Problem 1.50 in Jehle and Reny is stated in a way that is at best confusing and if taken literally.

The problem is stated as follows:

> *Someone consumes a single good and his indirect utility function is*
>
> $$v(p, y) = G\left(A(p) + \frac{\bar{y}^{\eta} y^{1-\eta}}{1-\eta}\right)$$
>
> *where*
>
> $$A(p) = \int_{p}^{p^0} x(\xi, \bar{y}) d\xi.$$
>
> *and $G(\cdot)$ is a positive monotonic function of one variable.*
>
> *a) Derive the consumer's demand for x and show that it has constant elasticity equal to η.*
>
> *b) Suppose the consumer has an income equal to $\bar{y}$ and the price rises from p to $p' > p$. Argue that the consumer's change in utility caused by this price change can be measured by*
>
> $$-\int_{p}^{p'} x(\xi, \bar{y}) d\xi < 0.$$
>
> *Interpret this measure.*

If we take the statement of the question literally, there is only one commodity. Where p is the price of that commodity and y is income, the budget is simply the scalar equation $px = y$. This implies that $x = y/p$ and if this commodity is desirable, then the function $u(x) = x$ would represent preferences. So the indirect utility function would be a monotone increasing function of y/p. If we want the separable form in the problem, we could write $v(p, y) = G(\ln y - \ln p)$, which means we would need $A(p) = -\ln p$, but instead of $\frac{\bar{y}^{\eta} y^{1-\eta}}{1-\eta}$, we would have something like $\frac{\bar{y}^{\eta} \ln y}{1-\eta}$.

So the single commodity interpretation of the problem doesn't make much sense. Suppose that instead, we assume that the writer of the problem meant that there are two commodities, where good 1 is *numeraire* and that the p is the ratio of the price of good 2 to good 1, while the y is the ratio of income to the price of good 1. Now we get a much more interesting problem. Suppose that the problem is interpreted as:

> *Someone consumes two goods. Let Good 1 be the* numeraire *with price 1, let p be the ratio of the price of good 2 to the price of good 1 and let y be the ratio of income to the price of good 1. The consumer's indirect utility function can be written as*
>
> $$v(1, p, y) = G\left(A(p) + \frac{\bar{y}^{\eta} y^{1-\eta}}{1-\eta}\right)$$

where

$$A(p) = \int_p^{p^0} x(\xi, \bar{y}) d\xi.$$

and $G(\cdot)$ *is a positive monotonic function of one variable.*

a) Derive the consumer's demand for good 2 and show that it has constant elasticity equal to η.

b) Suppose the consumer has an income equal to $\bar{y}$ *and the price rises from* p *to* $p' > p$. *Argue that the consumer's change in utility caused by this price change can be measured by*

$$-\int_p^{p'} x(\xi, \bar{y}) d\xi < 0.$$

Interpret this measure.

To answer a, we apply Roy's identity. Where $D_2(1, p, y)$ is the Marshallian demand for good 2, we have

$$\begin{aligned} D_2(1, p, y) &= -\frac{\partial v(1, p, 6)}{\partial p} \div \frac{\partial v(1, p, y)}{\partial y} \\ &= x(p, \bar{y}) \div \bar{y}^{\eta} y^{-\eta} \\ &= x(p, \bar{y}) \left(\frac{y}{\bar{y}}\right)^{\eta} \end{aligned}$$

This elasticity of demand for good 2 with respect to income is $\frac{d \log D_2(1,p,y)}{d \log y} = \eta$.

To answer b, note that since G is simply a monotone increasing function, indirect utility can also be represented by the function

$$v^*(p, y) = A(p) + \frac{\bar{y}^{\eta} y^{1-\eta}}{1-\eta}.$$

Then

$$v^*(p', \bar{y}) - v^*(p, \bar{y}) = A(p') - A(p) = -\int_p^{p'} x(\xi, \bar{y}) d\xi$$

We note that $D_2(1, p, \bar{y}) = x(p, \bar{y})$ for all p. So the integral in the previous expression is just the change in the area under the demand curve as the price is moved up from p to p'.

Further things to note:

We can show that if $x(p, \bar{y})$ is a decreasing function of p and if $0 < \eta < 1$, then the proposed function $v(1, p, y)$ satisfies all of the necessary conditions for v to be an indirect utility function. (You can prove that $v(1, p, y)$ is convex in p and y by looking at the Hessian. It has zero off-diagonals and positive diagonals. You can also show that if $v(1, p, y)$ is convex in p and y, and v is homogeneous of degree zero, then $v(p_1, p_2, y) = v(1, p_2/p_1, y/p_1)$ must be convex in p_1, p_2, and y.) The other conditions are pretty easy to check.

This means that we know there is some utility function that generates demand functions of the form

$$D_2(1,p,y) = x(p,\bar{y})\left(\frac{y}{\bar{y}}\right)^{\eta}.$$

Even if the direct utility function doesn't have a nice closed-form solution, we have seen what the indirect utility function looks like.

One can generate lots of useful special cases of this form. Just one example: Suppose that $x(\xi,\bar{y}) = A - B\xi$, $\eta = 1/2$, $\bar{y} = 100$ and $p^0 = A/B$. Then

$$\begin{aligned}
v(p,y) &= \int_p^{p^0} (A - B\xi)\, d\xi + \frac{10\sqrt{y}}{1/2} \\
&= A\xi - \frac{B}{2}\xi^2 \Big|_{\xi=p}^{x=A/B} + 20\sqrt{y} \\
&= \frac{A^2}{2B} - \left(Ap - \frac{B}{2}p^2\right) + 20\sqrt{y} \\
&= \frac{B}{2}p^2 - Ap + \frac{A^2}{2B} + 20\sqrt{y}
\end{aligned}$$

Then applying Roy's law, we find that the demand for good 2 is given by

$$D_2(1,p,y) = -\frac{(Bp - A)}{10y^{-1/2}} = (A - Bp)\,\frac{y^{1/2}}{10}.$$

To complete this example, we should also find the demand $D_1(1,p,y)$ for good 1 and make sure that we have an interior solution.

From the budget equation, we see that it must be that $D_1(1,p,y) + pD_2(1,p,y) = y$. Therefore

$$\begin{aligned}
D_1(1,p,y) &= y - pD_2(1,p,y) \\
&= y - (Ap - Bp^2)\frac{y^{1/2}}{10}
\end{aligned}$$

One last thing to notice. For this demand function to be well defined at $(1,p,y)$, we therefore need

$$y - (Ap - Bp^2)\frac{y^{1/2}}{10} \geq 0.$$

This will be the case for large enough y. Just for fun, we can play around and see how big y would need to be.

We have

$$y - (Ap - Bp^2)\frac{y^{1/2}}{10} \geq 0$$

if and only if

$$10\sqrt{y} \geq Ap - Bp^2.$$

Now $Ap - Bp^2$ is maximized when $p = A/2B$ and the maximal value of $Ap - Bp^2$ is $\frac{A^2}{4B}$. So we can be sure that

$$10\sqrt{y} \geq Ap - Bp^2$$

if

$$10\sqrt{y} \geq \frac{A^2}{4B}$$

which is equivalent to

$$y \geq \frac{A^4}{1600B^2}.$$