Translated Homothetic Utility and Stone-Geary Utility

Change of variables

Suppose that there are n commodities. Let $\beta^i = (\beta^i_1, \ldots, \beta^i_n)$ be a vector of n parameters, which may be either positive, negative, or zero. Suppose that preferences of consumer i can be represented by a strictly quasi-concave utility function of the functional form $u^i(x_1, \ldots, x_n) = u\left(x_1 - \beta^i_1, \ldots, x_n - \beta^i_n\right) = u(x - \beta^i)$ with domain $X^i = \{x | x \geq \beta^i\}$. We assume that all consumption bundles not in X^i are worse than any bundle in X^i for consumer i. Here we assume that the only differences between consumers take the form of differing β^i's.

If Consumer i can afford any bundles in X_i, then i's Marshallian demand function $x^i(p, m_i)$ is found by solving the following maximization problem:

Problem A: Choose $\bar{x}^i$ to maximize $u(x - \beta^i)$ subject to $px \leq m_i$, $x \geq \beta^i$ and $x \geq 0$.

A convenient way to solve this problem is by the "substitution of variables" trick. Define $z^i = x - \beta^i$. Then , since $x = z^i + \beta^i$, Problem A can be restated as an equivalent problem:

Problem B: Choose $\bar{z}^i$ to maximize $u(z^i)$ subject $pz^i \leq m_i - p\beta^i$, $z^i \geq 0$ and $z^i \geq -\beta^i$.

Translated homothetic utility and Gorman Polar form

Suppose that the function $u(z)$ is homogeneous of degree 1. Then preferences over the z vectors (but not necessarily over the x vectors) are homothetic. The solution for Problem B is then the solution to maximizing a homothetic utility subject to the constraint that income equals $m_i - p\beta^i$. Thus this solution can be written as $z(p, m_i) = f(p)(m_i - p\beta^i)$ (where $f(p)$ is an n vector). Since u is homogeneous of degree 1 it follows that

$$v(p, m) = u\left(f(p)(m_i - p\beta^i)\right) = (m_i - p\beta^i)u\left(f(p)\right).$$

4 But this means that indirect utility is of the Gorman polar form $A(p)m_i - B^i i(p)$ where $A(p) = u\left(f(p)\right)$ and $B^i(p) = p\beta^i u\left(f(p)\right)$

Stone-Geary utility

Suppose that utility of Consumer i is given by

$$u^i(x_1,\ldots,x_n)=\prod_{j=1}^{n}(x_j-\beta_j^i)^{\alpha_j}$$

where $\alpha_j>0$ for all i and $\sum_{j=1}^n \alpha_j=1$. If we let $z=(z_1,\ldots,z_n)$ where $z_j=x_j-\beta_j^i$, then Problem B becomes a simple Cobb-Douglas maximization problem in the $z's$ for a consumer with "income" $m-p\beta^i$

At an interior solution, we then have

$$z_j^i=\alpha_j\left(\frac{m_i-p\beta^i}{p_j}\right).$$

It then follows that

$$x_j^i=z_j^i+\beta_j^i=\alpha_j\left(\frac{m_i-p\beta^i}{p_j}\right)+\beta_j^i.$$

We then see that that consumer i's expenditure on good j is simply a linear function of prices and income. In particular, for each commodity j,

$$p_jx_j^i=\alpha_jm_i-\alpha_j\sum_k p_k\beta_k^i+\beta_j^ip_j. \tag{1}$$

Consider an economy with m consumers with utility of this type. Let $M=\sum_i m_i$ and $B_j=\sum_i\beta_j^i$ and let $X_j=\sum_i x_j^i$. Summing both sides of Equation 1, we have

$$p_jX_j=\alpha_jM-\alpha_j\sum_k p_kB_k+B_jp_j \tag{2}$$

You can see why this formulation would be popular with people who like to run regressions. On the left side we have aggregate expenditure on good j and on the right side we have the variables aggregate income and the prices of each good. The regression coefficients that are estimated are then the parameters α_j, and the B_k's. The system of equations that one gets by applying Equation 2 for each of the n goods is known as the *linear expenditure systerm* or sometimes as the Stone-Geary systerm.

An example

A consumer has income m and utility function

$$(x_1+1)^{1/2}(x_2-1)^{1/2}$$

defined on the set $X = \{(x_1, x_2)|x_1 \geq 0, x_2 > 1\}$. This is a Stone-Geary utility with $\beta_1 = -1$ and $\beta_2 = 1$. If we set $z_1 = x_1 + 1$ and $z_2 = x_2 - 1$, then $x_1 = z_1 - 1$ and $x_2 = z_2 + 1$. We can restate the consumer's maximization problem in terms of the z's as follows: Chooses z_1 and z_2 to maximize

$$z_1^{1/2} z_2^{1/2}$$

subject to the constraint that $p_1(z_1 - 1) + p_2(z_2 + 1) = m$ or equivalently,

$$p_1 z_1 + p_2 z_2 = m + p_1 - p_2.$$

At a constrained maximum interior to the set X, it must be that

$$z_i = \frac{m + p_1 - p_2}{2p_i}.$$

Then it must be that

$$x_1 = z_1 - 1 = \frac{m - p_1 - p_2}{2p_1}$$

and

$$x_2 = z_2 + 1 = \frac{m + p_1 + p_2}{2p_2}.$$

But these two equations are the quantities demanded only if (x_1, x_2) is in the interior of the set X. Now $x_1 = z_1 - 1 = \frac{m-p_1-p_2}{2p_1} \geq 0$ if and only if $m \geq p_1 + p_2$. and $x_2 > 1$ if and only if $m + p_1 + p_2 > 2p_2$, or equivalently, $m > p_2 - p_1$. For all $p_1 > 0$, we have $p_1 + p_2 > p_2 - p_1$. Therefore there will be an interior solution if and only if $m > p_1 + p_2$.

If $p_1 + p_2 > m > p_2 - p_1$, there will be a corner solution in which $x_1 = 0$ and $x_2 = m/p_2$. If $m < p_2 - p_1$, the consumer can not afford any bundles in X.