Vector space (aka Linear space) and convex combinations

- A collection of objects called vectors, which can be added together or multiplied by scalars.– Euclidean n-space is an example
- If S is a vector space, a convex combination of two elements $x \in S$ and $y \in S$ of a linear space is an element $\lambda x + (1 - \lambda) y$ of S where $\lambda \in [0, 1]$.
- A set A is said to be a convex set if for all x and y in A, every convex combination of x and y is also in A.

Convex preferences

- The preference relation $\succeq$ on X is defined to be *convex* if for all x and y in X, if $x \succeq y$, then $\lambda x + (1 - \lambda) y \succeq y$ for all $\lambda \in [0, 1]$.
- This is equivalent to the statement that: For all $y \in X$, $\succeq (y)$ is a convex set.

Strictly convex preferences

- The preference relation $\succeq$ is *strictly convex* if for all x and y in X, if $x \neq y$ and $x \geq y$, then $\lambda x + (1 - \lambda)y \succ y$ for all $\lambda \in [0, 1]$. (Every convex combination of two distinct vectors is strictly preferred to at least one of them.)
- The preference relation $\succeq$ is *semi-strictly convex* if for all x and y in X, if $x \succ y$, then $\lambda x + (1 - \lambda)y \succ y$ for all $\lambda \in [0, 1)$.
- Find an example of preferences that are semi-strictly convex, but not strictly convex.

Quasi-concave functions

- A function $f : A \to \Re$ is *quasi-concave* if for $x, y \in A$, $f(x) \geq f(y)$ implies that $f(\lambda x + (1-\lambda)y) \geq f(y)$ for all $\lambda \in [0,1]$.
- A function $f : A \to \Re$ is *strictly quasi-concave* if for $x, y \in A$, $f(x) \geq f(y)$ and $x \neq y$ implies that $f(\lambda x + (1-\lambda)y) > f(y)$ for all $\lambda \in (0,1]$.
- A function $f : A \to \Re$ is *semi-strictly quasi-concave* if for $x, y \in A$, if $f(x) > f(y)$ implies that $f(\lambda x + (1-\lambda)y) > f(y)$ for all $\lambda \in (0,1]$.

Concave functions

- A function $f : A \to \Re$ is *concave* if for $x, y \in A$, $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$ for all $\lambda \in [0,1]$.
- A concave function must be quasi-concave, but the converse is not true... Prove this.
- Rainshed property: The set $S = \{(x,y) | f(x) \geq y\}$ is convex. If x denotes the coordinates on the floor and $f(x)$ the height of the roof above, then S denotes alll of the points under the roof. If this set is convex, the building sheds rain pretty well.
- What do concave functions of single variable look like? What do quasi-concave functions of a single variable look like?