

0.1 Production functions with a single output

0.1.1 Homothetic and Homogeneous Production Functions

Homothetic production functions have the property that $f(x) = f(y)$ implies $f(\lambda x) = f(\lambda y)$.

Homogeneous production functions have the property that $f(\lambda x) = \lambda^k f(x)$ for some k . Homogeneity of degree one is constant returns to scale. Homogeneous implies homothetic, but not conversely. Example $f(x_1, x_2) = x_1 x_2 + 1$ is homothetic, but not homogeneous. Draw a picture.

Separable production function. Example $f(x_1, x_2, x_3) = F(g(x_1, x_2), x_3)$. Suppose factors 1 and 2 are skilled and unskilled labor and factor 3 is capital. What does separability mean?

Additively separable production function. $f(x_1, \dots, x_n) = F(\sum_i g(x_i))$

Abram Bergson's theorem, A function is additively separable and homothetic if and only if it is of one of the following two forms: $F(\sum_i a_i x_i^\rho)$ or $F(\sum_i a_i \ln x_i)$.

Corollary It is additively separable and homogeneous of degree k iff it is of the form

$$\left(\sum_i a_i x_i^\rho \right)^{k/\rho}$$

or of the form

$$\prod_i x_i^{a_i}$$

where $\sum_i a_i = k$.

The first of these cases is known as a constant elasticity of substitution production function (ces) and the second as a Cobb-Douglas production function.

Why is it called ces? The elasticity of substitution between two factors is defined to be the absolute value of the percent change in the ratio in which the factors are used that results from a one percent change in the ratio of the wages of the two factors. That is, it is the partial derivative of the log of the ratio of factor inputs with respect to the log of the ratio of the factor prices. (elasticity of this with respect to that is always -d log this /d log that.)

For a cost-minimizing firm, the ratio of the marginal products of two factors that are both used must equal the ratio of their wages. In this case the marginal rate of substitution between factors i and j equals the wage ratio when

$$\frac{a_i x_i^{\rho-1}}{a_j x_j^{\rho-1}} = \frac{w_i}{w_j}.$$

Take logs of both sides of this equation and you have

$$\ln \frac{w_i}{w_j} = \ln \frac{a_i}{a_j} + (\rho - 1) \ln \frac{x_i}{x_j}.$$

Rearrange this equation to write

$$\ln \frac{x_i}{x_j} = \frac{1}{(\rho - 1)} \ln \frac{a_i}{a_j} - \frac{1}{(\rho - 1)} \ln \frac{w_i}{w_j}$$

Therefore the elasticity of substitution between factors i and j is $\sigma = 1/(1 - \rho)$, which is a constant. Hence *ces*.

For the Cobb-Douglas utility, the elasticity of substitution between any two factors is 1. (Prove this yourself.) It can be proved that the Cobb-Douglas utility function is the limit as $\rho \rightarrow 0$ of the *ces* utility functions with parameter ρ .

Empirical economists find the *ces* form especially useful, since if they have data on changing factor prices and factor utilizations, they can run a log-log regression of factor utilization ratios on factor price ratios to estimate the elasticity of substitution as the slope and the ratio a_1/a_2 as the intercept of their regression.

If the production function is quasi-concave as well as *ces*, then it must be that $\rho \leq 1$. Notice that ρ can be negative. Notice also that if $\rho > 1$, equalizing the ratio of marginal products to the wage ratio does not minimize cost of production. Look at the example where $\rho = 2$. How do you minimize production costs?

It is instructive to graph the function $\sigma = f(\rho) = 1/(1 - \rho)$. Notice that if $\rho < 0$, then $0 < \sigma < 1$ and that $\lim_{\rho \rightarrow -\infty} f(\rho) = 0$. Notice also that if $0 < \rho < 1$, then $\sigma > 1$ and that the limit of $f(\rho)$ as ρ approaches 1 from below is ∞ . Finally, note that if $\rho > 1$, then $\sigma < 0$, with the limit of $f(\rho)$ as ρ approaches 1 from above equal to $-\infty$ and $\lim_{\rho \rightarrow \infty} f(\rho) = 0$.

The functional share of a factor is defined as its quantity times its wage divided by the value of output. Note that in the *ces* case, an increase in a factor's wage, holding other wages constant, will increase or decrease the factor's functional share depending on whether the elasticity of substitution is less than or greater than one.

0.1.2 Cost Function for C.E.S Production Function

It turns out that the cost function for a *c.e.s* production function is also of the *c.e.s.* form and if the production function has elasticity of substitution σ , the corresponding cost function has elasticity of substitution $1/\sigma$.

I leave the Cobb-Douglas case to you.

Let's simplify the *ces* to the case of two factors, constant returns to scale and where $a_1 = a_2 = 1$. Once we see how to find the cost function for this case, a couple of cheap tricks allow one to handle nonconstant returns differing a_i 's. Extending to more than two goods is also easily done and yields no surprises.

Where $\sigma = \frac{1}{1-\rho}$, we have $\rho = \frac{\sigma-1}{\sigma}$. We will focus on the class of quasi-concave *c.e.s.* functions, for which $\rho \leq 1$ and hence $\sigma \geq 0$. It turns out to be slightly more convenient to write the utility function by substituting σ for ρ .

Let

$$f(x) = \left(x_1^{\frac{\sigma-1}{\sigma}} + x_2^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}. \quad (1)$$

Let us find $c(w, 1)$, the cost of producing one unit in the cheapest possible way when wages are $w = (w_1, w_2)$. To do so, we begin by finding the conditional factor demand functions $x_1(w, 1)$ and $x_2(w, 1)$, which are the amounts of the

two factors used to produce one unit of product in the cheapest possible way when the wage vector is w . Having found the conditional factor demands, we can then solve for $c(w, 1) = w_1x_1(w, 1) + w_2x_2(w, 1)$.

Calculation shows that the ratio of marginal products of factors 1 and 2 is

$$\frac{f_1(x_1, x_2)}{f_2(x_1, x_2)} = \left(\frac{x_1}{x_2}\right)^{-\frac{1}{\sigma}} \quad (2)$$

Cost minimization requires that the ratio of marginal products equals the wage ratio. Therefore if $x_1 = x_1(w, 1)$ and $x_2 = x_2(w, 1)$, then

$$\left(\frac{x_1}{x_2}\right)^{-\frac{1}{\sigma}} = \frac{w_1}{w_2}. \quad (3)$$

From Equation 3 it follows that

$$x_1w_1^\sigma = x_2w_2^\sigma = K(w_1, w_2) \quad (4)$$

for some function $K(w_1, w_2)$. Therefore

$$x_1(w_1, w_2, 1) = K(w_1, w_2)w_1^{-\sigma} \text{ and } x_2(w_1, w_2, 1) = K(w_1, w_2)w_2^{-\sigma} \quad (5)$$

It must be that the input vector $(x_1(w, 1), x_2(w, 1))$, is sufficient to produce one unit of output. That is

$$x_1^{\frac{\sigma-1}{\sigma}} + x_2^{\frac{\sigma-1}{\sigma}} = 1 \quad (6)$$

Substituting the expressions for x_1 and x_2 from Equations 5 into Equation 6 and arranging terms, we find that

$$K(w_1, w_2) = (w_1^{1-\sigma} + w_2^{1-\sigma})^{\frac{\sigma}{1-\sigma}} \quad (7)$$

Now

$$\begin{aligned} c(w, 1) &= w_1x_1(w, 1) + w_2x_2(w, 1) \\ &= w_1K(w_1, w_2)w_1^{-\sigma} + w_2K(w_1, w_2)w_2^{-\sigma} \\ &= (w_1^{1-\sigma} + w_2^{1-\sigma})^{1+\frac{\sigma}{1-\sigma}} \end{aligned} \quad (8)$$

$$= (w_1^{1-\sigma} + w_2^{1-\sigma})^{\frac{1}{1-\sigma}} \quad (9)$$

Now we can show the remarkable fact that the elasticity of substitution of the cost function is just the inverse of that of the production function. Remember that when a c.e.s. function is of the form

$$(x_1^\rho + x_2^\rho)^{\frac{1}{\rho}}$$

its elasticity of substitution is $1/(1-\rho)$. In the case of our production function, we set $\rho = (\sigma-1)/\sigma$ and simple algebra shows that $\sigma = 1/(1-\rho)$. Our cost function turns out to be of the form of equation 0.1.2 with $\rho = 1-\sigma$ where σ is the elasticity of substitution of the production function. Now the elasticity of substitution of this cost function is $1/(1-(1-\sigma))$, which is just $1/\sigma$.