Rooftop Theorem for Concave functions

This theorem asserts that if f is a differentiable concave function of a single variable, then at any point x in the domain of f, the tangent line through the point $(x, f(x))$ lies entirely above the graph of f. You should draw a picture.

Theorem 1. *If f is a continuously differentiable concave function of a single variable, defined on a real interval I, then for all x_1 and x_2 in I,*

$$f(x_1) + (x_2 - x_1)f'(x_1) \geq f(x_2).$$

Geometrically, this theorem says that the tangent line to the graph of f passing through any point $(x_1, f(x_1))$ must lie entirely on or above the graph of f. You should draw a couple of pictures to convince yourself of this geometry.

Proof. Since f is a concave function, it must be that for all x_1 and x_2 in I, and all $t \in [0, 1]$,

$$f((1-t)x_1 + tx_2) \geq (1-t)f(x_1) + tf(x_2). \tag{1}$$

Rearranging terms, we see that Equation 1 is equivalent to

$$f\left(x_1 + t(x_2 - x_1)\right) - f(x_1) \geq t\left(f(x_2) - f(x_1)\right). \tag{2}$$

Dividing both sides of equation 2 by t, we have

$$\frac{f\left(x_1 + t(x_2 - x_1)\right) - f(x_1)}{t} \geq f(x_2) - f(x_1) \tag{3}$$

This implies that

$$(x_2 - x_1)\frac{f\left(x_1 + t(x_2 - x_1)\right) - f(x_1)}{t(x_2 - x_1)} \geq f(x_2) - f(x_1) \tag{4}$$

Then it must be that

$$(x_2 - x_1)\lim_{t \to 0}\frac{f\left(x_1 + t(x_2 - x_1)\right) - f(x_1)}{t(x_2 - x_1)} \geq f(x_2) - f(x_1) \tag{5}$$

But then we have

$$\lim_{t \to 0}\frac{f\left(x_1 + t(x_2 - x_1)\right) - f(x_1)}{t(x_2 - x_1)} = \lim_{h \to 0}\frac{f\left(x_1 + h\right) - f(x_1)}{h} = f'(x_1) \tag{6}$$

It follows that:

$$(x_2 - x_1)f'(x_1) \geq f(x_2) - f(x_1). \tag{7}$$

Rearranging Equation 8, we have the desired result, namely

$$f(x_1) + (x_2 - x_1)f'(x_1) \geq f(x_2). \tag{8}$$

□

Now an easy and important consequence of the Rooftop Theorem is the following.

Theorem 2. *If f is a continuously differentiable function of a single variable, defined on a real interval I, then f is a concave function if and only if $f''(x) \leq 0$ for all $x \in I$.*

One proof of this theorem is to apply Taylor's theorem and the Rooftop theorem. (Hint: Write the exact form of the second order Taylor's expansion.)

Here is another proof. Suppose that f is a concave function. Choose any two points x and y in I such that $x > y$. The Rooftop Theorem implies that $f(x) - f(y) \leq f'(y)(x - y)$ and also $f(y) - f(x) \leq f'(x)(y - x)$. The second inequality is equivalent to $f(x) - f(y) \geq f'(x)(x - y)$. It follows that $f'(x)(x - y) \leq f(x) - f(y) \leq f'(y(x - y)$ and hence that $f'(x) \leq f'(y)$ whenever $x > y$. But this means that f' is a non-increasing function and hence $f''(x) \leq 0$ for all $x \in I$.

A similar argument establishes the converse.