Lecture Notes on Separable Preferences

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When applied economists want to focus attention on a single commodity or on one commodity group, they often find it convenient to work with a two-commodity model, where the two commodities are the one that they plan to study and a composite commodity called "other goods". For example, someone interested in the economics of nutrition may wish to work with a model where one commodity is an aggregate commodity "food" and the other is "money left over for other goods." To do so, they need to determine which goods are foods and which are not and then define the quantity of the aggregate commodity, food, as some function of the quantities of each of the food goods.

In the standard economic model of intertemporal choice model of intertemporal choice, commodities are distinguished not only by their physical attributes, but also by the date at which they are consumed. In this model, if there are T time periods, and n undated commodities, then the total number of dated commodities is nT. Macroeconomic studies that focus on savings and investment decisions often assume that there is just one "aggregate good" consumed in each time period and that the only non-trivial consumer decisions concern the time path of consumption of this single good.

In these examples, and in many other applications of economics, the tactic of reducing the number of commodities by aggregation can make difficult problems much more manageable. In general, such simplifications can only be purchased at the cost of realism. Here we examine the "separability conditions" that must hold if this aggregation is legitimate.

To pursue the food example, suppose that the set of n commodities can be partitioned into two groups, foods and non-foods. Let there be m food commodities. We will write commodity vectors in the form $x = (x_F, x_{\sim F})$ where x_F is a vector listing quantities of each of the food goods and $x_{\sim F}$ is a vector quantities of each of the non-food goods. If we are going to be able to aggregate, it must be that consumers have preferences representable by utility functions u of the form

$$u\left(x_F, x_{\sim F}\right) = U^*\left(f(x_F), x_{\sim F}\right)$$

f is a real-valued function of m variables and where U^* is a strictly increasing function of its first argument. The function f is then a measure of the amount of the aggregate commodity food. Notice that the function u is a function of n variables, while U^* is an function of only $n-m+1$ variables, which denote quantities of each of the non-food variables and a quantity of the food aggregate f.

Economists' standard general model of intertemporal choice, is based on the use of dated commodities. Thus if there are n "undated commodities," and T time periods, we define x_{it} to be the quantity of commodity i consumed in period t. We define x_t to be the n-vector of commodities consumed in period t and we define $x = (x_1, \dots, x_T)$ to be the nT-vector listing consumption of each good in each period. This is sometimes called a time profile of consumption. We assume that individuals have preferences over time profiles of consumption that are representable by a utility function $u(x_1, \dots, x_T)$ where each x_i is an n-vector. Suppose that u takes the special form

$$u(x_1, \dots, x_T) = U^* \left(f_1(x_1), \dots, f_T(x_T)\right)$$

where each f_t is a real-valued function of n variables. Then U^* is a function of T variables, each of which is an aggregate of consumption in a single period.

Preference relations and separability

We can express these ideas a little more formally in terms of preferences and consumption sets. Let M be a subset of the commodity set $\{1, \dots, n\}$ with $m < n$ members and let $\sim M$ be the set of commodities not in M. Let the consumption set S be the Cartesian product $S_M \times S_{\sim M}$ of possible consumption bundles of the goods in M and of goods in $\sim M$.[1] Where $x \in S$, we write $x = (x_M, x_{\sim M})$ where x_M is an m vector listing a quantity of each good in M and $x_{\sim M}$ lists a quantity of each good in $\sim M$.

Definition 1. *Preferences R are separable on M if whenever it is true that*

$$(x_M, x_{\sim M})\, R\, (x'_M, x_{\sim M})$$

for some $x_{\sim M}$, it must also be that

$$(x_M, x'_{\sim M})\, R\, (x'_M, x'_{\sim M})$$

for all $x'_{\sim M} \in S_{\sim M}$.

[1] The assumption that S is the Cartesian product of S_M and $S_{\sim M}$ rules out the possibility that the possible consumptions in group M depend on what is consumed in $\sim M$ or vice versa.

In words, the separability condition says that if you like M-bundle, x_M, better than the M-bundle, x'_M, when each M-bundle is accompanied by the non-M bundle $x_{\sim M}$, then you will also like M-bundle x_M better than x'_M if each M-bundle is accompanied by any other bundle from the non-M group. Be sure to notice that in each of these comparisons with differing M-bundles, the non-M bundle is held constant.

Let consider an example in which preferences are not separable. There are 3 commodities, cars, bicycles, and gasoline. Let x_1 be the number of cars that a consumer consumes in a week, x_2 the number of bicycles, and x_3 the number of gallons of gasoline. Suppose that this consumer prefers to drive to work rather than ride her bicycle and that 10 gallons of gasoline will suffice to get her car to work every day of the week. Thus she prefers $(1,0,10)$ to $(0,1,10)$. If her preferences are separable between the commodity group $\{1,2\}$ and commodity 3, then she must also prefer $(1,0,0)$ to $(0,1,0)$. But unless she prefers sitting in a stopped car to commuting by bicycle, this latter preference does not seem likely. In this example, the third good, gasoline is a complement for good 1, cars, but not for good 2, bicycles. As a result, the commodity group $\{1,2\}$ is not separable from the rest of the commodity bundle.

The main theorem that relates separable preferences to the structure of utility functions is:

Theorem 1. *If there are n commodities and preferences are represented by a utility function $u(x)$ with $M \subset \{1,\dots,n\}$, then preferences are separable on M if and only if there exists a real valued aggregator function $f : S_M \to R$ such that $u(x_M, x_{\sim M}) = U^*(f(x_M), x_{\sim M})$ where $U^* : R^{n-m+1} \to R$ is a function of $n-m+1$ variables that is increasing in its first argument.*

Proof of Theorem 1:

First show that if utility has this form, then preferences are separable on M. Note that if $(x_M, x_{\sim M}) \, R \, (x'_M, x_{\sim M})$, then $U^*(f(x_M), x_{\sim M}) \geq U^*(f(x'_M), x_{\sim M})$. Since U^* is an increasing function of its first argument and since the same $x_{\sim M}$ appears on both sides of the inequality, it must be that $f(x_M) \geq f(x'_M)$. But if $f(x_M) \geq f(x'_M)$, then it must be that $U^*(f(x_M), x'_{\sim M}) \geq U^*(f(x'_M), x'_{\sim M})$ for all $x'_{\sim M} \in S_{\sim M}$. Therefore $(x_M, x'_{\sim M}) \, R \, (x'_M, x'_{\sim M})$ for all $x'_{\sim M} \in S_{\sim M}$, which means that R is separable on M.

Conversely, suppose that preferences are separable on M. Select any $\bar{x}_{\sim M} \in S_{\sim M}$. (It doesn't matter which one, just pick one.) Define $f(x_M) =$

$u(x_M, \bar{x}_{\sim M})$. For all $(x_M, x_{\sim M})$ define $U^*(f(x_M), x_{\sim M}) = u(x_M, x_{\sim M})$. Now this definition is legitimate if and only if for any two vectors x_M and x'_M of M-commodities such that $f(x_M) = f(x'_M)$ and for any $x_{\sim M} \in S_{\sim M}$, it must be that $u(x_M, x_{\sim M}) = u(x_{M}., x_{\sim M})$ (This definition would be illegitimate if there were some x_M and x'_M such that $f(x_M) = f(x'_M) = y$ but $u(x_M, x_{\sim M}) \neq u(x'_M, x_{\sim M})$, because then we would have two conflicting definitions for $U^*(y, x_{\sim M})$.) It is immediate from the definition of separability and the definition of the function f that if $f(x_M) = f(x'_M)$, then $u(x_M, x_{\sim M}) = u(x'_M, x_{\sim M})$ for all $x_{\sim M} \in S_{\sim M}$. All that remains to be checked is that U must be an increasing function of f. This should be easy for the reader to verify. □

Exercise 1. *Suppose that there are four commodities and that utility can be written in the form,* $U(x_1, x_2, x_3, x_4) = U^*(f(x_1, x_2), x_3, x_4)$ *where the functions* U^* *and* f *are differentiable.*

a) Show that the marginal rate of substitution between goods 1 and 2 is independent of the amount of good 3.

b) Construct an example of a utility function of this form such that the marginal rate of substitution between goods 3 and 4 depends on the amounts of goods 1 and 2.

The notion of separability extends in the obvious way to allow more than one aggregate commodity. Suppose, for example, that there are six commodities and that that utility can be represented in the form

$$U(x_1, \ldots, x_6) = U^*(f_1(x_1, x_2), f_2(x_3, x_4, x_5), x_6).$$

In this case we say that preferences are separable on the commodity groups $\{1, 2\}$ and $\{3, 4, 5\}$. (We could also say that preferences are separable on the singleton commodity group $\{6\}$.)

Exercise 2. *Where utility functions are of the functional form*

$$U(x_1, \ldots, x_6) = U^*(f_1(x_1, x_2), f_2(x_3, x_4, x_5), x_6)$$

a) show that the marginal rate of substitution between goods 3 and 4 is independent of the quantities of goods 1 and 2.

b) Construct an example of a utility of this form where the marginal rate of substitution between goods 3 and 6 depends on the quantity of good 1.

For the next two exercises, suppose that there are 3 time periods and two ordinary commodities, apples and bananas, and consumption of apples and bananas in period t are denoted by x_{at} and x_{bt}.

Exercise 3. *Where utilities of time profiles are of the functional form*

$$U\left(f_1(x_{a1}, x_{b1}), f_2(x_{a2}, x_{b2}), f_3(x_{a3}, x_{b3})\right)$$

a) show that a consumer's marginal rate of substitution between apples in period 2 and bananas in period 2 is independent of the amount of apples consumed in period 1.

b) construct an example of a utility function of this type where the consumer's marginal rate of substitution between apples in period 2 and apples in period 3 depends on consumption of apples in period 1.

One can also produce interesting nested structures of separable preferences

Exercise 4. *Where utility of time profiles of apple and banana consumption can be represented in the form*

$$U\left(f_1(x_{a1}, x_{b1}), v\left(f_2(x_{a2}, x_{b2}), f_3(x_{a3}, x_{b3})\right)\right)$$

a) show that the marginal rate of substitution between apples in period 2 and apples in period 3 does not depend on the quantity of apples in period 1.

b) construct an example of a utility function of this form where the marginal rate of substitution goods apples in period 1 and and apples in period 2 depends on the quantity of apples in period 3.

Exercise 5. *Consider teams of males and females who play mixed doubles tennis matches. Suppose that it is possible to assign numbers x_i to each female i and y_i to each male i in such a way that there is a real-valued function f of four variables, such that a team consisting of male i and female i will defeat a team consisting of male j and female j if and only if $f(x_i, y_i, x_j, y_j) > 0$. Explain why it might not be reasonable to assume that there exist real-valued functions F and g of two variables such that the team of i's would beat the team of j's if and only if $F(g(x_i, y_i), g(x_j, y_j)) > 0$? If such functions exist, how would you interpret them?*

Exercise 6. *A consumer has utility function*

$$U(x_1, x_2, x_3) = 2x_1^{1/2} + 2x_2^{1/2} + x_3.$$

Solve for this consumer's demand for each of the three goods as a function of prices (p_1, p_2, p_3) *and income* m.

Where there is separability, consumer choice problems can be simplified by recasting them as two-stage problems. For example, suppose that Sarah's utility function can be written as $U(x_1, \ldots, x_n) = F(f_C(x_1, ..., x_k)x_{k+1}, \ldots, x_n)$ where the first k commodities are items of clothing. Suppose that Sarah faces a budget constraint $\sum_{i=1}^n p_i x_i = m$. We can decompose the problem into two choices. For any given clothes budget how should she spend it? Knowing how she would spend her clothes budget, how much money should she allocate to clothes and how much to each of the other goods. Given the price vector for clothing, $p_C = (p_1, \ldots, p_k)$ and clothes budget y, she will choose a clothing bundle $x_C = (x_1, \ldots, x_k)$ that maximizes $f_C(x_1, \ldots, x_k)$ subject to $\sum_{i=1}^k p_i x_i = y$. Her indirect utility for clothes $v_C(p_C, y)$ is then defined as the maximum value of f_C that she can achieve with this budget. Now Sarah's choice of how much to spend on clothing can be expressed as choosing y to maximize $F\left(v_C(p_C, y), x_{k+1} \ldots, x_n\right)$ subject to the budget constraint $y + \sum_{i=k+1}^n p_i x_i = m$.

Exercise 7. *A consumer has utility function*

$$U(x_1, x_2, x_3, x_4) = 2x_1^{1/2} + 2x_2^{1/2} + x_3^{1/2} x_4^{1/2}.$$

a) If the prices of goods 3 and 4 are p_3 *and* p_4*, and a consumer spends a total of* y *on good 3 and 4, how much good 3 and how much good 4 should he buy?*

b) If the consumer spends y *on goods 3 and 4, and the prices of good 1 and good 2 are* p_1 *and* p_2*, how much good 1 should he buy? how much good 2?*

c) Write an expression for this consumer's utility if she spends her budget to maximize her utility given that she spends total of y *on goods 3 and 4.*

d) Now find the choice of y *that leads to the highest utility and the resulting quantities of goods 1,2, 3, and 4.*

Additively separable preferences

Suppose that there are n commodities and that preferences can be represented in the additive form

$$u(x_1, \ldots, x_n) = \sum_{i=1}^{n} v_i(x_i)$$

where the functions v_i are real valued functions of a single real variable. If this function is differentiable, then the marginal rate of substitution between any two commodities, j and k is independent of the quantities of any other goods since it is equal to the ratio of derivatives $v_j'(x_j)/v_k'(x_k)$. More generally, notice that if preferences are representable in this additive form, then for any subset M of the set of commodities, preferences must be separable on the set M of commodities and on its complement $\sim M$. If preferences can be represented by a utility function of this additive form, we say that preferences are "additively separable on all commodities".

When are preferences additively separable?

The most useful necessary and sufficient condition for preferences to be additively separable is that every subset of the set of all commodities is separable. The proofs that I know of for this proposition are a bit more elaborate than seems appropriate here. A somewhat more general version of this theorem can be found in a paper by Gerard Debreu [1]. Debreu's paper seems to be the first satisfactorily general solution to this problem. Other proofs can be found in [5] and [2]

Theorem 2. *Assume that preferences are representable by a utility function and that there are at least three preference-relevant commodities (where commodity i is said to be preference-relevant if there exist at least two commodity bundles x and y that differ only in the amount of commodity i and such that x is preferred to y). Then preferences are representable by an additively separable utility function if and only if every nonempty subset M of the set of commodities is separable.*

Additive separability with two goods

You might wonder whether it is always possible to write an additively separable utility function in case there are only two goods. The answer is no, and I will show you a counterexample in a minute.

If there are two goods and if preferences can be represented by a utility function of the form $U(x_1, x_2) = v_1(x_1) + v_2(x_2)$, then the following "double cancellation condition" must hold. If $(x_1, x_2)R(y_1, y_2)$ and $(y_1, z_2)R(z_1, x_2)$, then $(x_1, z_2)R(z_1, y_2$. To see that the double cancellation property is a necessary condition for additive separability, note that if $(x_1, x_2)R(y_1, y_2)$ and $(y_1, z_2)R(z_1, x_2)$, then

$$v_1(x_1) + v_2(x_2) \geq v_1(y_1) + v_2(y_2) \text{ and}$$

$$v_1(y_1) + v_2(z_2) \geq v_1(z_1) + v_2(x_2).$$

Adding these two inequalities and cancelling the terms that appear on both sides of the inequality, we have

$$v_1(x_1) + v_2(z_2) \geq v_1(z_1) + v_2(y_2),$$

which means that $(x_1, z_2)R(z_1, y_2)$.

Debreu [1] showed that the double cancellation condition is both necessary and sufficient for preferences to be additively separable when there are only two goods.

Theorem 3. *Assume that preferences are representable by a utility function. If there are two commodities, then preferences are representable by an additively separable utility function if and only if the double cancellation condition holds.*

It is not always easy to discern at first glance whether a given utility function can be converted by a monotonic transformation into additively separable form. Consider the utility function $U(x_1, x_2) = 1 + x_1 + x_2 + x_1 x_2$. In this form it is not additively separable. But note that $x_1 + x_2 + x_1 x_2 = (1+x_1)(1+x_2)$. The function $V(x_1, x_2) = \ln U(x_1, x_2) = \ln(1+x_1) + \ln(1+x_2)$ is a monotonic transformation of U and is additively separable. Therefore the preferences represented by U are additively separable.

On the other hand, consider preferences represented by the utility function $U(x_1, x_2) = x_1 + x_2 + x_1 x_2^2$. These preferences can not be represented by an additively separable utility function. How do we know this? We can show that these preferences violate the double cancellation condition. To show this, note that $U(1, 1) = U(3, 0) = 3$ and that $U(3, 2) = U(8, 1) = 17$. The double cancellation condition requires that $U(1, 2) = U(8, 0)$. But $U(1, 2) = 7$ and

$U(8,0) = 8$ and $8 \neq 7$. So these preferences violate the double cancellation condition and hence cannot be additively separable.

It is sometimes not easy to see whether a given utility function can be monotonically transformed to additively separable form. Checking that the double cancellation conditions are satisfied everywhere may take forever. Fortunately there is a fairly easy calculus test. We leave this as an exercise.

Exercise 8. *Show that if $U(x_1, x_2) = F(v_1(x_1) + v_2(x_2))$ for some monotonically increasing function F, then it must be that the log of the ratio of the two partial derivatives of U is the sum of two functions, one of which depends only on x_1 and one of which depends only on x_2.*

Cardinality and affine transformations

Additively separable representations turn out to be "unique up to affine transformations."

Theorem 4. *If preferences are continuous on R^n, and representable by an additively separable utility function $U = \sum_{i=1}^n u_i(x_i)$ such that $u_i(x_i)$ is not constant, then if $V = \sum_{i=1}^n v_i(x_i)$ represents the same preferences as U, it must be that for some real numbers $a > 0$ and b_i, $v_i(x_i) = au_i(x) + b_i$ for all $i = 1, \ldots, n$. In this case, $U(x) = aV(x) + b$, where $b = \sum_i b_i$.*

The proof of this proposition is easy if we add the assumptions that U and V are differentiable and strictly increasing in all goods. We show this proof here.

Proof. If $U = \sum_{i=1}^n u_i(x_i)$ and $V = \sum_{i=1}^n v_i(x_i)$ represent the same preferences and are differentiable, then

$$\sum_{i=1}^n v_i(x_i) = F\left(\sum_{i=1}^n u_i(x_i)\right)$$

for some strictly increasing function F. Differentiating both sides of this equation with respect to x_i, we have

$$v_i'(x_i) = F'\left(\sum_{i=1}^n u_i(x_i)\right) u_i'(x_i)$$

for all $x \in R^n$. Differentiate both sides of this identity with respect to x_j where $j \neq i$. This gives us the expression

$$0 = F''\left(\sum_{i=1}^{n} u_i(x_i)\right) u_i'(x_i) u_j'(x_j).$$

Since $u_i'(x_i) > 0$ and $u_j'(x_j) > 0$, it must be that $F''(u) = 0$ for all u in the range of the function U. Solving this differential equation, we have $F'(u) = a$ for some constant a, and hence for all i, $F(u(x_i)) = ax_i + b_i$ for some constants a and b_i.

$\square$

There is an old debate in economics about whether utility functions are "cardinal" or "ordinal". The ordinalist position is that utility functions are operationally meaningful only "up to arbitrary monotonic transformations." If all that economists are able to determine are ordinal preferences, then any two utility functions that represent the same ordinal preferences are equally suitable for describing preferences and a unit of utility has no operational meaning.

For many applications, it would be helpful to have a more complete measurement. Suppose for example, that you wanted to compare the happiness of one person with that of another. It would be handy to have a utility functions $U_i(\cdot)$ and $U_j(\cdot)$ for persons i and j such that you could say that person i is happier consuming x_i than person j is consuming x_j if $U_i(x_i) > U_j(x_j)$. But of course if utility is only unique up to monotonic transformations, one can choose different representations for i and j to make the answer to the "who is happier" question come out either way. 0 You have probably heard the saying "You can't make interpersonal welfare comparisons without cardinal utility." or maybe just plain, "You can't make interpersonal welfare comparisons." What can we make of such statements? Let's start with the second one. Anybody who has ever been a parent of more than one child has plenty of experience with making tradeoffs between the happiness of one person and another. Indeed, anyone who cares at all about the welfare of other human beings has to make some tradeoffs between different people's happiness. If one's preferences about the well being of others are transitive and complete, then one must have some kind of a utility function defined over the wellbeing of more than one person.

We may want to make some assumptions about these preferences. For example, you might assume that people have well defined utility functions over their own consumption bundles and also have "benevolent" inclinations toward others in their family so that if preferences of person j are represented by the function $u_j(x_j)$ where x_j is j's consumption bundle, then preferences of i might be represented by a utility function U of the separable form

$$U\left(u_1(x_1), \ldots, u_n(x_n)\right)$$

where U is an increasing function of each family member's utility for his or her own consumption. The form of the function U would of course depend on the particular utility representation chosen for each of the u_i's.

We might want to make the stronger assumption that U can be represented in additive form:

$$U\left(u_1(x_1), \ldots, u_n(x_n)\right) = \sum_{j=1}^{n} u_j(x_j)$$

where each u_j represents preferences of person j. Such a representation will be possible, if person i's preferences about the consumptions of all family members satisfy the conditions for additive separability. In particular, this requires a bunch of conditions like i's marginal rate of substitution between bananas for George and bananas for Hazel is independent of how happy Isobel is.

We note that if we have an additively separable representation of i's preferences over the utilities members, then this representation is "cardinal" in the sense that any other additively separable representation would simply be an affine transformation of the original one.

We also see that it wouldn't have to be that one's preferences over the utilities of others are additively separable. So cardinality of utility in the sense of uniqueness up to affine transformations is not a necessary condition for the ability to make interpersonal comparisons.

Even if we are not interested in comparing one person's utility to that of another, it would still be nice to be able to say things like Lucy cares more about the difference between bundle w and bundle x than she cares about the difference between bundle y and bundle z. To do this, we would need a utility function such that we could make meaningful statements of the form: If $u(w) - u(x) > u(y) - u(z)$ then Lucy cares more about the difference between w and x than she cares about the difference between y and z. If we can do arbitrary monotonic transformations on utility, then we can reverse the ruling on which difference is bigger by taking a monotone transformation.

For example, suppose that there are two goods and $u(x_1, x_2) = x_1 + x_2$. Consider the four bundles $w = (1.1, 1)$, $x = (0, 0)$, $y = (3, 3)$ and $x = (2, 2)$. Then $u(w) - u(x) = 2.1 > u(y) - u(z) = 2$. The utility function $v(x_1, x_2) = (x_1 + x_2)^2$ represents the same preferences as u, but $v(w) - v(x) = 4.41 < v(y) - v(z) = 7$.

On the other hand, suppose that the only utility transformations that we are willing to do are affine transformations, where we say that u is an affine transformation of v if $u(x) = av(x) + b$ for some positive number a and some real number b, then the ordering of utility differences is preserved under "admissable transforations."

Additively separable and homothetic preferences

The following result is known as Bergson's theorem (named after economist Abram Bergson (1914-2003) who is well known for work on social welfare functions and for studies of the Soviet economy.) This result tells us that if preferences are homothetic as well as additively separable, then they must belong to a very restricted (and very convenient) class of functions.

Theorem 5. *If preferences are representable by a continuous utility function, then they are additively separable and homothetic if and only they are representable by a utility function of one of these two forms:*

$$u(x) = \sum_i a_i x_i^b$$

or

$$u(x) = \sum_i a_i \ln x_i$$

A proof of this theorem can be found in Katzner [3]. Here is an outline of his proof in case you are curious.

Proof. If preferences are additively separable, then they can be represented by a utility function of the form

$$U(x_1, \ldots, x_n) = \sum_{i=1}^{n} u_i(x_i).$$

If preferences are also homothetic, then the marginal rate of substitution between i and j is not changed if the quantities of all goods are multiplied by a constant t. That is

$$\frac{u_i'(tx_i)}{u_j'(tx_j)} = \frac{u_i'(x_i)}{u_j'(x_j)}$$

for all $t > 0$. Differentiate both sides of this identity with respect to t. The derivative of the right side is zero. Thus the derivative of the left side must be zero. Therefore

$$u_j'(tx_j)x_i u_i''(tx_i) - u_i'(tx_i)x_j u_j''(tx_j) = 0.$$

Evaluate this expression at $t = 1$ and rearrange terms to find that

$$\frac{x_i u_i''(x_i)}{u_i'(x_i)} = \frac{x_j u_j''(x_j)}{u_j'(x_j)}.$$

Since the right hand side of this equation does not change as x_i changes, it must be that the left hand side is also constant when x_i changes. Therefore there must be some constant c such that for all i and j and all x_i and x_j,

$$\frac{x_i u_i''(x_i)}{u_i'(x_i)} = \frac{x_j u_j''(x_j)}{u_j'(x_j)} = c.$$

The differential equation

$$\frac{x_i u_i''(x_i)}{u_i'(x_i)} = c$$

is equivalent to

$$\frac{d}{dx_i} \ln u'(x_i) = \frac{c}{x_i}.$$

Integrating both sides of this equation, we have

$$\ln u_i'(x_i) = c \ln x_i + d_i$$

for some constant d_i. Taking exponentials of both sides of this equation, we find

$$u_i'(x_i) = k_i x_i^c$$

where $k_i = e^{d_i}$. Integrating both sides of this equation, we see that that there are two possible cases of interest.

If $c = -1$ then (ignoring additive constants),

$$u_i(x_i) = a_i \ln x_i$$

where $a_i = k_i$. If $c \neq -1$, then

$$u_i(x_i) = a_i x_i^{c+1}$$

where $a_i = k_i/(c+1)$. Since this reasoning applies for every i, the theorem is proved.

$\square$

Separability and Stationarity of Intertemporal Preferences

Suppose that a consumer has preferences over intertemporal streams of consumption. Let there be T time periods and n regular commodities. Then an intertemporal consumption stream is an nT dimensional vector $x = (x_1, \ldots, x_t, \ldots, x_n)$ where x_t is the n-vector of commodities consumed in period t.

Suppose that these preferences are additively separable over time, so that preferences over intertemporal consumption are represented by a utility function of the form

$$U(x_1, \ldots, x_n) = \sum_{i=1}^{n} u_i(x_i). \tag{1}$$

Prices in such a model can be written as $p = (p_1, \ldots, p_t \ldots, p_n)$ where p_t is the price vector in period t. Demand theory in this environment would be especially easy to work with if the budget constraint is

$$px = \sum_{t=1}^{n} p_t x_t = M \tag{2}$$

for some measure of wealth, M. Think of intertemporal prices in the following way. Let $\hat{p}_t$ be the vector of time t prices measured in time t dollars. Suppose that money borrowed in period t and returned in period $t+1$ pays back $1+r_t$ in period $t+1$ for each dollar borrowed in period t. Then a dollar in period 1 will be worth

$$\prod_{i=1}^{t}(1+r)^i$$

dollars in period t. Conversely, a dollar in period t is worth

$$\frac{1}{\prod_{i=1}^{t}(1+r)^i}$$

dollars in period 1. Then the p_t's will represent intertemporal prices in a budget constraint if we define

$$p_t = \frac{\hat{p}_t}{\prod_{i=1}^{t}(1+r)^i}$$

and if we define M to be the present value of all future income flows.

Of course if there are credit constraints and differences between borrowing and lending rates, the budget constraint becomes much more complex. Economists like to ignore these complexities and very frequently work with intertemporal models where the budget is of the form in Equation 2.

Note that this intertemporal budget becomes simpler if the interest rate r is constant over all time.

If preferences are representable in this way, then there are well-defined single-period demand functions for each period where demand in that period depends only on prices in that period and total expenditures in that period. If someone is maximizing a utility function of the form in Equation 1 subject to a budget constraint of the form in Equation 2, then it must be that for any t she is choosing x_t so as to maximize $u_t(x_t)$ subject to the constraint that $p_t x_t = e_t$ where e_t is her total expenditure in period t measured in the intertemporal prices. Of course, in general to find out how much she intends to spend in period t, we may need to know things about interest rates and prices in other periods.

There are some fairly plausible additional assumptions that impose interesting special structure on the utility functions in Equation 1. These results are due to Tjalling Koopmans. [4].

Consider any two time streams $(x_1, x_2, \ldots, x_n)$ and $(x_1, x_2', \ldots, x_n')$ that have the same consumption bundle in period 1, but possibly different bundles in other periods. Preferences are said to be *stationary over time* if the following is true: $(x_1, x_2, \ldots, x_n) \succeq (x_1, x_2', \ldots, x_n')$ if and only if $(x_2, \ldots, x_n, x_1) \succeq (x_2', \ldots, x_n', x_1)$.

Stationarity over time requires that "mere" position in time does not change the preference ordering over consumption stream. Thus if one prefers the stream $(x_2, \ldots, x_n)$ starting period 2 to the stream $(x_2, \ldots, x_n)$ starting in period 2, then one would feel the same way if this stream were moved up one period to start in period 1.

Additively separable preferences are said to be *impatient* if whenever you like single period bundle x_t better than single period bundle y_t, you would prefer to get the nice bundle earlier rather than later. That is, if $(x_1, x_2, \ldots, x_n) \succ (y_1, x_2, \ldots, x_n)$, then $(x_1, y_2, x_3, \ldots, x_n) \succ (y_1, x_2, \ldots, x_n)$.

Theorem 6. *If preferences over time streams are representable by a continuous utility function, then they will be additively separable and stationary over time if and only if they can be represented by a utility function of the form*

$$U(x_1, \ldots, x_n) = \sum_{t=1}^{n} \alpha^t u(x_t) \tag{3}$$

for some $\alpha > 0$. Such preferences are impatient if and only if $\alpha < 1$.

Proof: By assumption, preferences are represented by a utility function of the form 1. Suppose that $(x_1, x_2, \ldots, x_n) \succeq (x_1, x'_2 \ldots, x'_n)$. Then $\sum_{t=2}^{n} u_t(x_t) \geq \sum_{t=2}^{n} u_t(x'_t)$. Stationarity implies that in this case, $(x_2, \ldots, x_n, x_1) \succeq (x'_2, \ldots, x'_n, x_1)$ and therefore $\sum_{t=2}^{n} u_{t-1}(x_t) \geq \sum_{t=2}^{n} u_{t-1}(x'_t)$. Therefore the utility functions $\sum_{t=2}^{n} u_t(x_t)$ and $\sum_{t=2}^{n} u_{t-1}(x_t)$ represent the same preferences. Since two additively separable utility functions that represent the same preferences must be affine transformations of each other, it follows that $\sum_{t=2}^{n} u_t(x_t) = \alpha \sum_{t=2}^{n} u_{t-1}(x_t)$ for some $\alpha > 0$. But if this is the case, it must be that $u_2(x) = \alpha u_1(x)$ and $u_3(x) = \alpha u_2(x)$ and so on. But this means that for all $t = 1, \ldots, n$, $u_t(x) = \alpha^t u_1(x)$.

□

What happens if people age?

What happens if survival probability varies over lifetime?

What about an intergenerational interpretation? Suppose that x_t is the lifetime consumption of generation t.

"Time Inconsistent" Preferences

Psychologists and some economists have suggested that instead of discounting preferences exponentially, as is implied by Koopmans' stationarity assumption, the discount factor is proportional to the time delay. This way of weighting the future is sometimes knows as "hyperbolic discounting". While exponential discounting implies that payoffs t periods in the future are weighted at δ^t times payoffs today for some $\delta < 1$, a hyperpolic discounter values a payoff t periods into the future at $\frac{1}{1+dt}$ for some $d > 0$.

Another form of discounting, sometimes known as β, δ discounting or as quasi-hyperbolic discounting, values benefits received at time $t > 0$ at $\beta\delta^t$ times benefits received immediately.

Exercise: Find behavioral implications of each of these assumptions and compare these implications with those of exponential discounting.

Still more structure

Suppose that in addition to assuming additive separability and stationarity, we assume that there is only one commodity and that preferences over time streams of consumption are homothetic, then by Bergson's theorem, we have preferences representable by one of the following forms:

$$U(x_1, \ldots, x_n) = \sum_{t=1}^{n} \alpha^t \frac{1}{b} x_t^b \tag{4}$$

$$U(x_1, \ldots, x_n) = \sum_{t=1}^{n} \alpha^t \ln x_t \tag{5}$$

where x_t is the amount of the commodity consumed in time t.

Suppose that there is a constant interest rate r and that "nominal price" of the commodity in period t is p_t. Then the budget equation becomes

$$\sum p_t x_1 \frac{1}{(1+r)^t} = W$$

where W is the present value of wealth.

If p_t is constant and equal to 1, use the first order condition for maximization to find optimal consumptions in each period.

One can generalize this to the case where there are many commodities in each period, using the indirect utility function.

Uncertainty and separability

The assumption that decision-makers are expected utility maximizers implies that they have additively separable preferences across events. This assumption generates a von Neumann-Morgenstern utility function $U(x)$ that is determined up to affine transformations. The assumption that preferences are additively separable and stationary over time generates a single period utility function $u(x)$ that is also determined up to an affine transformation. The question arises: What would it mean for the von Neumann-Morgenstern utility function and the intertemporal utility function to be the same function u. Another question that arises is: How could they not be the same?

Let us consider a toy example. There are three time periods and two equally likely events, heads and tails. The only reason that anyone cares which event occurs is that some bets are riding on the outcome. There is only one consumption good. The consumer chooses lotteries over allocations of the good across three periods. We any possible lottery by a vector $x = (x_{H1}, x_{H2}, x_{H3}, x_{T1}, x_{T2}, x_{T3})$, where x_{Hi} is consumption in period i if event H happens and x_{Ti} is consumption in period i if event T happens. Such a vector is called a "contingent commodity bundle." If the consumer satisfies the axioms of expected utility theory, then her preferences can be represented by a utility function of the form

$$U(x_{H1}, x_{H2}, x_{H3}, x_{T1}, x_{T2}, x_{T3}) = \frac{1}{2}u(x_{H1}, x_{H2}, x_{H3}) + \frac{1}{2}u(x_{T1}, x_{T2}, x_{T3})$$

where the function $u(\ldots)$ is the von Neumann-Morgenstern utility.

The theory von Neuman-Morgenstern axioms imply that her preferences over alternative time streams of consumption contingent on one event are independent of what her prospects are if the other event happens. These preferences are represented by the utility function $u(x_1, x_2, x_3)$. Now suppose that in addition, conditional on the event, this consumer's preferences over time are additively separable and stationary. This would be the case if the von Neumann-Morgenstern functions are of the special form: $u(x_1, x_2, x_3) = u(x_1) + au(x_2) + a^2u(x_3)$ for some $a > 0$ and some function of a single variable $u(\cdot)$. In this case, preferences over contingent commodity bundles could be represented by the simple function

$$U(x) = \frac{1}{2}\Big(u(x_{H1}) + au(x_{H2}) + a^2u(x_{H3}) + u(x_{T1}) + au(x_{T2}) + a^2u(x_{T3})\Big).$$

To answer the first question: Suppose that these two functions are the same. Individuals choosing alternative lotteries over time streams of consumption would then choose so as to maximize an expected utility function of the form $E(\sum_t \alpha^t u(x_t)) = \sum_t \alpha^t E(u(x_t))$.

For the second question. We could consider alternative theories in which preferences were additively separable over time and stationary where utility is represented by $\sum_t \alpha^t f(E(u(x_t)))$ for some nonlinear function f. What difference would this make?

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