) $\{\emptyset, \{1\}\}$. (k) $\{\emptyset\}$. (l) $\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$. (m) Two points, 3 and -3. (n) All points to right of -1. (o) All points to left of 3 and all points to the right of 4. (p) All points between 3 and 4. (q) All points to the left of -5 and all points to the right of 1. (r) Null set. (s) The set of all linear graphs. (t) $\{x | 2 < x < 4\}$; points between 2 and 4. (w) $a < x \wedge x \leq b$. (z) $\{x | a < x \leq b\}$. (bb) $\{x | x \geq a\}$. (ii) The closed interval from $a - e$ to $a + e$.

Answers to Problems

1. $\{\emptyset, \{0\}\}$. 3. $(\underline{-1\_\_1})$. 5. $\{-2\}$. 7. $(-\infty\underline{\_\_1/3}\}$. 9. $\emptyset$. 11. $2 < x < 3$. 13. $2 < x \leq 3$. 15. $x < 3$. 17. $x < 2 \vee x > 8$. 19. $x \leq -1$. 21. F. 23. F. 25. $\emptyset$.

3–5 Ordering of the real numbers. In Sections 1–7 and 3–4 we dealt with inequalities in terms of their geometric interpretation without definitions or proof. In this section we use simple set theoretic ideas to give a more logical theory.

Let Re be the set of all real numbers. We assume that there is a subset of Re, which we call Re^+, with the following properties.

(1) **Ax.** $(a \in Re^+ \wedge b \in Re^+) \rightarrow (a + b) \in Re^+$,

(2) **Ax.** $(a \in Re^+ \wedge b \in Re^+) \rightarrow ab \in Re^+$,

(3) **Ax.** $a \in Re \rightarrow [(a \in Re^+) \veebar (a = 0) \veebar (-a \in Re^+)]$,

(4) **Def.** $a < b = (b - a) \in Re^+$,

(5) **Def.** $a > b = b < a$,

(6) $a > 0 = a \in Re^+$.

(7) $a < 0 = -a \in Re^+$.

(a) Prove (7).

From (6), which follows immediately from (4)$(a{:}0, b{:}a)$, we see that Re^+ is just the set of positive real numbers. We may read "$a > 0$" $(a \in Re^+)$ as "a is positive" and "$a < 0$" $(-a \in Re^+)$ as "a is negative." With this intuitive idea of Re^+, (1) through (3) are most plausible. However, the proofs that follow do not depend on any interpretation.

(b) Justify (8) through (13).

(8) $(a > 0 \wedge b > 0) \rightarrow (a + b > 0)$,

(9) $(a > 0 \wedge b > 0) \rightarrow (ab > 0)$,

(10) $(a < 0) \veebar (a = 0) \veebar (a > 0)$,

(11) $(a > 0) = (-a < 0)$,

(12) $(a < 0) = (-a > 0)$,

(13) $(a < b) = (b - a > 0)$.

We can now prove the following frequently used laws.

(14) $[(a < b) \wedge (b < c)] \rightarrow (a < c)$,

(15) $(a < b) \leftrightarrow (a + c < b + c)$,

(16) $c > 0 \rightarrow (a < b \leftrightarrow ac < bc)$,

(16′) $c < 0 \rightarrow (a < b \leftrightarrow ac > bc)$.

The proof of (14) rests directly on (4). By hypothesis we have $a < b$ and $b < c$, that is, $b - a \in Re^+$ and $c - b \in Re^+$. From (1) we have, then, $(b - a) + (c - b) \in Re^+$ or $c - a \in Re^+$. But this last is just our conclusion by (4).

★(c) Write out the preceding proof, with steps and reasons, in formal style.
(d) Prove (15). (*Suggestion:* Write hypothesis and conclusion in terms of Re^+.)

To prove (16) and (16′), we begin with $a < b$, that is, $b - a \in Re^+$, by hypothesis. Now, if in addition $c > 0$, that is, $c \in Re^+$, we have $(b - a)c \in Re^+$, $bc - ac \in Re^+$, or $ac < bc$, as desired in (16). But if $c < 0$, $-c > 0$ or $-c \in Re^+$ by (12), and we have $(b - a)(-c) \in Re^+$, $ac - bc \in Re^+$, or $ac > bc$, as required in (16′).

An understanding of (16′) is essential for work with inequalities. Because of it, inequalities cannot be treated as though they were equations!

(e) Illustrate (16) and (16′) for $(a{:}2, b{:}4, c{:}2)$, $(a{:}2, b{:}4, c{:}-2)$, $(a{:}-2, b{:}4, c{:}-1)$. (f) Restate (8) through (16′) in geometric terms by reading $a < b$ as "a lies to the left of b."

Laws (15) through (16′) can be used to solve simple inequalities. For example,

(17) $3x + 4 < 5x + 1 \leftrightarrow -2x < -3$ (15)

(18) $\leftrightarrow x > 3/2$ (16′).

(g) Show that $mx + b < 0 \leftrightarrow x < -b/m$ is not a law, and write a correct law giving the solution of $mx + b < 0$. (h) Solve $1 - 5x > 2x + 4$ as in (17) and (18). (i) What is the geometric interpretation of $a \leq b$? of $a \geq b$?

(19) **Def.** $(a \leq b) = (a < b \vee a = b)$,

(20) **Def.** $(a \geq b) = (a > b \vee a = b)$.

We can now easily prove that the square of any real number is nonnegative; that is,

(21) $x \in Re \rightarrow x^2 \geq 0.$

(j) According to (10), there are only three possible cases: $x < 0$, $x = 0$, $x > 0$. If $x < 0$, $-x > 0$, and $(-x)(-x) > 0$ by (12) and (9). But $(-x)(-x) = x^2$, so $x^2 > 0$ in this case. Dispose of the other two cases.

Suppose we wish to solve the inequality $x^2 - 4 < 0$. We have $(x + 2)(x - 2) < 0$. If $x + 2 > 0$, we may divide by it to get $x - 2 < 0$. Hence if both $x + 2 > 0$ and $x - 2 < 0$, that is, if $x > -2$ and $x < 2$, x is a solution. Evidently any point in the open interval $(-2\_\_2)$ is a solution. But suppose $x + 2 < 0$. Then division yields $x - 2 > 0$. Hence if both $x + 2 < 0$ and $x - 2 > 0$ are satisfied, x is a solution. But this means $x < -2 \wedge x > 2$, which is impossible. Another way of looking at the inequality is to observe that for $(x + 2)(x - 2)$ to be negative, one factor must be positive and the other negative, which yields the same result.

(k) Solve $x^2 - 1 \leq 0$ in this way. (l) Solve $x^2 - 4 > 0$.

The following theorems are helpful in solving quadratic inequalities.

(22) $ab > 0 \leftrightarrow [(a > 0 \wedge b > 0) \vee (a < 0 \wedge b < 0)]$,

(23) $ab < 0 \leftrightarrow [(a > 0 \wedge b < 0) \vee (a < 0 \wedge b > 0)]$.

Use (22) and (23) to solve: (m) $2x^2 - 1 < 0$, (n) $x^2 \geq 17$, (o) $x^2 + 3x + 2 < 0$, (p) $x^2 + 3x + 1 > 0$, (q) $(x - 3)(x + 4) < 0$.

Problems

1. Show from (10) that if $a \neq 0$, then $a > 0$ or $a < 0$.
2. Consolidate in a single axiom the entire assumption ending with (1) through (3), including all quantifiers required.
3. Show that $a + 1 > a$ is a law by proving that it is logically equivalent to $1 > 0$.
4. Show that if $c > 0$, $a + c > a$.
5. Show that $a - 1 < a$.
6. Interpret the following verbally, argue for their plausibility, and give examples.

(24) $\sim(a < a)$,

(25) $a < b \rightarrow \sim(b < a)$,

(26) $a \leq a$,

(27) $(a \leq b \wedge b \leq a) \rightarrow (a = b)$,

(28) $(a \leq b \wedge b \leq c) \rightarrow (a \leq c)$,

(29) $(a < b) \leftrightarrow (-a > -b)$,

(30) $(a < b) \leftrightarrow (a - c < b - c)$,

(31) $(a < 0) \leftrightarrow (1/a < 0)$,

(32) $(a > 0) \leftrightarrow (1/a > 0)$,

(33) $(a < b) \veebar (a = b) \veebar (a > b)$,

(34) $(ab > 0) \leftrightarrow (a/b > 0)$,

(35) $(ab > 0) \rightarrow (a < b \leftrightarrow 1/a > 1/b)$,

(36) $(ab < 0) \rightarrow (a < b \leftrightarrow 1/a < 1/b)$,

(37) $a^2 + b^2 \geq 2ab$,

(38) $(a \geq 0 \wedge b \geq 0) \rightarrow \left[\frac{a + b}{2} \geq \sqrt{ab}\right]$,

(39) $(a > 1) \rightarrow (a^2 > a)$,

(40) $(0 < a < 1) \rightarrow (a^2 < a)$,

(41) $(0 < a < b \wedge 0 < c < d) \rightarrow (ac < bd)$,

(42) $(a < b \wedge c < d) \rightarrow (a + c < b + d)$,

(43) $(0 \leq a \wedge 0 \leq b) \rightarrow [a \leq b \leftrightarrow a^2 \leq b^2]$.

★7. Any sentence involving $<, >, \leq$, or $\geq$ can be rewritten by using the definitions (5), (19), and (20). For example, (29) yields $b > a \leftrightarrow -b < -a$. Also, laws concerning $\leq$ and $\geq$ can be constructed by combining laws of equality and inequality. For example, (29), (2–6–11), and (2–7–37) yield $a \leq b \leftrightarrow -a \geq -b$. Use these and other methods to find further theorems about inequalities.

★8. The *arithmetic mean* of a and b is $(a+b)/2$, and the *geometric mean* is $\sqrt{ab}$. State (38) verbally. Prove (37) by showing that it is equivalent to $(a-b)^2 \geq 0$. Prove (38) by a substitution in (37).

In Problems 9 through 12 identify the set.

9. $\{x \mid x^2 - 2 < 0\}$.
10. $\{x \mid x^2 - 2x + 4 > 0\}$.
11. $\{x \mid (x-7)(x+4) \leq 0\}$.
12. $\{x \mid x(x+1)(x-2) \geq 0\}$.

13. Show that (21) is equivalent to the statement that no negative number has a real square root.

Problems 14 through 17 refer to the discriminant of a quadratic equation and the quadratic formula discussed in Section 2–8.

14. Without solving, determine whether the following have real roots: $x^2 - x + 1 = 0$, $2x^2 + 5 + 1 = 0$, $x^2 - 10x + 25 = 0$.
15. For what values of k are the roots of $x^2 + 2x + k = 0$ real?
16. For what values of k are the roots of $2kx^2 + x - 1 = 0$ not real?
17 For what values of k are the roots of $x^2 + kx + 2 = 0$ real?

★18. Show that if $A^2 > K$, $B^2 > K$, $A > 0$, and $B > 0$, then $AB > K$.
★19. Show that $0 < a < b \wedge x > 0 \rightarrow a/b < (a+x)/(b+x) < 1$.

★20. Suppose that $Y = C + S$, when Y = national income, C = consumption, and S = savings. Show that with national income constant, a decrease in savings means an increase in consumption.
★21. Prove $0 < v < 1 \rightarrow 1 - v < 1 - v^2$.

★22. Prove $\left(\dfrac{a}{a+b} > \dfrac{c}{c+d}\right) \leftrightarrow \left(\dfrac{a}{a+c} > \dfrac{b}{b+d}\right)$.

★23. Prove $\left(-1 \leq \dfrac{r_{12} - k_1}{k_2} \leq 1\right) \rightarrow (k_1 - k_2 \leq r_{12} \leq k_1 + k_2)$.

24. Under what conditions is $4.36\sigma > 2.65\sigma$?
★25. Prove that $(ax + by)^2 \leq (a^2 + b^2)(x^2 + y^2)$.
26. Solve for x: $ax - 2 + (2 - a) < x + a + (2 - a)$.
27. Solve $\dfrac{3}{x} + 4 \leq \dfrac{2}{x}$.
28. "The square, divided by 10, of any number larger than 0 and smaller than 10 . . . is a positive quantity smaller than the number itself." ("Predicting Supreme Court Decisions Mathematically," by Fred Kort, in the *American Political Science Review*, Vol. LI, No. 1, March, 1957.) Prove this.

Answers to Exercises

(a) $(a < 0) = (0 - a) \in Re^+ = -a \in Re^+$. (b) From (6) and the axioms. (d) $(a < b) \leftrightarrow (b - a) \in Re^+ \leftrightarrow (c - c) + (b - a) \in Re^+ \leftrightarrow b + c - (a + c) \in Re^+ \leftrightarrow a + c < b + c$. (f) For example, (11): If a is to the right of the origin, $-a$ is to the left. (g) False for any $m < 0$; $m > 0 \rightarrow (mx + b < 0 \leftrightarrow x < -b/m) \wedge m < 0 \rightarrow (mx + b < 0 \leftrightarrow x > -b/m)$.
(h) $x < -3/7$. (i) $a \leq b$: the point a coincides with or is to the left of b. (j) $x = 0 \rightarrow x^2 = 0$; $x > 0 \rightarrow x^2 > 0$. (k) $(x+1)(x-1) \leq 0 \leftrightarrow [(x+1) \leq 0 \wedge (x-1) \geq 0] \vee [(x+1) \geq 0 \wedge (x-1) \leq 0] \leftrightarrow [x \leq -1 \wedge x \geq 1] \vee (x \geq -1 \wedge x \leq 1] \leftrightarrow -1 \leq x \leq 1$; and the graph of $x^2 - 1 \leq 0$ is the closed interval $(\underline{-1\ \ 1})$. (l) $x < -2 \vee x > 2$. (m) $-\dfrac{1}{\sqrt{2}} < x < \dfrac{1}{\sqrt{2}}$. (n) $x \leq -\sqrt{17} \vee x \geq \sqrt{17}$. (o) $-2 < x < -1$. (p) $x < (-3 - \sqrt{5})/2 \vee x > (-3 + \sqrt{5})/2$. (q) $-4 < x < 3$.

Answers to Problems

1. From (2–2–11), (2–7–48), and other laws of logic. 3. (15)(a:0, b:1, c:a). 5. (15)(a:−1, b:0, c:a). 9. $(-\sqrt{2}\,\underline{\ \ }\,\sqrt{2})$. 11. $(\underline{-4\ \ 7})$. 13. $\forall x\,[x \in Re \rightarrow x^2 \geq 0] = \sim \exists x \sim [x \in Re \rightarrow x^2 \geq 0] = \sim \exists x\,[x \in Re \wedge x^2 < 0]$.
15. $\{k \mid 2^2 - 4k \geq 0\} = \{k \mid k - 1 \leq 0\} = (-\infty\,\underline{\ \ 1})$. 17. $k \leq -\sqrt{8} \vee k \geq \sqrt{8}$.

3–6 Operations on sets. We consider a universe of discourse U together with all its subsets. If A is the subset, the *complement of* A, A' (read "A prime"), is the set of all elements of U that do not belong to A. Formally,

(1) Def. $$A' = \{x \mid \sim(x \in A)\}.$$

For example, if $U = \{0, 2, 4, 6, 8, 10\}$, $\{0, 4, 8\}' = \{2, 6, 10\}$.

(a) With the same U, $\{2, 4, 6\}' = ?$ (b) Justify (2) and (3).

(2) $$x \in A' = \sim(x \in A),$$

(3) $$\{x \mid f(x)\}' = \{x \mid \sim f(x)\}.$$

If A and B are subsets, their *intersection*, $A \cap B$, is the set of elements of U that belong to both of them.

(4) Def. $$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$

For example, $\{0, 2, 6, 8\} \cap \{2, 8, 10\} = \{2, 8\}$.

(c) $\{a, b, c, d\} \cap \{a, c, e, f\} = ?$ (d) Justify (5) and (6).

(5) $x \in (A \cap B) = x \in A \ \wedge \ x \in B,$

(6) $\{x \mid f(x)\} \cap \{x \mid g(x)\} = \{x \mid f(x) \wedge g(x)\}.$

If A and B are subsets, their *union*, $A \cup B$, is the set of elements that belong to either or both of them.

(7) **Def.** $A \cup B = \{x \mid x \in A \ \vee \ x \in B\}.$

For example, $\{0, 2, 6, 8\} \cup \{2, 8, 10\} = \{0, 2, 6, 8, 10\}.$

(e) $\{a, b, c, d\} \cup \{a, c, e, f\} = ?$ (f) Justify (8) and (9).

(8) $x \in (A \cup B) = x \in A \ \vee \ x \in B,$

(9) $\{x \mid f(x)\} \cup \{x \mid g(x)\} = \{x \mid f(x) \ \vee \ g(x)\}.$

(g) The symbols $\cap$ and $\cup$ are sometimes read "cap" and "cup." Suggest a reason for the choice of these symbols.

The intersection, union, and complement may be conveniently visualized geometrically if we think of the universe of discourse as a region in a plane and the other sets as regions within it. In Fig. 3–3 we show the intersection as the region common to A and B, the union as the region consisting of those points in either or both A and B, and the complement as the region outside A. Note that A' is not everything not in A, but everything in U that is not in A. Diagrams like Fig. 3–3 are called *Venn diagrams.* They are useful aids to the imagination; however, like the truth tables in Chapter 2, they are not to be confused with proofs.

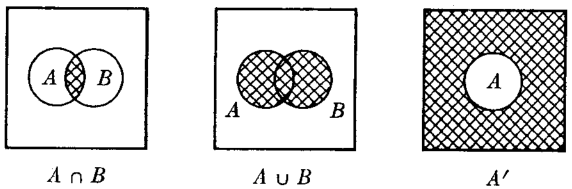


FIGURE 3–3

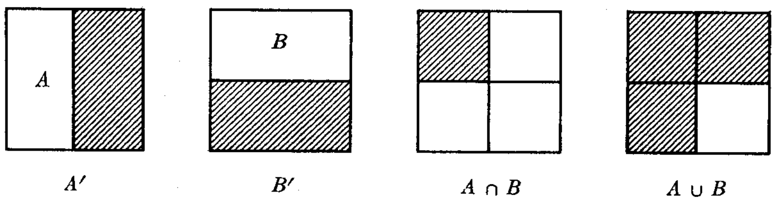


FIGURE 3–4

A special kind of Venn diagram, due to Lewis Carroll, is illustrated in Fig. 3–4. It is very convenient for visualizing the subsets formed from a universe by two sets.

(h) Copy the diagram for $A \cap B$ in Fig. 3–4 and label the three unshaded squares properly, using the following labels: $A' \cap B'$, $A' \cap B$, $A \cap B'$. On a similar figure, shade (i) $A' \cup B'$, (j) $A' \cup B$, (k) $A \cup B'$, (l) $(A \cap B)'$. (m) Repeat Exercises (i) through (l) using a Venn diagram like that of Fig. 3–3.

Let U = the human race, A = the residents of New York City, and B = the citizens of the U. S. A. Express the following symbolically and draw Venn diagrams: (n) the American citizens who live in N. Y., (o) aliens, (p) people who live outside N. Y., (q) citizens who do not live in N. Y., (r) the set of all people, (s) residents of N. Y. together with all aliens.

With symbols defined as in the preceding paragraph, express in words: (t) $A' \cap B$, (u) $A \cap B$, (v) $A \cup B$, (w) $A' \cap B'$, (x) $A' \cup B'$.

Complements, unions, and intersections of intervals are very convenient for specifying the solutions of inequalities. For example, we found in Section 3–5 that $\{x \mid x^2 - 4 < 0\} = (-2 \_ 2)$. Then

(10) $\{x \mid x^2 - 4 \geq 0\} = (-2 \_ 2)' = \{x \mid x \leq -2 \ \vee \ x \geq 2\}$

(11) $= \{x \mid x \leq -2\} \cup \{x \mid x \geq 2\}$

(12) $= (-\infty \_ -2) \cup (2 \_ \infty).$

By combining intervals we can get a very large variety of sets of points on the axis of real numbers. Among these sets are those consisting of single points, since $\{x\} = (a \_ x) \cap (x \_ b)$ for any $a < b$. Thus we may think of a set consisting of a single point as a "degenerate" interval. We might even write $\{x\} = (x \_ x)$.

Find a simple expression for and sketch the following sets: (y) $(2 \_ \infty)'$, (z) $(3 \_ 5) \cap (4 \_ 6)$, (z′) $(3 \_ 5) \cup (4 \_ 6)$.

PROBLEMS

An insurance company with policyholders U is studying characteristics of certain kinds of policyholders. Let A = adult policyholders, B = male policyholders, C = married policyholders. Describe in words:

1. A'.
2. B'.
3. C'.
4. $A \cap B$.
5. $A \cup B$.
6. $B \cap C'$.
7. $B' \cap C'$.
8. $B' \cup C'$.
9. $A \cap (B \cap C)$.
10. $(A \cap B) \cap C$.
11. U'.
12. $A' \cap (B \cap C)$.
13. $A' \cup (B \cap C)$.
14. $(B' \cap C) \cap A$.
15. $(A \cup B)'$.
16. $B \cup B'$.

Using three circles, draw Venn diagrams of the following:

17. $A \cap (B \cap C)$. 18. $A \cup (B \cup C)$. 19. $(A \cap B')'$.
20. $(A \cup B) \cap C$. 21. $B \cup (C \cap A')$. 22. $A' \cap (B' \cap C')$.

★23. Suggest ways in which diagrams of the type of Fig. 3–4 could be adapted to representing three or more sets.

Sketch:

24. $(2 - \epsilon \_ 2 + \epsilon)$ where $\epsilon > 0$. 25. $(\underline{3 \_ 5}) \cup (\underline{5 \_ 6})$.
26. $(\underline{3 \_ 4}) \cap (\underline{4 \_ 5})$. 27. $(-\underline{1 \_ 3}) \cup (\underline{0 \_ 2})$.
28. $(-\underline{1 \_ 3}) \cup (\underline{2 \_ 5})$. 29. $(-\underline{1 \_ 3}) \cap (\underline{2 \_ 5})$.
30. $(-\underline{1 \_ 3}) \cap (\underline{0 \_ 2})$. 31. $\{1\} \cup (\underline{3 \_ 4})$.

Show that

32. $(-\infty \_ b)' = (\underline{b \_ \infty})$. 33. $(b \_ \infty)' = (-\infty \_ \underline{b})$.
34. $(a \_ b)' = (-\infty \_ \underline{a}) \cup (\underline{b} \_ \infty)$. 35. $(\underline{a \_ b})' = (-\infty \_ a) \cup (b \_ \infty)$.
36. $(a \_ \underline{b})' = (-\infty \_ \underline{a}) \cup (b \_ \infty)$. 37. $(\underline{a} \_ b)' = (-\infty \_ a) \cup (\underline{b \_} \infty)$.
38. $(\underline{a \_ a}) = \{a\}$. 39. $(a \_ a) = \emptyset$.
40. $a > b \rightarrow [(\underline{a \_ b}) = \emptyset]$.
41. $(\underline{a \_ b}) \cap (\underline{c \_ d}) = (\underline{c \_ b})$ is not a law.

For Problems 42 through 45, let $A \underline{\cup} B$ be the elements in A or B, but not both.

★42. Give a formal definition.
★43. Draw a Venn diagram and a Lewis Carroll type.
★44. Show that $A \underline{\cup} B = (A \cup B) \cap (A \cap B)'$.
★45. Under what conditions is $A \underline{\cup} B = A \cup B$?

The *relative complement* $B - A$ of A with respect to B is defined as the set of elements of B that are not in A.

★46. Give a formal definition.
★47. Show that the complement of a set is its relative complement with respect to the universe of discourse.

Express 48 through 55 in terms of intervals.

48. $\{x \mid (x+1)(x-3) \geq 0\}$. 49. $\{x \mid (x+1)(x-3) \leq 0\}$.
50. $\{x \mid x^2 - 3 < 0\}$. 51. $\{x \mid x^2 - 3 > 0\}$.
52. $\{x \mid x^2 + 5x - 6 \geq 0\}$. 53. $\{x \mid x^2 + x + 2 > 0\}$.
54. $\{x \mid x^2 + 2x + 5 \leq 0\}$. 55. $\{x \mid x^2 + 2x + 1 < 0\}$.

Answers to Exercises

(a) $\{0, 8, 10\}$. (b) (3–3–3). (c) $\{a, c\}$. (d) (3–3–3). (e) $\{a, b, c, d, e, f\}$. (f) (3–3–3). (g) Similarity to $\wedge$ and $\vee$ in terms of which they are defined. (h) See Fig. 3–5. (i) Fig. 3–6. (j) Fig. 3–7. (k) Fig. 3–8. (l) See Fig. 3–6. (n) $A \cap B$. (o) B'. (p) A'. (q) $A' \cap B$. (r) U. (s) $A \cup B'$. (t) Citizens who do not live in N. Y. (u) Citizens who live in N. Y. (v) Those who are citizens or who live in N. Y., i.e., citizens and New Yorkers. (w) Aliens

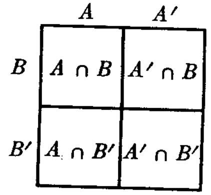


Figure 3–5

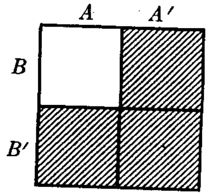


Figure 3–6

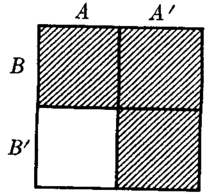


Figure 3–7

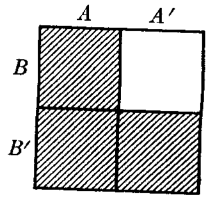


Figure 3–8

who live outside N. Y. (x) Those who live outside N. Y. or who are aliens. (y) $(-\infty \_ 2)$. (z) $(\underline{4 \_ 5})$. (z′) $(3 \_ 6)$.

Answers to Problems

1. Minor policyholders. 3. Unmarried policyholders. 5. All adults and all males. 7. Unmarried females. 9. Married adult males. 11. Empty set. 13. Minors and all married males. 15. Those who are neither adults nor males, i.e., the minor females. 25. $(\underline{3 \_ 6})$. 27. $(\underline{-1 \_ 3})$. 28. $(\underline{-1 \_ 5})$. 31. A point and an interval. 32 through 41. Use definitions from Section 3–4. 42. $\{x \mid x \in A \underline{\vee} x \in B\}$. 45. $A \cap B = \emptyset$. 46. $B - A = A' \cap B$. 49. $(\underline{-1 \_ 3})$. 51. $(-\infty \_ -\sqrt{3}) \cup (\sqrt{3} \_ \infty)$. 53. $(-\infty \_ \infty)$. 55. $\emptyset$.

★3–7 Algebra of sets. Since operations on sets are defined in terms of the corresponding logical operations, we might expect that the sets would satisfy laws similar to those of logic.

(1) $(A \cup B)' = A' \cap B'$
(2) $(A \cap B)' = A' \cup B'$ (*De Morgan's laws*),
(3) $A \cup B = B \cup A$
(4) $A \cap B = B \cap A$ (*Commutative laws*),
(5) $A \cup (B \cup C) = (A \cup B) \cup C$
(6) $A \cap (B \cap C) = (A \cap B) \cap C$ (*Associative laws*),
(7) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(8) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (*Distributive laws*),
(9) $A \cup A = A$
(10) $A \cap A = A$ (*Idempotent laws*),
(11) $(A')' = A$ (*Involution law*).

Diagrams are helpful in appreciating these and other laws. By making separate diagrams for each side, we can see that different procedures are

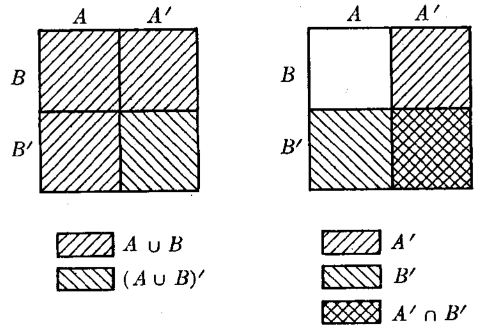


FIGURE 3–9

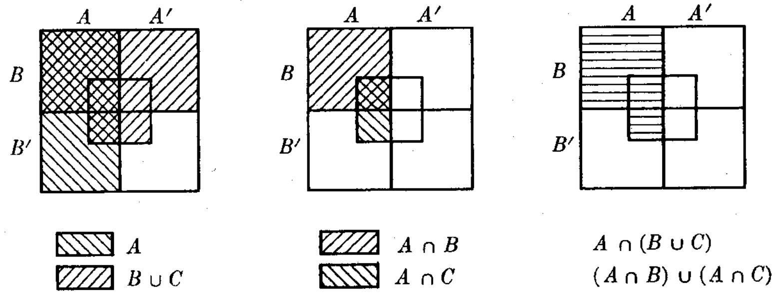


FIGURE 3–10

involved and that the final result is the same. Different kinds of shading should be used for each set that is of importance, and the sets should be clearly labeled. Figure 3–9 shows Lewis Carroll diagrams of the two sides of (1). In Fig. 3–10 we sketch the sides of (8), letting the central square represent C.

(a) Let $U =$ people, $A =$ students, $B =$ males, $C =$ females, so that $A \cap (B \cap C) =$ students who are both male and female, and $A \cup (B \cup C) =$ those who are either students or men or women. State each law above in words in terms of these sets.

The proofs of the above laws are easy. One simply applies the definitions and then the identities of Section 2–5. For example,

(12) Proof of (3):

(α) $A \cup B = \{x \mid x \in A \vee x \in B\}$ (3–6–7)

(β) $= \{x \mid x \in B \vee x \in A\}$ $(p \vee q = q \vee p)(p{:}x \in A, q{:}x \in B)$

(γ) $= B \cup A$ (3–6–7)$(A{:}B, B{:}A)$.

(b) What rules of proof were used in step (β)? (c) in step (γ)?

One can also argue informally for these identities by showing, as called for by (3–3–4), that every element of one set belongs to the second, and conversely. To argue for (3) in this way we suppose that an element belongs to $A \cup B$. Then it must belong to A or to B or both. But then it must belong to B or to A or to both, and hence to $B \cup A$. Similarly, beginning with an element belonging to $B \cup A$, we argue that it must belong to $A \cup B$. Of course this is merely an informal paraphrase of (12) with the logical identities remaining implicit.

(d) Argue informally for (1) and prove it.

(13) **Def.** $A \cup B \cup C = A \cup (B \cup C)$,

(14) **Def.** $A \cap B \cap C = A \cap (B \cap C)$,

(15) $A \cup \emptyset = A$,

(16) $A \cap \emptyset = \emptyset$,

(17) $A \cup U = U$,

(18) $A \cap U = A$,

(19) $A \cup A' = U$,

(20) $A \cap A' = \emptyset$,

(21) $U' = \emptyset$,

(22) $\emptyset' = U$,

(23) $(A = B') \leftrightarrow (A' = B)$,

(24) $A = (A \cap B) \cup (A \cap B')$,

(25) $A \cup B = (A \cap B') \cup (A' \cap B) \cup (A \cap B)$,

(26) $U = (A \cap B) \cup (A' \cap B) \cup (A \cap B') \cup (A' \cap B')$,

(27) $A \cup (A \cap B) = A$,

(28) $A \cap (A \cup B) = A$.

(e) Argue for the above from diagrams and informally from the meanings of the terms. (f) Prove (15). (g) Prove (23).

Simplify: (h) $B \cap (A \cup A')$, (i) $A \cup (C \cup C')$, (j) $(S \cap T) \cap (T \cap T')$, (k) $(C \cap C') \cap (C \cap C')$, (l) $(B \cup C) \cup (B' \cap C')$, (m) $(A \cup B \cup C) \cup (A' \cap B' \cap C')$, (n) $U \cap \emptyset'$, (o) $U' \cup \emptyset$, (p) $(B \cap C) \cup (B \cap C')$, (q) $(S \cap T) \cup (S \cap T') \cup (S' \cap T)$, (r) $A' \cup (B \cap C)'$.

Using a diagram like Fig. 3–10, sketch each of the following: (s) $A' \cap (B \cup C)$, (t) $A \cap B \cap C$, (u) $A' \cap B' \cap C'$, (v) $A' \cup B' \cup C'$, (w) $A \cap B' \cap C$, (x) $A \cup (B' \cap C')$, (y) the set in Exercise (r).

PROBLEMS

In the introduction to a recent dictionary there appears a diagram similar to the one in Fig. 3–11, in which X = formal literary English, Y = colloquial English, and Z = illiterate English.

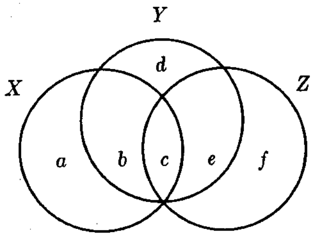


FIGURE 3–11

1. Name a, b, c, d, e, f, $a \cup b$, $b \cup c$, $a \cap X$, and $b \cup d \cup e$ in terms of X, Y, and Z and verbally.

2. What assumption is implicit in the diagram? Express it in symbols and words.

3. For A, B, and C defined as in Problem 4, find the sets of Exercises (s) through (y).

4. Illustrate the laws of this section for $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $C = \{2, 4, 5\}$.

5. Find the complement of $(A' \cup B') \cap (A' \cup C')$.

6. Prove some of the laws of this section.

7. Derive (26) by applying the distributive law to the right member.

★8. Discover and prove some laws involving the relative complement.

★9. Do the same for $A \uplus B$.

★10. Interpret the following definitions:

(29) **Def.** $\bigcup S = \{x \mid \exists y\ y \in S \wedge x \in y\}$,

(30) **Def.** $\bigcap S = \{x \mid \forall y\ y \in S \rightarrow x \in y\}$.

★11. Show that $\bigcup\{A, B\} = A \cup B$, and $\bigcap\{A, B\} = A \cap B$.

★12. Let K = the set of points in a plane within and on the boundary of a square. Let S consist of all circular regions that include K. Argue that $\bigcap S = K$.

★13. Let C = the set of all points in a plane within and on the circumference of a circle. Let T be the set of all polygons inscribed in the circle. Argue that $\bigcup T = C$.

14. Aristotle expressed the fundamental laws of logic as follows: (1) All A is A. (2) Nothing is both A and not A. (3) Everything is either A or not A. Express these laws in set terminology.

15. Prove (28).

ANSWERS TO EXERCISES

(a) (1) Those who are not either students or males = those who are nonstudents and nonmales. (2) Those who are not male students = those who are either not students or not males. (3) Those who are either students or males = those who are either males or students. (4) Student males = male students. (5) Those who are either students, or males or females = those who are students or males, or females. An alternative: Students together with males and females = students and males together with females. (6) Students who are males and females = student males who are females. (7) Students together with the male females = those who are both student males and student females. (8) Students who are males or females. (9) Students and students = students. (10) Students who are students = students. (11) Those who are not nonstudents = students. *Note:* Either "and" or "or" may be involved in a correct translation of $\cup$. Thus $A \cup B$ = the A's and the B's = those who are in A or in B = those who are in A and those who are in B. But $A \cap B$ = the A's in B = the B's in A = those who are in A and in B. It is best not to rely on rules or memory, but to understand the meaning and translate into the best English in the particular context.

(b) Rule of Substitution in $p \vee q = q \vee p$ to get $x \in A \vee x \in B = x \in B \vee x \in A$, then Rule of Replacement to insert the second for the first. (c) Rule of Substitution in (3–6–7), then Rule of Replacement, to replace the right member of (β) by (γ). (d) If an element belongs to $(A \cup B)'$, it does not belong to $A \cup B$. Hence it cannot belong to A or to B. Hence it does not belong to A and it does not belong to B, that is, it belongs to $A' \cap B'$. The converse is shown similarly. The proof goes as follows: $(A \cup B)' = \{x \mid x \in (A \cup B)'\} = \{x \mid \sim(x \in A \cup B)\} = \{x \mid \sim(x \in A \vee x \in B)\} = \{x \mid \sim(x \in A) \wedge \sim(x \in B)\} = \{x \mid x \in A' \wedge x \in B'\} = \{x \mid x \in (A' \cap B')\} = A' \cap B'$. An alternative way of working the proof is to write $x \in (A \cup B)' = \sim(x \in (A \cup B)) = \ldots = x \in (A' \cap B')$, and then use (3–3–4).

(e) For example, (15) says that the objects that belong to a set or to the null set are just those that belong to the set. (f) $A \cup \emptyset = \{x \mid x \in A \vee x \in \emptyset\}$. By (3–3–2), $\sim(x \in \emptyset)$, and by (2–7–31) $\sim(x \in \emptyset) \rightarrow [(x \in A \vee x \in \emptyset) \leftrightarrow (x \in A)]$. Hence by the Rule of Inference $(x \in A \vee x \in \emptyset) \leftrightarrow (x \in A)$. Hence $\{x \mid x \in A \vee x \in \emptyset\} = \{x \mid x \in A\}$ by (3–3–6), and $\{x \mid x \in A\} = A$. (g) By hypothesis $A = B'$. Hence $A' = (B')'$. But $(B')' = B$ by (11)$(A{:}B)$. Hence $A = B' \rightarrow A' = B$. The converse is proved similarly. (h) B. (i) U. (j) $\emptyset$. (k) $\emptyset$. (l) U. (m) U. (n) U. (o) $\emptyset$. (p) B. (q) $S \cup T$. (r) $(A \cap B \cap C)'$.

ANSWERS TO PROBLEMS

1. $a = X \cap Y' \cap Z'$, $b = X \cap Y \cap Z'$, $c = X \cap Z$, $d = X' \cap Y \cap Z'$, $e = X' \cap Y \cap Z$, $f = X' \cap Y' \cap Z$, $a \cup b = X \cap Z'$, $b \cup c = X \cap Y$, $a \cap X = a$, $b \cup d \cup e = Y \cap (X \cap Z)'$. 2. $X \cap Y' \cap Z = \emptyset$. There is no formal literary illiterate English that is not colloquial; i.e., all formal literary English that is illiterate is also colloquial. 3. (s) $\{4, 5\}$. (t) $\{2\}$. (u) $\emptyset$. (v) $\{2\}'$. (w) $\emptyset$.

(x) $\{1, 2, 3\}$. (y) $\{1, 3, 4, 5\}$. 5. $A \cap (B \cup C)$. 7. Right member $= [A \cap (B \cup B')] \cup [A' \cap (B \cup B')] = (A \cap U) \cup (A' \cap U) = A \cup A' = U$. 15. $A \cap (A \cup B) = (A \cup \emptyset) \cap (A \cup B) = A \cup (\emptyset \cap B) = A \cup \emptyset = A$.

★3–8 Relations between sets. In Section 3–4 we defined and illustrated briefly the relation of inclusion. With the aid of set algebra we can analyze inclusion further and consider some other possible relations between sets.

From the definition (3–4–1) we have immediately

(1) $$A \subseteq A,$$

(2) $$[A \subseteq B \wedge B \subseteq C] \rightarrow A \subseteq C,$$

(3) $$\emptyset \subseteq A,$$

(4) $$A \subseteq U.$$

From (3–3–4),

(5) $$[A \subseteq B \wedge B \subseteq A] \rightarrow A = B.$$

Additional laws are

(6) $$(A \cap B) \subseteq A,$$

(7) $$A \subseteq (A \cup B),$$

(8) $$A \subseteq B \leftrightarrow B' \subseteq A',$$

(9) $$A \subseteq B \rightarrow (A \cap C) \subseteq (B \cap C),$$

(10) $$A \subseteq B \rightarrow (A \cup C) \subseteq (B \cup C),$$

(11) $$A \subseteq B \leftrightarrow [A \cap B = A],$$

(12) $$A \subseteq B \leftrightarrow [A \cup B = B],$$

(13) $$A \subseteq (B \cap C) \leftrightarrow [A \subseteq B \wedge A \subseteq C],$$

(14) $$(A \cup B) \subseteq C \leftrightarrow [A \subseteq C \wedge B \subseteq C],$$

(15) $$[A \subseteq B \wedge \sim (A \subseteq C)] \rightarrow \sim (B \subseteq C),$$

(16) $$A \subseteq B \leftrightarrow [A \cap B' = \emptyset].$$

These theorems are intuitively evident if they are expressed verbally or illustrated by Venn diagrams. They are also easy to prove by simply using the definitions and applying theorems from Chapter 2. For example, to prove (6), we assume by hypothesis that $x \in A \cap B$. Then $x \in A \wedge x \in B$. But by the law $[p \wedge q] \rightarrow p$ and the Rule of Inference

it follows that $x \in A$. Hence $\forall x\,(x \in A \cap B \rightarrow x \in A)$, which is equivalent to (6) by the definition of inclusion. Or we may argue informally that if an element lies in both A and B, it must lie in A, and hence every member of $A \cap B$ is a member of A.

(a) Argue informally for the above laws in terms of Venn diagrams and the meaning of terms. (b) Prove (7).

To prove (15) we note that it is equivalent to (2) by the identity

$$[(p \wedge q) \rightarrow r] = [(p \wedge \sim r) \rightarrow \sim q]$$

with

$$(p{:}A \subseteq B,\ q{:}B \subseteq C,\ r{:}A \subseteq C).$$

(c) Prove the logical identity (2–5–35). (d) Make the substitution $(A{:}C, B{:}A, C{:}B)$ in (2) and then prove (17).

(17) $$[A \subseteq B \wedge \sim(C \subseteq B)] \rightarrow \sim(C \subseteq A).$$

To show the plausibility of (16) we note that A is a subset of B if and only if every member of A is a member of B, that is, if and only if no members of A are in B', that is, $A \cap B' = \emptyset$.

To prove it, we write

$$\begin{aligned} A \subseteq B &= [\forall x\,(x \in A \rightarrow x \in B)] \\ &= [\sim\exists x \sim(x \in A \rightarrow x \in B)] \\ &= [\sim\exists x\,(x \in A \wedge x \in B')] \\ &= [\sim\exists x\, x \in (A \cap B')] \leftrightarrow [A \cap B' = \emptyset]. \end{aligned}$$

(e) Justify each step in the preceding argument.

If two sets have no members in common we say that they are *disjoint*. This relation may be symbolized by $A \,//\, B$, the notation being suggested by the fact that two straight lines in a plane are disjoint if and only if they are parallel.

(18) **Def.** $$A \,//\, B = [A \cap B = \emptyset].$$

(f) Draw a Venn diagram showing two disjoint sets. (g) Show

(19) $$A \,//\, B \rightarrow B \,//\, A.$$

(h) Prove (20).

(20) $$A \,//\, B \leftrightarrow A \subseteq B'.$$

(i) Prove that $A//B \leftrightarrow B \subseteq A'$.

$$(21) \qquad [A \mathbin{//} B \wedge A \mathbin{//} C] \to A \mathbin{//} (B \cup C).$$

(j) Show that $[A//B \wedge B//C] \to A//C$ is not a law.

We define two further relations, leaving their properties to be worked out by the reader.

(22) Def. $\quad A \subset B = [A \subseteq B \wedge A \neq B]$.

When $A \subset B$ we say that A is a *proper subset* of B.

(k) Give an example for which $A \subset B \wedge A \subseteq B$.

(23) Def. $\quad A \between B = \sim[A \subseteq B \vee B \subseteq A \vee A \mathbin{//} B]$.

When $A \between B$ we say that A *overlaps* B.

(l) Draw a Venn diagram illustrating $A \between B$ and several examples in which $\sim(A \between B)$. (m) Illustrate the definitions and theorems of this section using $U = \{1, 2, 3, 4, 5\}$ and its subsets.

Problems

1. Complete the reasons in the following proofs and supply informal arguments for the laws.

(24) Proof of (2):

(a) $x \in A \to x \in A$	(?),
(b) $A \subseteq A$	(3–4–1)$(B{:}A)$.

(25) Proof of (3):

(a) $A \subseteq B \wedge B \subseteq C$	Hyp,
(b) $(x \in A \to x \in B) \wedge (x \in B \to x \in C)$	(1), (2–10–14),
(c) $x \in A \to x \in C$	(?),
(d) $A \subseteq C$	(?),
(e) (3)	Q.E.D., (a), (d).

2. Prove (8) by using (2–5–28).
3. Prove (9) by using (2–7–25).
4. Prove (10) by using (2–7–11).
5. Prove (11) by noting that $A \cap B \subseteq A$ by (6), and proving that $A \subseteq A \cap B$ from the hypothesis $A \subseteq B$ and (9)$(C{:}A)$. Why is this sufficient?
6. Prove (12) from (7) and (10).

7. Prove that $A \subseteq B \leftrightarrow A//B'$.
8. Prove (26).

$$(26) \qquad A \subseteq B \leftrightarrow [A \subset B \veebar A = B].$$

9. Discover, argue for, and prove other laws that involve proper inclusion.
10. Prove (27).

$$(27) \qquad [A//B \wedge C \subseteq A] \to C//B.$$

11. Prove that the sets in the right member of (3–7–25) are disjoint two by two.
12. Do the same for (3–7–26).
13. Prove (28).

$$(28) \qquad A \between B \to B \between A.$$

14. Do Problem 9 for overlapping.
15. Show that $\exists A\ \exists B\ A//B \wedge A \subseteq B$!
16. Show that $A = B \veebar A \subset B \veebar B \subset A \veebar A//B \veebar A \between B$ is not a law. Show that it is a law if the hypothesis $A \neq \emptyset \wedge B \neq \emptyset$ is added.

★17. Prove $[\{x \mid a \leq x \leq b\} = \{x \mid c \leq x \leq d\}] \to (a = c \wedge b = d)$.
★18. Prove $\{x \mid a \leq x < b\} \neq \{x \mid a \leq x \leq b\}$.
★19. Prove $\{x \mid 1 \leq x \leq 2\} \subseteq \{x \mid 0.5 \leq x \leq 7\}$.

Under what conditions does

★20. $(a\_b) \between (c\_d)$?
★21. $(a\_b)//(c\_d)$?
★22. $(a\_b) \subset (c\_d)$?

★23. Prove $A \subseteq \emptyset \to A = \emptyset$.
★24. Consider the following sets of figures in plane geometry: T = triangles, O = obtuse triangles, R = right triangles, E = equilateral triangles, S = scalene triangles, I = isosceles triangles, A = acute triangles. What relation holds between each pair? (Pair each one in order with every following one.)
★25. Prove $\exists x\ \exists y\ x \in y \wedge x \subset y$.
★26. Prove $[A \cup X = U \wedge A \cap X = \emptyset] \to [X = A']$.

Answers to Exercises

(a) For example, (7) is plausible, since any member of A is certainly in A or in B. Again, (8) is true because if every element of A is in B, any element not in B cannot be in A. (b) $x \in A \to x \in A \vee x \in B \to x \in (A \cup B)$ by $(p \to p \vee q)(p{:}x \in A, q{:}x \in B)$. (c) $[(p \wedge q) \to r] = [\sim(p \wedge q) \vee r] = \sim p \vee \sim q \vee r$; $[(p \wedge \sim r) \to \sim q] = [\sim(p \wedge \sim r) \vee \sim q) = \sim p \vee \sim\sim r \vee \sim q = \sim p \vee \sim q \vee r$. (d) It yields $C \subseteq A \wedge A \subseteq B \to C \subseteq B$; then by (2–5–35) $(p{\cdot}A \subseteq B, q{:}A \subseteq B, r{:}C \subseteq B)$ we have (17).
(e) (3–4–1); (2–10–3) $(f(x){:}\sim(x \in A \to x \in B)$; (2–5–24); (3–6–4); (3–3–8).

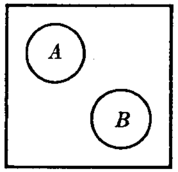
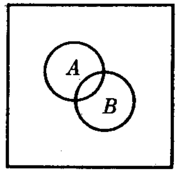


FIGURE 3–12



$A \between B$

FIGURE 3–13

(f) See Fig. 3–12. (g) Since $[A \cap B = \emptyset] = [B \cap A = \emptyset]$. (h) (16), (18). (i) (19), (20). (j) $(A{:}\{1, 2\}, B{:}\{3, 4\}, C{:}\{1, 2\})$. (k) $(A{:}\{1, 2\}, B{:}\{1, 2, 3\})$. (l) See Fig. 3–13.

ANSWERS TO PROBLEMS

1. (24a) $p \to p$ (25c) $[(p \to q) \land (q \to r)] \to (p \to r)$. (25d) (3–4–1). Note that quantifiers are omitted. 3. $(p{:}x \in A, q{:}x \in B, r{:}x \in C)$. 5. (5). 7. $A \subseteq B \leftrightarrow A \cap B' = \emptyset \leftrightarrow A//B'$ by (16) and (18). 9. Some examples: $\sim(A \subset A)$, $A \subset B \to \sim(B \subset A)$, $A \subset B \land B \subset C \to A \subset C$. 11. Any product of two of these sets has $A \cap A'$ or $B \cap B'$ as a factor. 13. (23). 15. $(A{:}\emptyset.\ B{:}\emptyset)$. 21. $b \leq c \lor a \geq d$.

3–9 Descriptions. It is often convenient to refer to something as the object that satisfies a certain condition. For example, "the set that has no members" is a description of the empty set. Such descriptions take the form "the x such that . . . ," where the dots stand for some sentence involving x that is satisfied by one and only one value of x. We introduce the symbol $\ni$ to stand for "such that" and adopt the formula $x \ni f(x)$ to stand for the unique solution of $f(x)$. For example, $[x \ni 2x = 1] = 1/2$. We read $x \ni f(x)$ as "the x such that $f(x)$." Expressions of this form have meaning if and only if there is one and only one solution of the condition $f(x)$, that is, if and only if $\exists! x\, f(x)$. The essential property of this undefined formula is given by the following axiom.

(1) **Ax.** $[a = x \ni f(x)] = [f(a) \land \forall y\, f(y) \to y = a]$.

(a) Read (1) in words. (b) Why is "x" in "$x \ni f(x)$" a dummy? (c) Justify $a = (x \ni x = a)$. (d) Comment on "$x \ni x^2 = 1$." Justify the following: (e) $1 = (x \ni \forall z\, xz = z)$, (f) $0 = (x \ni \forall y\, y + x = y)$, (g) $a/b = (x \ni xb = a)$, (h) $a - b = (x \ni x + b = a)$, (i) $\{x | f(x)\} = S \ni \forall x\, [x \in S \leftrightarrow f(x)]$.

We illustrate the utility of the description notation by using it to formulate several definitions.

Given a set of real numbers S, we say that x is the *maximum* of S if x is in S and is larger than any other member of S. Similarly we call x the *minimum* of S if it is in S and is smaller than any other member of S.

Geometrically the maximum of S is the point in S farthest to the right, and the minimum is the point farthest to the left.

(2) **Def.** $\max S = x \ni x \in S \land \forall y\, [y \in S \to x \geq y]$,

(3) **Def.** $\min S = x \ni x \in S \land \forall y\, [y \in S \to x \leq y]$.

Find the following: (j) $\max\{1, 2, 3\}$; (k) $\max\{5\}$; (l) $\min\{4, -3, 8\}$; (m) $\max\{-3, 3\}$; (n) $\min\{x | x^2 \leq 5\}$.

In Section 1–4 the absolute value of x, symbolized by $|x|$, was introduced intuitively as the length of the vector x. In (1–13–17) we defined it by considering separately the cases $x \geq 0$ and $x < 0$. We now can give a simpler definition.

(4) **Def.** $|x| = \max\{x, -x\}$.

(o) Use (4) to find $|0|$, $|-3|$, $|10|$.

We can now easily prove the following, which are equivalent to (1–13–17).

(5) $x \geq 0 \leftrightarrow |x| = x$,

(6) $x \leq 0 \leftrightarrow |x| = -x$.

The reader should check that the definition is consistent with the intuitive concept. Recalling from Section 1–4 that $b - a$ is interpreted as the vector from a to b or as the directed distance from a to b, we see that $|b - a|$ is the length of this vector. Accordingly we describe $|b - a|$ as the *undirected distance between a and b*. In particular, $|x|$, which is equal to $|x - 0|$, is the undirected distance between the origin and the point x. These geometric interpretations are very helpful in dealing with absolute values.

The following laws are evident from the geometric interpretation. They can, of course, be proved from (4).

(7) $|a| \geq 0$,

(8) $(|a| = 0) \leftrightarrow (a = 0)$,

(9) $|a| = |-a|$,

(10) $(|a| = |b|) \leftrightarrow (a = b \lor a = -b)$,

(11) $|a + b| \leq |a| + |b|$,

(12) $|a| - |b| \leq |a + b|$,

(13) $|ab| = |a| \cdot |b|,$

(14) $\max\{a, b\} = (1/2)[a + b + |b - a|],$

(15) $\min\{a, b\} = (1/2)[a + b - |b - a|].$

The most useful law for solving equations involving absolute values is (10). Thus $(|x - 1| = 3) \leftrightarrow (x - 1 = 3 \vee x - 1 = -3) \leftrightarrow (x = 4 \vee x = -2)$. A still easier way to solve this equation is to note that it requires that x lie at an undirected distance of 3 from 1. Hence it must lie at 4 or -2.(!) We sketch this in Fig. 3–14.

Solve each of the following and sketch as in Fig. 3–14 [note that $x + 2 = x - (-2)$]: (p) $|x - 3| = 1$, (q) $|4 - x| = 3$, (r) $|x + 2| = 1$.

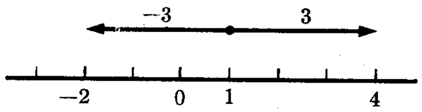


FIGURE 3–14

Solving inequalities involving absolute values is most easily done from the geometric interpretation. Thus to solve $|x - 3| \leq 2$, we note that it requires x to lie within an undirected distance of 2 from 3. Hence it lies between 1 and 5 inclusive, i.e., in the closed interval $(\underline{1\_5})$, which is the solution. Similarly, the solution of $|x - 3| \geq 2$ is $(-\infty\_\underline{1}) \cup (\underline{5}\_\infty)$. Also, since

$$|x + 3| \leq 1 \leftrightarrow |x - (-3)| \leq 1 \leftrightarrow -4 \leq x \leq -2,$$

then

$$\{x \mid |x + 3| \leq 1\} = (\underline{-4\_-2}).$$

Solve and sketch the solution for: (s) $|x| \leq 3$, (t) $|x| \geq 3$, (u) $|x - 5| \leq 3$, (v) $|x - 8| > 2$.

The following laws generalize the results of the previous exercises.

(16) $\{x \mid |x| < b\} = (-b\_b),$

(17) $\{x \mid |x| > b\} = (-\infty\_-b) \cup (b\_\infty),$

(18) $\{x \mid |x - a| < b\} = (a - b\_a + b),$

(19) $\{x \mid |x - a| > b\} = (-\infty\_a - b) \cup (a + b\_\infty).$

Use absolute values to write brief defining sentences for: (w) $(-\epsilon\_\epsilon)$, (x) $(3 - \delta\_3 + \delta)$, (y) $(\underline{2\_4})'$, (z) $(a - \delta\_a + \delta)'$.

PROBLEMS

1. In the light of Exercise (g), explain briefly why "$a/0$" is meaningless.
2. Comment on "Whenever they disagree, I choose the version which, in my judgment, is the more credible and at the same time the more interesting of the two."
3. Show that $[a = x \ni f(x)] = [\{a\} = \{x \mid f(x)\}]$.
4. Comment on "$\max\{x \mid x < 1\}$."
5. Let $T(x)$ = the truth value of x. Show that $T(p \vee q) = \max\{T(p), T(q)\}$ and $T(p \wedge q) = \min\{T(p), T(q)\}$.
6. What is the range of significance of "S" in "$\max S$"?
7. Cite several sets S such that $\max S$ or $\min S$ does not exist.

Solve and sketch:

8. $|2x| = 4$.
9. $|2x + 3| = 7$.
10. $|x| = -1$.
11. $|x| = 0$.
12. $|x + 4| < 1$.
13. $|x + 4| \geq 5$.
14. $|x - 3| > -2$.
15. $|2x - 1| < 2$.
16. $|x - 5| < 0.01$.
17. $|x - 2| < 0.02$.
18. $|x + 0.01| < 5$.
19. $|1 - 3x| = |x|$.
20. Find c and δ such that $(a\_b) = \{x \mid |x - c| < \delta\}$.
21. Use absolute values to write brief defining sentences for (3_8), $(\underline{-1\_5})$, and (2_4).
22. Often results of scientific measurements are given in the form $x = a \pm e$, meaning that x probably differs from a by at most e. Express this in each of the forms $\alpha \leq x \leq \beta$, $|x - \gamma| \leq \delta$, and $x \in (\underline{r\_s})$.
23. $x \ni \forall y \sim (y \in x) = ?$
24. $x \ni \forall y\, y \in x = ?$
25. Use a description to define $\sqrt{x}$. (See 1–13–16.)

★26. Prove (5) by proving that $x \geq 0 \leftrightarrow x \geq -x$.

★27. Prove (6) similarly.

★28. Prove (7) and (8) by brief arguments.

★29. Prove (9) by showing that $\max\{a, -a\} = \max\{-a, -(-a)\}$.

★30. Prove (10) by noting its equivalence to $\max\{a, -a\} = \max\{b, -b\} \leftrightarrow a = b \vee a = -b$.

★31. Writing (11) in the form $|a - (-b)| \leq |a| + |b|$, interpret it geometrically, sketch, and argue for its plausibility.

★32. Prove (11) by considering the cases $a > 0, b > 0$; $a < 0, b < 0$; $a > 0, b < 0$; and $a < 0, b > 0$.

★33. Treat (12) according to Problems 31 and 32.

★34. Symbolize by a description "the greatest good for the greatest number." Does it exist?

ANSWERS TO EXERCISES

(a) "a is the x such that f of x" means that f of a is true, and if f of y is true, then y equals a; that is, a is the one and only solution of $f(x)$. (b) $x \ni f(x)$ stands for the solution for the variable x of the sentence $f(x)$. Without a variable present, this would make no sense. Hence x cannot be replaced by a constant. (c) By (1), since $(a = a) \wedge \forall y\,(y = a \rightarrow y = a)$. (d) Nonsense, since the sentence has two solutions. (e) $\forall z\, 1 \cdot z = z \wedge \forall y\,[\forall z\, yz = z \rightarrow y = 1]$. (f) To avoid confusion due to y appearing in two quantifiers, we substitute $(y{:}z)$ here to get $0 = (x \ni \forall z\, z + x = z)$. Then $\forall z\, z + 0 = z \wedge \forall y\,[\forall z\, z + y = z \rightarrow y = 0]$. (g) $(a/b)b = a \wedge \forall y\, yb = a \rightarrow y = a/b$. Note that this holds only if $b \neq 0$. (i) $[x \in \{x | f(x)\} \rightarrow f(x)] \wedge \forall y\,[(\forall x\, x \in y \leftrightarrow f(x)) \rightarrow y = \{x | f(x)\}]$.

(j) 3. (k) 5. (l) -3. (m) 3. (n) $-\sqrt{5}$. (o) 0, 3, 10. (p) $\{2, 4\}$. (q) $\{1, 7\}$. (r) $\{-1, -3\}$. (s) $(-\underline{3 \quad 3})$. (t) $(-\infty \underline{\quad -3}) \cup (\underline{3 \quad \infty})$. (u) $(\underline{2 \quad 8})$. (v) $(-\infty \underline{\quad} 6) \cup (10 \underline{\quad} \infty)$. (w) $|x| < \epsilon$. (x) $|x - 3| < \delta$. (y) $|x - 3| > 1$. (z) $|x - a| \geq \delta$.

Answers to Problems

1. $x \cdot 0 = a$ has no solution or more than one according to $a \neq 0$ or $a = 0$, as explained in Section 1–14. 3. To say that $\{x | f(x)\}$ is the singleton $\{a\}$ is to say that a is the unique solution of $f(x)$. 5. Refer to truth tables and consider cases. 7. $(\underline{2 \quad 5})$ has neither max nor min. 9. $\{2, -5\}$. 11. $\{0\}$. 13. $(-\underline{9 \quad 1})'$. 15. $(-\underline{0.5 \quad 1.5})$. 17. $(\underline{1.98 \quad 2.02})$. 19. $\{1/4, 1/2\}$.

21. $|x - 5.5| < 2.5$, $|x - 2| \leq 3$, $|x - 3| < 1$. 23. $\emptyset$. 25. $y \ni y^2 = x \wedge y \geq 0$. 27. $x \leq 0 \rightarrow -x \geq 0 \rightarrow -x \geq x \rightarrow \max\{x, -x\} = -x$.

29. Since $\{-a - (-a)\} = \{-a, a\} = \{a, -a\}$! 31. The distance between $-b$ and a is not greater than the sum of the distances between the origin and a and b.

★3–10 Sets and sentences. When we say that a number is prime, we are asserting that it belongs to the set of prime numbers or that it has the property of being an integer greater than 1 whose only factors are itself and 1. In this way every statement asserting set membership can be reformulated as a claim that the object has the property that is peculiar to members of the set. This correspondence between sets and properties is embodied in (3–3–3). Because of it we can formulate any sentence in terms of set membership, or, conversely, we can formulate any sentence about set membership in terms not explicitly involving sets.

(a) Consider the statement "3 is odd." Complete the following equivalent sentences: "______ $\in$ ______"; "3 is an integer ______." (b) Consider the statement "He is my father." Complete the following equivalent sentences: "______ = my father"; "______ $\in$ ______"; "______ $x \ni x$ ______." (c) Consider "There is an even prime number." Complete the following equivalent sentences: "$\{x | x$ is prime and x is even$\}$ $\neq$ ______"; $\exists x$ ______."

Exercises (a), (b), and (c) suggest the variety of ways of expressing ideas in terms of or without reference to sets. To systematize the relations between sets and sentences, let $P = \{x | p(x)\}$, $Q = \{x | q(x)\}$. Below we give a table of statements in terms of the sets and the corresponding synonymous statements involving the defining sentences.

(1)	$a \in P$	$p(a)$,
(2)	$P = Q$	$\forall x\, p(x) \leftrightarrow q(x)$,
(3)	$P \subseteq Q$	$\forall x\, p(x) \rightarrow q(x)$,
(4)	$P // Q$	$\forall x \sim (p(x) \wedge q(x))$,
(5)	$a \in P'$	$\sim p(a)$,
(6)	$a \in P \cap Q$	$p(a) \wedge q(a)$,
(7)	$a \in P \cup Q$	$p(a) \vee q(a)$,
(8)	$P = \emptyset$	$\forall x \sim p(x)$, or $\sim \exists x\, p(x)$,
(9)	$P = U$	$\forall x\, p(x)$.

Translate into set terminology: (d) He is neither happy nor wealthy. (H = happy people; W = wealthy people.) (e) He is healthy, wealthy, and wise. (S = wise people.) (f) He is a happy man. (M = men.) (g) If $\angle ABC$ is a right angle, then $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$, and conversely. ($R$ = right triangle; P = triangles such that the square of the length of one side is the sum of the squares of the lengths of the other two sides.) (h) A number cannot be both even and odd. (i) If x is a man then x is mortal. (j) No one lives forever. (k) If a triangle has two equal angles then it has two equal sides.

There are always many ways of stating an idea verbally. As the previous exercises illustrate, the essential thing is to understand the meaning and then restate in precise terms. Sometimes the set language is simpler. This is evidently the case in (2), (3), (4), (8), and (9). There the set language enables us to dispense with explicit use of quantifiers. For example, it is easier to say that all men are mortal (men $\subseteq$ mortals) than to say that if x is a man then x is mortal. On the other hand, the set terminology is sometimes less convenient. This appears to be the case in (1), (5), (6), and (7). For example, it is easier to say that John loves Mary than to say that John belongs to the set of people that love Mary. In many cases, either terminology is used conveniently. For example, we say that 3 is prime or that 3 is a prime number.

(l) Review Section 2–10, especially (10) and (11) and Problems 28 through 43. State the following in terms of sets. (m) All triangles are polygons. (n) All equiangular triangles are equilateral, and conversely. (o) No oblique triangles are right triangles. (p) Some isosceles triangles are equilateral. (q) Not all odd numbers are prime. (r) Every integer has one as a factor.

In classical (presymbolic) logic, much reasoning was carried on in terms of "all," "some," and "none," a kind of primitive reasoning in terms of sets, which was called "syllogistic reasoning." The rules for such reasoning are complicated, and all such problems can be handled more easily in modern symbolic terms. The easiest method is to express all statements in terms of set inclusion, then use the laws of this chapter.

The following table indicates the appropriate translations:

(10)	All P are Q,	$P \subseteq Q$.
(11)	Not all P are Q, *or* Some P are not Q,	$\sim(P \subseteq Q)$.
(12)	No P are Q, *or* All P are not Q,	$P \subseteq Q'$.
(13)	Some P are Q,	$\sim(P \subseteq Q')$.

In the typical syllogistic problem, we are given two premises of the forms (10) and (11) and asked to derive a third if this is possible. For example, given that all collies are dogs ($C \subseteq D$) and that all dogs are mortal ($D \subseteq M$), we have immediately that all collies are mortal ($C \subseteq M$) by (3–8–2). All syllogistic reasoning can be reduced to the application of (3–8–2) or its variants (3–8–15) and (3–8–17). For example, given that all primes greater than 2 are odd ($P \subseteq D$) and that some numbers greater than two are not odd ($\sim(N \subseteq D)$), it follows that some numbers greater than 2 are not prime ($\sim(N \subseteq P)$) by (3–8–17).

To test a syllogistic argument for validity, simply express its premises and conclusion in set terms and see whether it follows from the laws of set inclusion. For example, given that all primes greater than 2 are odd and that some numbers greater than 2 are not prime, does it follow that some numbers greater than 2 are not odd? In symbols,

$$[(P \subseteq D) \wedge \sim (N \subseteq P)] \rightarrow \sim(N \subseteq D)?$$

The reader can easily check that this is not in the form (3–8–15) or (3–8–17). Recalling that a valid argument must be in the form of a law, we can easily show that the above is invalid by the counterexample (P:{1, 2}, D:{1, 2, 3, 4}, N:{3, 4}).

Of course we could handle syllogistic problems directly in terms of laws of logic, using quantifiers, or we could express the inclusion relations in various ways by using (3–8–16) or (3–8–20). In any case, by symbolizing such problems we avoid both the complexity and ambiguity of the traditional logic.

(s) "If the first premise is the proposition that all human beings are motivated by self-interest in their actions, the conclusion that all rulers tend to serve their interests can readily be obtained by means of a syllogism." ("The Issue of a Science of Politics in Utilitarian Thought," by F. Kort, *American Political Science Review*, December, 1952) Do this.

What conclusion can be drawn from: (t) all A is B and no B is C, (u) all A is B and some A is C, (v) no A is B and some C is B?

(w) Some laws are complicated; no confusing laws are satisfactory; every complicated law is confusing. Draw all conclusions you can. (x) No A is B, no B is C. Can you draw a conclusion?

Problems

Express Problems through 12 in terms of sets and using quantifiers.

1. All dogs are animals.
2. No dogs are able to talk.
3. No one can be both a man and a woman.
4. Not all men are good men.
5. Some women are blond and blue-eyed, some are blond and not blue-eyed, and some are blue-eyed and not blond.
6. Some men are both rich and happy.
7. If a triangle has two equal sides, it has two equal angles.
8. If a triangle has no equal sides, it has no equal angles.
9. Not all primes are odd.
10. No perfect squares are primes. (Is it true?)
11. Some even numbers are prime.
12. Some A is not B.

With P, Q, R defined as for (1) through (9), continue the table of synonyms by inserting missing entries in 13 through 18.

13. $P \subset Q$,	?	14. $P \between Q$,	?
15. $P \neq \emptyset$,	?	16. $P \neq U$,	?
17. ?	$p(a) \veebar q(a)$.	18. ?	$p(a) \wedge q(a)$.

★19. When two sentences cannot both be true, they are said to be *contradictory*. Suppose $p(x)$ and $q(x)$ are contradictory. Express this in terms of quantifiers and in terms of sets.

20. How is the discussion of this section illustrated in Section 3–5?

In Problems 21 through 26, draw any conclusion you can or decide on the validity of the reasoning.

21. All A is B, some A is C.
22. No A is B, all C is A. ∴ No C is B.
23. All A is B, no C is A. ∴ No C is B.
24. No A is B, no B is C.

25. "Of the prisoners who were put on their trial at the last Assizes, all, against whom the verdict 'guilty' was returned, were sentenced to imprisonment; some, who were sentenced to imprisonment, were also sentenced to hard labor. Hence, some, against whom the verdict 'guilty' was returned, were sentenced to hard labor." [This and Problem 26 are among the many highly realistic and practical logical problems composed by Lewis Carroll and published in his *Symbolic Logic* in 1897.]

26. "No kitten that loves fish is unteachable; no kitten without a tail will play with a gorilla; kittens with whiskers always love fish; no teachable kitten has green eyes; no kittens have tails unless they have whiskers." Draw all possible conclusions.

27. Go through this chapter, formulating the laws of logic corresponding to each law of set theory.

28. Do the reverse, translating laws of Chapter 2 into set terminology.

29. In this section we have violated our agreement about the meaning of "equals." Where?

30. Read "The Maneuvers in Set Thinking," by W. L. Duren, in *The Mathematics Teacher*, May, 1958, for an interesting review of the ideas of this chapter and a preview of some ways in which they will be used later in the book.

Answers to Exercises

(a) 3, odd numbers; that is not divisible by 2. (b) He; He, {my father}; He, is my father. (c) $\emptyset$; x is prime and x is even. (d) He $\in (H' \cap W')$ (e) He $\in (H \cap W \cap S)$. (f) He $\in (H \cap M)$. (g) $R = P$. (h) Evens $\cap$ odds $= \emptyset$. (i) Men $\subseteq$ mortals. (j) $F = \emptyset$ where $F = \{x \mid x$ lives forever$\}$. (k) $A \subseteq B$ where $A =$ triangles with two equal angles, $B =$ triangles with two equal sides.

(m) Tr $\subseteq$ Po. (n) Eq = El. (o) Ob $\subseteq$ R$'$. (p) $\sim$(Is $\subseteq$ Eq$'$). (q) $\sim$(Od $\subseteq$ P). (r) $J \subseteq F$, where $J =$ integers, $F =$ numbers having one as a factor. (s) [Kings $\subseteq$ humans $\wedge$ humans $\subseteq$ beings motivated by self-interest] $\to$ [kings $\subseteq$ BMSI]. (t) No A is C, by $[A \subseteq B \wedge B \subseteq C'] \to (A \subseteq C')$. (u) Some B is C, by $[A \subseteq B \wedge \sim(A \subseteq C')] \to \sim(B \subseteq C')$ by (3–8–15). (v) Not all C is A, by $[A \subseteq B' \wedge \sim(C \subseteq B')] \to \sim(C \subseteq A)$ by (3–8–17). (w) Some laws are not satisfactory. We have $\sim$(Comp $\subseteq$ L$'$), (Conf $\subseteq$ Sat$'$), (Comp $\subseteq$ Conf). From the last two by (3–8–2), (Comp $\subseteq$ Sat$'$). From this and the first by (3–8–15), $\sim$(Sat$'$ $\subseteq$ L$'$) or $\sim$(L $\subseteq$ Sat). (x) No.

Answers to Problems

1. $D \subseteq A$, $\forall x\, x \in D \to x \in A$. 3. $M \cap W = \emptyset$, $\sim\exists x\, x \in M \wedge x \in W$. 5. $W \cap \text{Blo} \cap \text{Blu} \neq \emptyset \wedge W \cap \text{Blo} \cap \text{Blu}' = \emptyset \wedge W \cap \text{Blo}' \cap \text{Blu} = \emptyset$. $[\exists x\, x \in W \wedge x \in \text{Blo} \wedge x \in \text{Blu}] \wedge [\exists x\, x \in W \wedge x \in \text{Blo} \wedge x \in \text{Blu}'] \wedge [\exists x\, x \in W \wedge x \in \text{Blo}' \wedge x \in \text{Blu}]$. 7. Es $\subseteq$ Ea. $\forall x\, [x$ has two equal sides $\to x$ has two equal angles$]$. 9. $\sim(P \subseteq D)$. 11. $\sim(E \subseteq P')$. 12. $\sim(A \subseteq B)$.

13. $\forall x\, [p(x) \to q(x)] \wedge \sim \forall x\, [q(x) \to p(x)]$. 15. $\exists x\, p(x)$. 17. $a \in P \cup Q$. 19. (4). 21. Some B is C. 22. Valid. 23. Invalid. 24. No conclusion. 25. Invalid! No conclusion [see *Symbolic Logic* by Lewis Carroll (C. L. Dodgson) p. 63]. 26. No kitten with green eyes will play with a gorilla. 29. In Problems 7 and 8, and in Exercise k.

CHAPTER 4

PLANE ANALYTIC GEOMETRY

4–1 Ordered pairs. Only rarely are we concerned with a single object in isolation; usually we are interested in objects related to other objects. In the simplest case we have two objects paired. For example, with each finite set, we may pair the number of its members; with each item in a store, we may pair its price; with each time, we may pair the velocity of a moving body; and so on. For definiteness, let us think of the items in a store and their prices. We number the item and give price in cents, so that item 1 has price 5, item 2 has price 103, and so on. We could make a table showing this information. Each pair, consisting of an item and its price, makes up a set of two members, for example {1, 5}. However, suppose that the number and price of an item were the same, for example, that item 6 has price 6. Then we should have for this item {6, 6} or {6}, which is no longer a pair. Or suppose that item 10 has price 15 and item 15 has price 10. Then we wish to distinguish these two pairs, but {10, 15} = {15, 10}. Accordingly, we see that the simple concept of a pair of numbers is not adequate in this situation. We have to be able to distinguish which member of the set is the item and which the price.

A pair in which we distinguish one of the members as the first and the other (which need not be different) as the second is called an *ordered pair.* The ordered pair whose first member is x and whose second member is y is symbolized by (x, y) and is read "x, y" or "the ordered pair x, y." The essential property of an ordered pair is given by the following axiom.

(1) Ax.
$$[(a, b) = (c, d)] = [a = c \wedge b = d].$$

(a) Show that $[\{a, b\} = \{c, d\}] = [a = c \wedge b = d]$ is not a law. (b) Prove (2). ★(c) We could define (x, y) by $(x, y) = \{\{x, y\}, \{x\}\}$ and then prove (1). Do this.

(2)
$$[(a, b) = (b, a)] = [a = b].$$

In Section 1–2 we associate single numbers with points on an axis in such a way that to each real number there corresponds one and only one point, and to a point one and only one real number. In Section 1–4 and later sections we make use of this one-to-one correspondence to interpret numbers and operations on them in terms of points and vectors. For example, we interpret $b - a$ as the directed distance from a to b and its