Name ______________________

# Midterm Examination: Economics 210A

## November 7, 2012

Answer Question 1 and any 4 of the other 6 questions. Good luck.

1) Let $f$ be a real-valued concave function whose domain is a convex subset of $\Re^n$. Let $g$ be a function from the reals to the reals and define the composite function $h(x) = g(f(x))$.

State whether each of the following claims about the function $h$ is true or false. If true, give a proof, justifying each claim made in your proof. If false, give a counterexample and prove that your counterexample is a counterexample.

*A) If $g$ is a strictly increasing function, then $h$ is a concave function.*

False: Here is a counterexample. Let $f$ be a function of a single variable, $f(x) = x$. Then $f$ is a concave function. Let $g(y) = y^3$. Then $y$ is an increasing function. Then $h(x) = x^3$. But $h$ is not a concave function, as you can verify by taking the second derivative of $h$.

*B) If $g$ is an increasing, concave function, then $h$ is a concave function.*

True: Take any $x$ and $y$ in the domain of $f$. Then since $f$ is a concave function, $f(\lambda x+(1-\lambda)y) \geq \lambda f(x)+(1-\lambda)f(y)$. Since $g$ is a strictly increasing function, it must then be that

$$g\left(f(\lambda x + (1-\lambda)y)\right) \geq g\left(\lambda f(x) + (1-\lambda)f(y)\right).$$

Since $g$ is a concave function, it must be that

$$g\left(\lambda f(x) + (1-\lambda)f(y)\right) \geq \lambda g(f(x)) + (1-\lambda)g\left(f(y)\right).$$

The previous two inequalities imply that

$$g\left(f(\lambda x + (1-\lambda)y)\right) \geq \lambda g(f(x)) + (1-\lambda)g\left(f(y)\right).$$

Therefore since $g\left(f(\lambda x + (1-\lambda)y)\right) = h(\lambda x + (1-\lambda)y$ and $h(x) = g(f(x))$ and $h(y) = g\left(f(y)\right)$, it follows from the previous inequality that $h(\lambda x + (1-\lambda)y \geq \lambda h(x) + (1-\lambda)h(y)$, which means that $h$ is a concave function.

*C) If $g$ is a concave function, then $h$ is a concave function.*

False: Here is a counterexample. Let $f$ be a function with domain the non-negative real numbers and let $f(x) = x^{1/2}$. Let $g(y) = -y$. Then $h(x) = -x^{1/2}$. This is not a concave function, since its second derivative is positive.

*2) Rocky consumes two goods. He prefers any bundle such that $x_1 > 0$ and $x_2 > 1$ to any bundle for which these two inequalities are not satisfied. His* *preferences over bundles such that $x_1 > 0$ and $x_2 > 1$ can be represented by the utility function*

$$u(x_1, x_2) = \ln(x_1 + 1) + \ln(x_2 - 1)$$

*for all $(x_1, x_2)$ .*

*A) For what price-income combinations does Rocky choose positive amounts of both goods?*

*B) Find Rocky's Marshallian demand function.*

*C) Find Rocky's indirect utility function.*

If Rocky consumes positive amounts of both goods, it must be that his marginal rate of substitution is equal to the price ratio of the two goods. This implies that

$$\frac{p_1}{p_2} = \frac{x_2 - 1}{x_1 + 1},$$

which in turn implies that

$$p_1 x_1 + p_1 = p_2 x_2 - p_2.$$

From the budget equation we have $p_1 x_1 + p_2 x_2 = m$. Solving these two equations in two unknowns, we find

$$x_1(p, m) = \frac{m - p_1 - p_2}{2p_1}$$

and

$$x_2(p, m) = \frac{m + p_1 + p_2}{2p_2}.$$

From these equations, we see that he consumers positive amounts of both goods if and only if $m > p_1 + p_2$ with $p_1 > 0$ and $p_2 > 0$.

When $m > p_1 + p_2$, the Marshallian demand function is given by the two equations above. If $p_1 + p_2 \geq m \geq p_2$, then $x_1(p, m) = 0$ and $x_2(p, m) = m/p_2$.

Rocky's indirect utility function is

$$\begin{aligned} v(p, m) &= \ln\left(\frac{m + p_1 - p_2}{2p_1}\right) + \ln\left(\frac{m + p_1 - p_2}{2p_2}\right) \\ &= 2\ln(m + p_1 - p_2) - \ln 2p_1 - \ln 2p_2 \\ &= \ln\left(\frac{m + p_1 - p_2}{4p_1 p_2}\right) \end{aligned} \quad (1)$$

if $p_1 + p_2 < m$.

$$v(p, m) = \ln\left(\frac{m}{p_2} - 1\right) = \ln(m - p_2) - \ln p_2,$$

if $p_1 + p_2 > m > p_2$.

*3) Rocky, from the previous problem, is one consumer in an economy in which there are n people, $i = 1, \dots n$. Person i has a utility function of the form*

$$u_i(x_1, x_2) = A_i \ln(x_1 + b_i) + \ln(x_2 - 1).$$

*What restrictions, if any, do we need to put on the parameters $A_i$ and $b_i$ so that aggregate demands for goods 1 and 2 are determined by prices and the sum of incomes and do not depend on the distribution of income? Relate your answer to the Gorman polar form.*

If Consumer $i$ buys positive amounts of both goods, it must be that

$$\frac{A_i(x_{2i} - 1)}{x_{1i} + b_i} = \frac{p_1}{p_2}$$

or equivalently,

$$p_1 x_{1i} + p_1 b_i = A_i p_2 x_{2i} - A_i p_2.$$

We also have the budget equation

$$p_1 x_{1i} + p_2 x_{2i} = m_i.$$

Solving these two linear equations, we find that if $i$ consumes positive amounts of both goods,

$$x_{1i}(p, m_i) = \left(\frac{A_i}{A_i + 1}\right)\left(\frac{m_i - b_i p_1 - p_2}{p_1}\right) = \left(\frac{A_i}{A_i + 1}\right) m_i + \left(\frac{A_i}{A_i + 1}\right)\left(\frac{-b_i p_1 - p_2}{p_1}\right),$$

and

$$x_{2i}(p, m_i) = \left(\frac{1}{A_i + 1}\right)\left(\frac{m_i + b_i p_1 + p_2}{p_2}\right)) = \left(\frac{1}{A_i + 1}\right) m_i + \left(\frac{1}{A_i + 1}\right)\left(\frac{b_i p_1 + p_2}{p_2}\right).$$

Now aggregate demand for good 1 is

$$\sum_i x_{1i}(p, m_i) = \sum_i \left(\frac{A_i}{A_i + 1}\right) m_i + \sum_i \left(\frac{A_i}{A_i + 1}\right)\left(\frac{-b_i p_1 - p_2}{p_1}\right),$$

If $A_1 = A_2 = \ldots = A_n = A$, then aggregate demand for good 1 is

$$\sum_i x_{1i}(p, m_i) = \left(\frac{A}{A + 1}\right) \sum_i m_i - \frac{A}{A + 1}\left(\sum_i b_i - n\frac{p_2}{p_1}\right).$$

In this case, aggregate demand for good 1 is determined by prices and aggregate income and does not depend on income distribution. If any two $A_i$'s are different, it is easy to verify that income redistribution form a person with higher $A_i$ to one with lower $A_i$ would increase demand for good 1. A very similar argument shows that the same is true for good 2.

So for prices and incomes such that all consumers buy positive amounts of both goods, aggregate demand is determined by prices and aggregate income if and only if the $A_i$'s are all the same. (The $b_i$'s can be different.)

How does this relate to the results about Gorman polar form? The theorem is that aggregate demand depends on prices and aggregate income if and only if indirect utilities can be represented in the form $A(p)m_i + B_i(p)$. Rocky and his friends have utility functions that are monotonic transformations of utilities of the Stone-Geary form. In particular, if you exponentiate their utility functions and take them to the power $1/(1+A_i)$, you see that they have the same preferences as someone with utility function

$$(x_{1i}+b_i)^{\alpha_i}(x_2-1)^{(1-\alpha_i)}$$

where $\alpha_i$ is between 0 and 1. Indirect utility for someone with this utility is

$$\frac{m_i + p_1 b_i - p_2}{p_1^{-\alpha_i} p_2^{-\alpha_i}} = \frac{1}{p_1^{-\alpha_i} p_2^{-\alpha_i}} m_i + \frac{p_1 b_i - p_2}{p_1^{-\alpha_i} p_2^{-\alpha_i}}.$$

If the $A_i$'s are all the same, then the $\alpha_i'$s are all the same and we can write the previous expression as

$$F(p)m_i + G_i(p)$$

where

$$F(p) = \frac{1}{p_1^{-\alpha_i} p_2^{-\alpha_i}}$$

and

$$G_i(p) = \frac{p_1 b_i - p_2}{p_1^{-\alpha_i} p_2^{-\alpha_i}}.$$

Note that when the $A_i$'s are all the same, $F(p)$ is the same for all $i$, but when the $b_i$'s differ, the $G_i$'s differ, which is ok with the Gorman polar form.

*4) A) What restrictions must $\alpha_1$, $\alpha_2$, and $f(y)$ satisfy for the following to be a legitimate indirect utility function? Explain.*

$$v(p_1, p_2, y) = f(y) p_1^{\alpha_1} p_2^{\alpha_2}$$

We know that a function $v(p, m)$ is an indirect utility function for a continuous utility function if and only if $v$ is continuous, homogeneous of degree 0 in $(p, m)$, decreasing in each $p_i$, increasing in $m$, and quasi-convex.

If the above function is decreasing in $p_i$, it must be that $\alpha_i < 0$. If it is homogeneous of degree zero, it must be that $f(ty)(tp_1)^{\alpha_1}(tp_2)^{\alpha_2} = f(ty)t^{\alpha_1+\alpha_2}p_1^{\alpha_1}p_2^{\alpha_2}$ is constant as $t$ changes. This implies that $f(t*1)t^{\alpha_1+\alpha_2}$ is a constant. This in turn implies that $f(y) = Ky^{-(\alpha_1+\alpha_2)}$ for some positive constant $K$. Recalling that $\alpha_i$'s are negative, we see that $f(y)$ is increasing in $y$. It is not hard to show that whenever $\alpha_1 < 0$ and $\alpha_2 < 0$, it must be that $v(p, m)$ is quasi-convex. It can also be shown that if $v(p, m)$ is homogeneous off degree 0 and quasi-convex in $p$ then it is quasi-convex in $(p, m)$.

*B) What restrictions must the functions $w(p_1, p_2)$ and $z(p_1, p_2)$ satisfy for the following to be a legitimate indirect utility function? Explain.*

$$v(p_1, p_2, y) = w(p_1, p_2) + z(p_1, p_2)y$$

The function $z$ must be homogeneous of degree -1 in $p_1$ and $p_2$. The function $w(p_1, p_2)$ must be homogeneous of degree zero and non-increasing in both its arguments. But the only way this can happen is if $w(p_1, p_2)$ is a constant. So it has to be that $v(p_1, p_2, y) = k + z(p_1, p_2)y$ where the function $z$ is homogeneous of degree -1.

5) A consumer has utility function

$$u(x_1, x_2) = \left(x_1^{1/2} + 2x_2^{1/2}\right)^2.$$

A) How is the ratio $x_1/x_2$ in which this consumer consumes the two goods related to the ratio of the prices of goods 1 and 2? }

$$\frac{x_1}{x_2} = \frac{1}{4}\left(\frac{p_2}{p_1}\right)^2$$

*B) What is the elasticity of substitution between the two goods?*

$\sigma = 2$

*C) Find the consumer's Marshallian demands for goods 1 and 2.*

$$x_1 = \frac{p_2}{p_1}\left(\frac{m}{p_2 + 4p_1}\right)$$
$$x_2 = \frac{4p_1}{p_2}\left(\frac{m}{p_2 + 4p_1}\right)$$

You can find this by brute force substitution, but it is a bit easier if you notice that since the utility function is homothetic, the demands must be proportional to income. From the answer to part A, it follows that

$$\frac{p_1x_1}{p_2x_2} = \frac{p_2}{4p_1}$$

From these two facts it follows that

$$\frac{p_1x_1}{m} = \frac{p_2}{p_2 + 4p_1}$$

and

$$\frac{p_2x_2}{m} = \frac{4p_1}{p_2 + 4p_1}$$

Multiply both sides of the first equation by $\frac{m}{p_1}$ to find $x_1$. Multiply both sides of the second equation by $\frac{m}{p_2}$ to find $x_2$.

*D) Find the consumer's indirect utility function.*

$$v(p,m) = \left( \left(\frac{p_2}{p_1}\right)^{1/2} + 2\left(\frac{p_1}{p_2}\right)^{1/2} \right)^2 \left(\frac{m}{p_2 + 4p_1}\right)$$

*6) Define the lexicographic preference ordering on $\Re^n$. Which of the following properties does the lexicographic ordering have? Completeness, transitivity, strict monotonicity, strict convexity, continuity.*

Let $\succ_L$ be the ordering defined as follows. For any two vectors $x = x_1, \ldots, x_n$, and $x' = x'_1, \ldots, x'_n$ let $k$ be the smallest index such that $x_k \neq x'_k$. Then $x \succ_L x'$ if and only if $x_k > x'_k$.

**This ordering is complete** since if $x \neq x'$, then they must differ in at least one coordinate and thus there will be a first coordinate $k$ in which they differ. Since the ordering of true real numbers is complete, either $x_k > x'_k$ or $x'_k > x_k$. Hence if $x \neq x'$, either $x \succ_L x'$ or $x' \succ_L x$.

**This ordering is transitive.** If $x \succ_L y$ and $y \succ_L z$, let $k$ be the first coordinate in which $x$ and $y$ differ and $k'$ be the first coordinate in which $y$ and $z$ differ. Suppose that $k < k'$. Then $z_k = y_k < x_k$ and $k$ is the first coordinate on which $x$ and $z$ differ, so $x \succ z$. Suppose $k' < k$. Then $z_k < y_k = x_k$ and so $x \succ_L z$. Finally if $k' = k$, we have $z_k < y_k < x_k$, so $x \succ_L z$. Therefore it must be that if $x \succ_L y$ and $y \succ_L z$ then $x \succ_L z$.

**This ordering is strictly monotonic**. If $x \geq y$ then either $x \succ y$ or $x = y$. If $x >> y$, then $x \succ y$.

**This ordering is strictly convex.** If $x \succ y$, then $x_k > y_k$ where $k$ is the first component in which they differ. Then $k$ is the first component in which $\lambda x + (1-\lambda)y$ differs. The $k$th component of $\lambda x + (1-\lambda)y$ is $\lambda_k x + (1-\lambda_k)y > y_k$, which implies that $\lambda x + (1-\lambda)y \succ_L y$.

**This ordering is not continuous if $n \geq 2$.** Let $x \succ y$, where $x_1 = y_1$ and $x_2 > y_2$. Then every open neighborhood of $x$ contains a vector $z$ such that $z_1 < x_1 = y_1$. For such a vector $z$, $y \succ z$. Therefore the set of points preferred to $y$ cannot be an open set. This means that $\succ$ is not continuous.

*7) A consumer buys two goods. The more of a good that the consumer buys the higher the price per unit he must pay. (Resale of the good is impossible-think of haircuts and other services–or airplane tickets that require you to show an id. ) The cost of buying $x_1$ units of 1 is $p_1 x_1^2$ and the cost of buying $x_2$ units of good 2 is $p_2 x_2^2$. The consumer's income is $m$. Suppose this consumer's utility function is $u(x_1, x_2) = x_1^a + x_2^a$. For what values of $a$ does this consumer buy positive amounts of both goods? (Hint: You may want to draw a diagram.) Find the "demand functions" $x_i(p_1, p_2, m)$ showing quantities that he will buy given $p_1$, $p_2$, and $m$ and "indirect utility function" $v(p_1, p_2, m)$ , showing his utility for what he buys. Does Roy's law still apply? Justify your answer.*

The consumer's budget equation is $p_1 x_1^2 + p_2 x_2^2 = m$. Notice that the budget set is bounded not buy a straight line, but by a curve that bulges out. Therefore a point of tangency with an indifference curve may be a maximum even if the utility function is not quasi-concave. All we need is that the utility function be "less convex than the budget equation". For example, the thick curve in Figure 1 shows the boundary of the budget set for the case where $p_1 = p_2 = 1/2$. The thin curves are indifference curves for someone with utility function $u(x_1, x_2) = x_1^{1.5} + x_2^{1.5}$. There is a tangency at the point $x_1 = x_2 = 1$ (marked with a dot) and as you can see from the diagram, this tangency is the best point the consumer can afford.

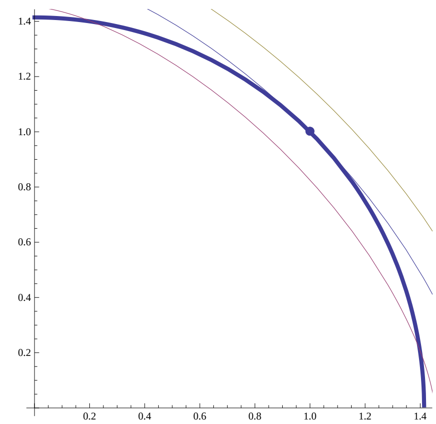


Figure 1: Interior maximum, $\alpha = 1.5$

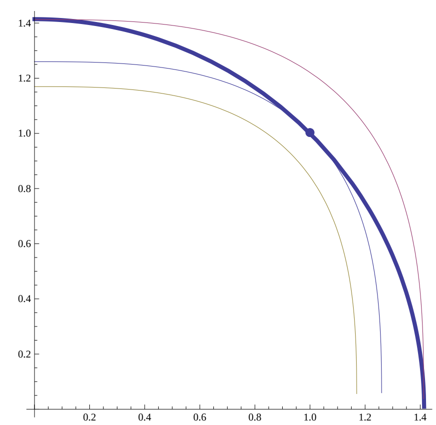


Figure 2: Corner maxima, $\alpha = 3$

In Figure 2 we have $\alpha = 3$ and the indifference curves bow even more sharply than the budget line. Again the budget line is shown as the thick line and the indifference curves as thin lines. There is at tangency at $x_1 = x_2 = 1$, but this time, as we see from the picture, this point is the consumer's worst, not his best point on the boundary of the budget set. His best point on the budget is at either of the extreme points where $x_1 = 0$ or $x_2 = 0$.

The consumer's best point will be interior if $\alpha < 2$ and a boundary point if $\alpha > 2$. One way to see this is to make a change of variables. Let $y_i = x_i^2$. The consumer's maximization problem can be written as Maximize $u^*(y_1, y_2) = y_1^{\alpha/2} + y_2^{\alpha/2}$ subject to $p_1y_1 + p_2y_2$. Now this is a standard consumer choice problem with a linear budget. For this problem, if $\alpha/2 < 1$, $u^*$ is quasi-concave and the consumer will choose an interior point with $y_1 > 0$ and $y_2 > 0$. If $\alpha/2 > 1$, the consumer will always choose a corner solution. Both $x_i$'s will be positive if and only if both $y_i$'s are positive. Therefore there will be an an interior solution if$\alpha \leq 2$ and a corner solution if $\alpha > 2$. (What if $\alpha = 2$?)

You could go ahead and solve for the Marshallian demands, using Lagrangians and brute force. That is not too hard, but lets try to do it an easier way. Let's stick with our change of variables. We can solve for the $y$'s and then solve for $x$'s from the $y$'s. Let $\beta = \alpha/2$. We know that

$$\frac{y_1}{y_2} = \left(\frac{p_1}{p_2}\right)^{-\sigma}$$

where $\sigma = \frac{1}{1-\beta}$. Multiplying both sides of this expression by $p_1/p_2$, we have

$$\frac{p_1y_1}{p_2y_2} = \left(\frac{p_1}{p_2}\right)^{1-\sigma}.$$

We also have from the budget equation,

$$\frac{p_1y_1}{m} + \frac{p_2y_2}{m} = 1$$

From the previous two equations, it follows that

$$\frac{p_1y_1}{m} = \frac{p_1^{1-\sigma}}{p_1^{1-\sigma} + p_2^{1-\sigma}}$$

Therefore

$$y_1(p, m) = \frac{p_1^{-\sigma}}{p_1^{1-\sigma} + p_2^{1-\sigma}} m$$

and

$$y_2(p, m) = \frac{p_2^{-\sigma}}{p_1^{1-\sigma} + p_2^{1-\sigma}} m.$$

Now $x_i(p, m) = \sqrt{y_i(p, m)}$, so we have

$$x_1(p, m) = \sqrt{\frac{p_1^{-\sigma}}{p_1^{1-\sigma} + p_2^{1-\sigma}}} \sqrt{m}$$

and

$$x_2(p, m) = \sqrt{\frac{p_2^{-\sigma}}{p_1^{1-\sigma} + p_2^{1-\sigma}}} \sqrt{m}.$$

The only thing left to do to get the expression back in terms of our original variables is to recall

$$\sigma = \frac{1}{1-\beta} = \frac{1}{1-\frac{\alpha}{2}} = \frac{2}{2-\alpha}.$$

We could substitute $2/(2-\alpha)$ for $\sigma$ in the above expressions and even get a bit of simplification, but we note that it will still be true that the $x_i$'s are proportional to the square root of income.

Does Roy's Law still apply? Before answering this, let's work out the general case found in the Extra Credit portion. I recommend taking a close look at this solution, since it illustrates the workings of the "envelope principle" that finds many uses in economics.

*For Extra Credit: Suppose that there are $n$ goods. A consumer has utility function $u(x)$ and a budget constraint of the form $\sum_i p_i g_i(x_i) = m$, where the $g_i$'s are all strictly increasing functions. Where $x(p, m)$ is the bundle that solves this constrained maximization problem with parameters $p$ and $m$, let $v(p, m) = u(x(p, m))$ be the corresponding "indirect utility function". Show that you can use a rule that is similar to Roy's Law to recover the demand functions (over the range of interior solutions) from the indirect utility function $v$.*

The first order condition for an internal maximum is that

$$\lambda p_i g_i'(x_i) = \frac{\partial u\,(x(p,m))}{\partial x_i} \tag{2}$$

for some $\lambda$ and all $i$. We know that $v(p, m) = u(x(p, m))$. Differentiating both sides of this equation with respect to $p_j$, we have

$$\frac{\partial v(p,m)}{\partial p_j} = \sum_i \frac{\partial u(x(p,m))}{\partial x_i} \frac{\partial x_i(p,m)}{\partial p_j} \tag{3}$$

Substituting from Equation 2 into Equation 3, we have

$$\frac{\partial v(p,m)}{\partial p_j} = \lambda \sum_i p_i g_i'\,(x_i(p,m))\, \frac{\partial x_i(p,m)}{\partial p_j} \tag{4}$$

The budget constraint requires that

$$\sum_i p_i g_i(x_i((p,m)) = m$$

for all $p$ and $m$. Differentiating both sides of the budget constraint with respect to $p_j$, we have

$$\sum_i p_i g_i'\,(x_i(p,m))\, \frac{\partial x_i(p,m)}{\partial p_j} + g_j\,(x(p,m)) = 0 \tag{5}$$

and hence

$$\sum_i p_i g_i'\,(x_i(p,m))\, \frac{\partial x_i(p,m)}{\partial p_j} = -g_j\,(x(p,m)) \tag{6}$$

Substituting from Equation 6 into Equation 4, we have

$$\frac{\partial v(p,m)}{\partial p_j} = -\lambda g_j\left(x(p,m)\right) \tag{7}$$

Now all we have left to do is relate $\lambda$ to the partial of $v$ with respect to $m$. To do this, we differentiate both sides of the equation $v(p,m) = u\left(x(p,m)\right)$ with respect to $m$.

$$\frac{\partial v(p,m)}{\partial m} = \sum_i \frac{\partial u\left(x(p,m)\right)}{\partial x_i}\frac{\partial x_i(p,m)}{\partial m} \tag{8}$$

Substituting from Equation 3 into Equation 8, we have

$$\frac{\partial v(p,m)}{\partial m} = \lambda \sum_i p_i g_i'\left(x(p,m)\right)\frac{\partial x_i(p,m)}{\partial m} \tag{9}$$

Finally, differentiate both sides of the budget equation with respect to $m$. We find that

$$\sum_i p_i g_i'\left(x(p,m)\right)\frac{\partial x_i(p,m)}{\partial m} = 1 \tag{10}$$

It follows from Equations 9 and 10 that

$$\frac{\partial v(p,m)}{\partial m} = \lambda \tag{11}$$

Now from Equations 7 and 11, it follows that

$$g_j\left(x_j(p,m)\right) = \frac{-\frac{\partial v(x(p,m))}{\partial p_j}}{\frac{\partial v(p,m)}{\partial m}} \tag{12}$$

Therefore we have the following analog to Roy's Law:

$$x_j(p,m) = g_j^{-1}\left(\frac{-\frac{\partial v(x(p,m))}{\partial p_j}}{\frac{\partial v(p,m)}{\partial m}}\right) \tag{13}$$

In the special case presented in the regular-credit part of the problem, we had $g_j(x_j) = x_j^2$. Thus we would have

$$x_j^2(p,m) = \frac{-\frac{\partial v(x(p,m))}{\partial p_j}}{\frac{\partial v(p,m)}{\partial m}}$$

and

$$x_j(p,m) = \sqrt{\frac{-\frac{\partial v(x(p,m))}{\partial p_j}}{\frac{\partial v(p,m)}{\partial m}}}.$$