When Was Coase Right?

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1 Introduction

Ronald Coase [10] argued that a proper economic analysis of harmful externalities should account for the fact that causation is usually two-sided. The amount of damage typically depends on the actions of the impacted parties as well as the actions of the externality generators. Coase maintained that, regardless of the way in which the law assigns liability, if the offended and offending parties are able to bargain freely, they are likely to reach an efficient solution where the value of marginal gains to activity by the externality generator are equal to the value of marginal losses to the damaged parties. Coase also asserted [10] (p 10) that "With costless market transactions, the decision of the courts concerning liability for damage would be without effect on the allocation of resources."

Coase's article consists of a series of examples and some insightful discussion. Coase made no claim to a formal theorem based on explicit assumptions. The commonly-used term "Coase Theorem" originated with George Stigler, who explained Coase's ideas in his textbook *The Theory of Price*. [16] (pp 110-114) Stigler claimed (without proof) that the Coase theorem establishes two results:

1. Under perfect competition, private and social costs will be equal.
2. The composition of output will not be affected by the manner in which the law assigns liability for damage.

Result 1 of Stigler's version of the Coase theorem can be interpreted as a statement that private bargaining "in the absence of transaction costs" will lead to a Pareto optimal level of externality-producing actions. Result 2 would follow from Result 1 only if it were true that in every Pareto optimal allocation, the amount of externality-producing activities would be the same, regardless of how private goods are distributed.

In the view of Coase and Stigler, the way in which liability is assigned affects "only" the distribution of private income, while the level of externalities to which bargaining leads is always be the same efficient amount.

In a paper called *What is the Coase Theorem?*, Leo Hurwicz [13] sought a genuine "Coase Theorem," with assumptions and a proof. Hurwicz referred to Stigler's Result 2 as the *Coase independence phenomenon* and investigated assumptions on preferences that would be necessary as well as sufficient for an economy to satisfy the Coase independence property. Hurwicz pointed out that the "no income-effects" condition, which requires utility functions be of quasi-linear form, linear in private goods, is *sufficient* for the Coase independence phenomenon. Hurwicz showed that if at least one consumer has quasi-linear preferences, then there will be Coase

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independence only if *all* consumers have quasi-linear preferences. He then argued that the so-called Coase theorem is of severely limited applicability, since it requires that consumers' willingness to pay for reductions in harmful externalities do not depend on their incomes.

In two earlier papers [4, 5], Richard Cornes and I posed the question: "When are Pareto efficient quantities of public goods independent of income distribution?" We found such independence for a more general class of preferences than the quasilinear family. While this class remains restrictive, preferences of this type allow willingness to pay for public goods to depend on income, and also allow for preferences to differ between individuals. We did not explictly consider *harmful* externalities, but interpreted our result as applying to the provision of public goods regarded as beneficial (or at least not harmful) by all consumers.

John Chipman and Guoqian Tian [8] revisited Hurwicz's contribution.They examined a model with two agents, a polluter and a pollutee,and and two commodities, a consumer good and pollution. They define Coase independence in a less demanding form than did Hurwicz, and when they do so, they found that for their polluter and pollutee, Coase independence is essentially equivalent to having preferences belong to the family of preferences found by Bergstrom and Cornes. Thus it is possible for there to be Coase independence even if people are willing to pay more to reduce pollution as they get richer.

Interesting applications of Coase independence have also appeared in discussions of the family. Gary Becker [2] (page 331) suggested that the "Coase theorem" implies that changes in divorce law, such as requiring divorce by mutual consent rather than unilateral withdrawal, would not affect divorce rates, though they might affect the division of family resources within marriages. Becker reasoned that if a married couple will always reach efficient bargains with each other about the terms of marriage, then the Coase theorem implies that they will divorce if and only if they can both be better off divorced than they would be under any arrangement of benefits within marriage.

Chiappori *et al* [9] argue that Becker's assumption of Coasian independence may not be appropriate in the case of divorce, because the utility possibility frontier for a married couple could quite plausibly cross the utility possibility frontier for these two if they were divorced, in which case some Pareto efficient outcomes would leave them married and others would have them divorced. If this is the case, the authors point out that the decision of a couple to divorce or not "depends on the *location* of the final outcome on the efficiency frontier," which in turn depends on the allocation of rights assigned by divorce law.

This paper revisits the contributions of Bergstrom and Cornes, Hurwicz, Chipman and Tian, and Chiappori et al. We extend the Bergstrom-Cornes results to include the case of harmful externalities, and show that Coase independence applies to a class of economies which, while still restrictive, allows there to be income effects on the valuation of externalities. We present what we believe to be a more thorough and clearer treatment of the issue of Coasian independence than can be found elsewhere.

2 Technology and Preferences

2.1 Feasible Allocations

Consider a community with n consumers, one private good and m public variables. We refer to public *variables* rather than public *goods*, since we allow the possibility that some of these variables represent levels of externalities that are disliked by some or all consumers. We refer to a vector y that specifies the level of each public variable as a *public choice*. We assume that there is an initial aggregate endowment of W units of private goods and a cost function $c(\cdot)$ such that given the public choice y, the amount of private goods available to be distributed among the n consumers is $W - c(y)$. Stating this more formally:

Assumption 1 (Endowment and Technology). *There is an initial endowment $W > 0$ of private goods and a compact set $Y \subset \Re^m_+$ of possible public choices. There is a continuous cost function $c(\cdot)$ defined on Y such that $x = W - c(y)$ is the amount of private goods available when the public choice is y.*

Definition 1 (Feasible allocations). *The set of feasible allocations is*

$$\mathcal{F} = \left\{ (x_1, \ldots, x_n, y) | (x_1, \ldots, x_n) \geq 0, y \in Y, \text{ and} \sum_i x_i = W - c(y) \right\}.$$

Definition 2 (Feasible interior allocations). *A feasible allocation $(x_1, \ldots, x_n, y)$ is said to be* interior *if $x_i > 0$ for all consumers i.*

2.2 Preferences that are uniformly affine in private goods

Each consumer has preferences over outcomes $(x_i, y) \in \Re_+ \times Y$, where x_i is i's consumption of private goods and y is the vector of public variables. These preferences are representable by continuous utility functions of the form $u_i(x_i, y)$. We consider the following family of preferences, which generalize the notion of quasi-linear utility.[1]

Definition 3 (Uniformly affine in private goods). *Preferences over the domain (X_i, Y) are uniformly affine in private goods if for all consumers i, these preferences can be represented by a utility function of the form:*

$$u_i(x_i, y) = A(y)x_i + B^i(y),$$

where $A(y)$ and $B^i(y)$ where $A(y) > 0$ for all $y \in Y$.

Notice that the function $A(y)$ is common to all i, while the functions $B^i(y)$ may differ between individuals. If preferences are uniformly utility affine in private goods, then Consumer i's marginal rate of substitution between the public variable j and private goods is

$$m^i_j(x_i, y) = \frac{A_j(y)}{A(y)} x_i + \frac{B^i_j(y)}{A(y)}. \tag{1}$$

where $A_j(y)$ and $B^i_j(y)$ are partial derivatives of A and B^i with respect to y_j.

The familiar example of quasi-linear utility is a special case of uniformly affine preferences where $A(y) = 1$. If consumers have quasi-linear utility, then each consumer i's marginal rate of substitution between public variable j and the private good is simply $B^i_j(y)$, which is independent of private consumption x_i.

From Equation 22, we see that consumers' marginal rates of substitution between public variable j and the private good increases or decrease with i's private consumption depending on whether the partial derivative, $A_j(y)$, is positive or negative. Since the sign of $B^i_j(y)$ may differ between consumers, it may be that some consumers prefer more and some consumers prefer less of a public variable j.

3 Utility Possibility Sets

The notion of *utility possibility set*, which was introduced by Paul Samuelson [14], can be usefully adapted to explore the relation between wealth distribution and efficient allocations.

[1] This class of utilities was introduced by Bergstrom and Cornes [5]. Chiappori *et al* [9] refer to these as "generalized quasi-linear utilities."

3.1 Contingent Utility Possiblity Sets

We define the y-contingent utility possibility set for any vector $y \in Y$ as the set of utility distributions that are possible if the public choice is y and the corresponding amount of private goods, $W - c(y)$, is somehow divided among consumers.

Definition 4 (y-Contingent Utility Possibility Set)**.** *The y-contingent utility possibility set for an economy with initial wealth W and cost function $c(\cdot)$ x is*

$$UP(W, y) = \left\{ (u_1(x_1, y), \ldots, u_n(x_n, y)) \,|\, (x_1, \ldots, x_n) \geq 0, \text{ and } \sum_i x_i = W - c(y) \right\}.$$

The full utility possibility set is then the union of the y-conditional utility possibility sets over all $y \in Y$.

Definition 5 (Utility possibility set)**.** *The utility possibility set for an economy with initial wealth W and the set Y of possible public choices is*

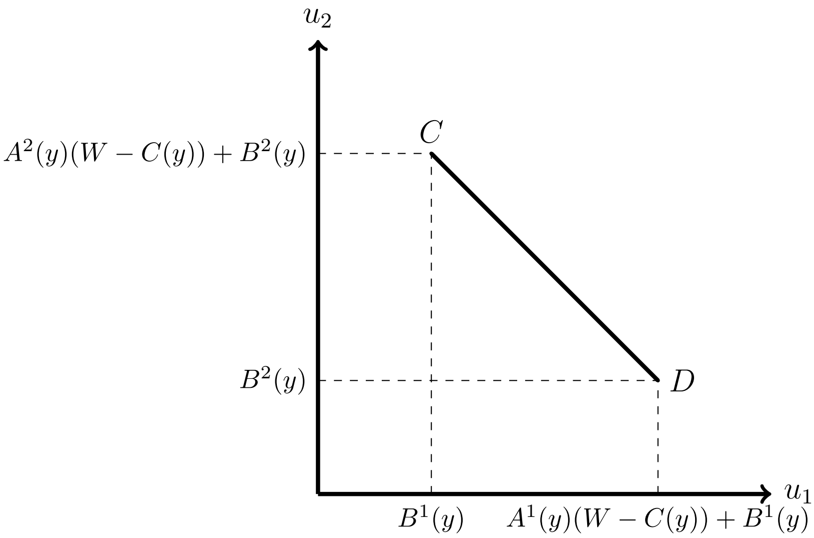
$$UP^*(W, Y) = \cup_{y \in Y} UP(W, y).$$

Figure 1 shows a y-contingent utility possibility set for an economy with two consumers whose preferences are uniformly affine in private goods, with $u_i(x_i, y) = A(y)x_i + B^i(y)$ for $i = 1, 2$. Adding these equations and recalling that $x_1 + x_2 = W - c(y)$, we have the equation

$$u_1(x_1, y) + u_2(x_2, y) = A(y)\,(W - c(y)) + B^1(y) + B^2(y).$$

It follows that the y-contingent utility possibility set is the line segment consisting of all (u_1, u_2) such that $u_1 + u_2 = W - c(y) + B_1(y) + B_2(y)$ and such that $u_i \geq B_i(y)$ for $i = 1, 2$.

Figure 1: A y-Contingent Utility Possibility Set



3.2 Parallel Contingent Utility Possibility Sets

We say that y-contingent utility possibility sets are parallel over the set $\mathcal{F}$ of feasible allocations if there exist utility functions $u_i(x_i, y)$ for each consumer i such that for all $(x_1, \ldots, x_n, y) \in \mathcal{F}$, the y-contingent utility possibility sets lie in parallel hyperplanes, with the sum of utilities constant for each y. Formally, we state this condition as follows.

Definition 6 (Parallel contingent utility possibility sets). *Contingent utility possibility sets are said to be* parallel *on a set $\mathcal{F}$ of feasible allocations if there is a real-valued $F(\cdot)$ with domain Y and for every consumer i there exists a utility function $u_i(x_i, y)$ that represents preferences of i such that the y-contingent utility possibility set $UP(W, y)$ is contained in the hyperplane*

$$\{(u_1, \ldots, u_n) | \sum_i u_i = F(y)\}.$$

Proposition 1 tells us that if preferences are uniformly affine in private goods, then y-contingent utility possibility sets must be parallel on Y. Proposition 2 tells us that the converse statement is also true.

Proposition 1. *If preferences of all consumers are uniformly affine in private goods, then the y-contingent utility possibility sets are parallel and take the form*

$$UP(W, y) = \Big\{(u_1, \ldots, u_n) \geq \big(B^1(y), \ldots, B^n(y)\big) \,|\, \sum u_i = F(y)\Big\}$$

where $F(y) = A(y)\,(W - c(y)) + \sum_i B^i(y)$.

Proof. If preferences are uniformly affine in private goods, then preferences of each consumer i are represented by $u_i(x_i, y) = A(y)x_i + B_i(y)$. Then it must be that

$$\sum_i u_i(x_i, y) = A(y) \sum_i x_i + \sum_i B^i(y).$$

An allocation $(x_1, \ldots, x_n, y)$ is feasible if and only if $\sum x_i = W - c(y)$ and $x_i \geq 0$ for all i. Therefore the y-contingent utility possibility set is the set

$$\{(u_1, \ldots, u_n) \geq \big((B^1(y), \ldots, B^n(y)\big) \,|\, \sum u_i = F(y)\}$$

where $F(y) = A(y)\,(W - c(y)) + \sum_i B^i(y)$. Hence the y-contingent utility possibility sets are parallel on Y. □

In order to prove that uniformly affine preferences are necessary as well as sufficient for parallel y-contingent utility possibility sets, we use a standard result from the theory of functional equations. An equation that satisfies the equation $f(x + y) = g(x) + h(y)$ for all real-valued x and y is known as a Pexider functional equation.(See Aczel [1], page 142). Aczel shows that if f, g, and h are continuous and satisfy the Pexider functional equation, then there must be real numbers a, b, and c, such that $f(x) = ax + b + c$, $g(x) = ax + b$ and $h(x) = ax + c$.

This result generalizes in a straightforward way to the case of sums of n terms. Although the usual statement of the result deals with functions whose domain is the entire real line, the proof that Aczel uses for the Pexider result shows that that this result holds if the domain is the non-negative reals.[2] We have the following lemma.

Lemma 1 (Pexider functional equations). *Let f_i, $i = 1, \ldots, n$, be continuous functions with domain $\Re_+$. If there is a function f such that $\sum_i f_i(x_i) = f(\sum x_i)$ for all $x_i \geq 0$, then there must exist constants a and $b_1, \ldots, b_n$ such that $f_i(x) = ax + b_i$ for $i = 1, \ldots, n$ and $f(x) = ax + \sum_i b_i$.*

Proof. A proof (which is quite elementary) can be found in Aczel [1] or in Diewert [11]. □

Proposition 2. *Contingent utility possibility sets are parallel on $\mathcal{F}$ if and only if preferences of all individuals are uniformly affine in private goods.*

[2]Diewert [11] also notices this in his notes on functional equations.

Proof. If y-contingent utility possibility sets are parallel, preferences of each i can be represented by a utility function $u_i(x_i, y)$, where for all allocations in $UP(W, y)$, $\sum_i u_i(x_i, y) = F(y)$. The y-contingent utility possibility set consists of all vectors, $(u_1(x_1, y), \ldots, u_n(x_n, y))$ such that $\sum_i x_i = W - c(y)$ and $x_i \geq 0$ for all i. It follows that if $\sum x_i = \sum x_i'$ and $x_i \geq 0$ for all i, then $\sum u_i(x_i, y) = \sum u_i(x_i', y)$. From Lemma 1, it then follows that for each $y \in Y$, the function $u_i(x,y)$ must be of the form $u_i(x_i, y) = A(y)x_i + B_i(y)$.

If preferences are uniformly affine in private goods, then they can be represented by $u_i(x_i, y) = A(y)x_i + B_i(y)$. The y contingent utility possibility frontier consists of all utility distributions possible with public choice y some distribution $(x_1, \ldots, x_n)$ of private goods such that $\sum_{i=1}^n x_i = W - c(y)$. This is the set

$$\{(u_1, \ldots, u_n) \geq (B_1(y), \ldots, B_n(y)) | \sum_{i=1}^n u_i = A(y)\,(W - c(y)) + \sum_{i=1}^n B_i(y)\}.$$

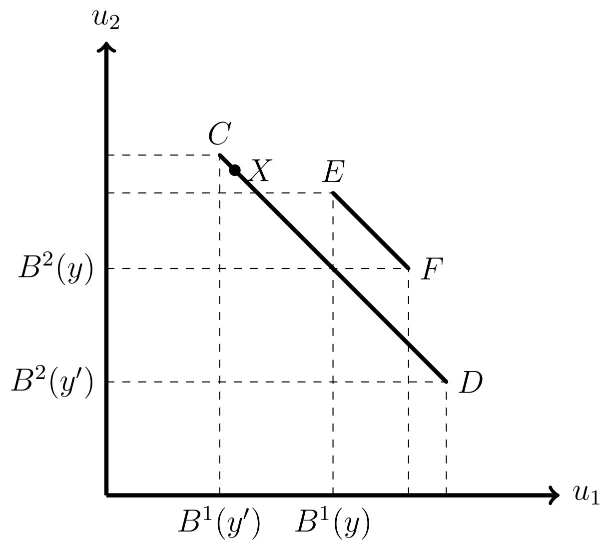
It follows that all y contingent utility possibility sets are parallel.

□

Example 1

Let there be a single public good that is desirable for both consumers. There are two possible public choices, y and y' where $y > y'$ and $c(y) > c(y')$. As Figure 2 shows, there are two parallel utility possibility sets, corresponding to public choices y and y'. The y-contingent utility possibility is the line segment EF and the y'-contingent utility possibility set is the line segment CD. These segments are parallel with slope -1.

Figure 2: A Utility Possibility Set With Two Possible y Vectors



The endpoints of CD and EF correspond to allocations in which all of the private goods go to one or the other of the two consumers. The line segment CD is longer than the segment EF because when $y' < y$, there are more private goods to be distributed and hence greater inequality of private income is possible with public choice y' than with public choice y.

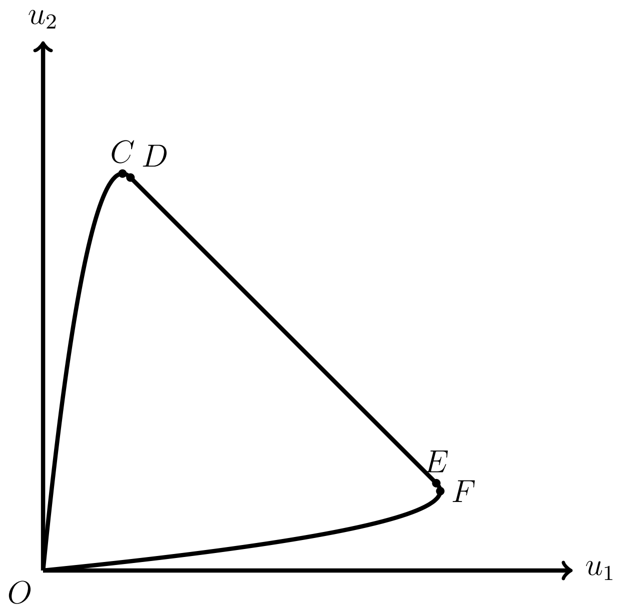
Example 2

Figure 3 shows a utility possibility set for two consumers with preferences that are uniformly affine in private goods and where there is a continuum of possible amounts of a single public good. The public good is produced at constant unit cost c and the set of possible quantities is $Y = [0, y^+]$.[3] If private wealth is W, and $cy < W$, the y-contingent utility possibility set is a line segment such that

$$u_1 + u_2 = F(y) = A(y)\,(W - c(y)) + B^1(y) + B^2(y) \tag{2}$$

and lying between the points $(F(y) - B_2(y), B_2(y))$ and $(B_1(y), F(y) - B_1(y))$ and that correspond to the utility distributions attained with y when all of the private goods are given to consumers 1 and 2, respectively.

Figure 3: A utility possibility set with a continuum of possible y's



Let $\bar{y}$ be the value of y that maximizes $F(y)$ on the set $[0, y^+]$. The $\bar{y}$-contingent utility possibility set is the line segment DE in Figure 3. This line segment is tangent to the curves OCD and OFE at its endpoints D and E. The full utility possibility set includes all points lying between the curves OCD and OFE and below the line CD. Points in the interior of the utility possibility set are reached with values of $y < \bar{y}$ and $x_i > 0$ for $i = 1$ and 2.

The utility possibility frontier, containing all of the Pareto efficient points, includes all points on the line CD as well as all points on the curved line segments, CD and EF. The Pareto optimal points on these curved line segments are achieved with income distributions in which one of the two consumers receives no private goods. In this example, the highest possible utility for Consumer 2 occurs at point C. Other points on the curve CD represent outcomes in which Consumer 2 continues to receive all of the private goods and where increased amounts of the public goods paid for by Consumer 2 benefit both consumers.

4 Coase independence

The notion of Coase independence, as presented by Stigler and by Hurwicz can take two possible forms, which we will call *weak Coase independence* and *strong Coase independence*.

Let us define a public choice y to be *always Pareto efficient* for an economy if every feasible allocation in which y is the public choice is Pareto optimal; no matter how the private goods are divided. More formally:

[3]We have drawn this figure the case where the utility functions are $u_1(x_1, y) = x_1 y + y$, $u_2(x_2, y) = x_2 y + y$, $c(y) = y$ $W = 9$, and $Y = [0, 9]$.

Definition 7 (Always Pareto efficient)**.** *For an economy with a set $\mathcal{F}$ of feasible allocations and a set Y of possible public variables, a public choice $y^* \in Y$ is* always Pareto efficient *if every allocation $\{(x_1, \ldots, x_n, y^*) \geq 0 | \sum x_i = W - c(y^*)\}$ is Pareto optimal.*

Definition 8 (Weak Coase independence)**.** *An economy with a set $\mathcal{F}$ of feasible allocations and a set Y of possible public choices, satisfies* weak Coase independence *if there is some feasible public choice $y^* \in Y$ that is always Pareto efficient.*

In Example 1 of the previous section, the set Y has two elements, y and y', and the utility allocations achievable with y' are represented by the line segment EF. In this example, y' is always Pareto optimal, since every feasible allocation with public choice y' is Pareto optimal. Thus the economy in this example satisfies weak Coase independence

Although y' is always Pareto optimal, there are Pareto optima that can not be reached with public choice y'. In Figure 2, the line segement CD shows the utility distributions possible with public utility distribution y. The point X on line CD is strictly preferred by Consumer 2 to any point on EF and must be Pareto optimal.

A stronger form of Coase independence requires that there is some public choice y^* such that y^* is the only Pareto optimal public choice for allocations in which every consumer receives a positive amount of private goods.

Definition 9 (Strong Coase Independence)**.** *An economy with a set $\mathcal{F}$ of feasible allocations and a set Y of possible public choices satisfies* strong Coase independence *if there is a unique public choice $y^* \in Y$ such that an interior allocation is Pareto optimal if and only if the public choice is y^*.*

Proposition 3. *If technology satisfies Assumption 1 and if consumer preferences are uniformly affine in private goods, then there is weak Coase independence, and there is an always Pareto efficient public choice $y^* \in Y$ such that $F(y^*) = A(y^*)\,(W - c(y^*)) + \sum_i B^i(y*) \geq F(y)$ for all $y \in Y$.*

Proof of Proposition 3. Assumption 1 requires that the set Y is compact. Continuity of the functions $A(y)$, $c(y)$, and $B^i(y)$ implies that $F(\cdot)$ is continuous. Therefore there exists $y^* \in Y$ such that $F(y^*) \geq F(y)$ for all $y \in Y$.

We next show that if $\sum_i x_i = W - c(y^*)$ and $x_i \geq 0$ for all i, then the allocation $(x_1, \ldots, x_n, y^*)$ is Pareto optimal. Suppose that the allocation $(x'_1, \ldots, x'_n, y)$ is Pareto superior to $(x_1, \ldots, x_n, y^*)$. Then it must be that $A(y)x'_i + B^i(y) \geq A(y^*)x_i + B^i(y^*)$ for all i, with strict inequality for some i. This implies that

$$A(y) \sum_i x'_i + \sum B^i(y) > A(y^*) \sum x_i + \sum B^i(y^*) \tag{3}$$

But $\sum_i x_i = W - c(y^*)$ and if $(x'_1, \ldots, x'_n, y)$ is feasible, it must also be that $\sum_i x'_i = W - c(y)$. Therefore if $(x'_1, \ldots, x'_n, y)$ is feasible and Pareto superior to $(x_1, \ldots, x_n, y^*)$, it must be that

$$F(y) = A(y)\,(W - c(y)) + \sum B^i(y') > A(y^*)\,(W - c(y^*)) + \sum B^i(y^*) = F(y^*). \tag{4}$$

But this is impossible, since y^* maximizes $F(\cdot)$ on Y. It follows that the public choice y^* is always Pareto optimal. □

The following corollary is immediate from Propositions 3 and 2.

Corollary 1. *If technology satisfies assumption 1 and if contingent utility possibility sets are parallel, then there is weak Coase independence.*

Proposition 3 and its corollary make no assumptions about convexity of preference or about convexity of the set of feasible allocations, and it tells us that there is some feasible public choice y^*, such that every distribution of private goods that is feasible with public choice y^* is Pareto optimal.

Example 1 shows that not every economy with weak Coase independence satisfies strong Coase independence. However, if we add the assumptions of strict convexity of preferences and convexity of the cost function for public choices,then there will be strong Coase independence whenever preferences are uniformly affine in private goods.

Assumption 2 (Convexity assumption)**.** *Preferences of all consumers i are strictly convex. The set Y is a convex set and the function $c(y)$ is a convex function.*

Proposition 4. *If technology satisfies assumption 1, if utility is uniformly affine in private goods, and if the convexity assumption 2 is satisfied, then there is strong Coase independence.*

Proof of Proposition 4. According to Proposition 3, the assumption that preferences are uniformly affine in private goods implies that there exists a public choice $y^* \in Y$ that is always Pareto efficient and such that

$$F(y^*) = A(y^*)\left(W - c(y^*)\right) + \sum_i B^i(y^*) \geq F(y) \tag{5}$$

for all $y \in Y$. If preferences are strictly convex, then the function $F(\cdot)$ is strictly quasi-concave and hence y^* is the unique maximizer of F on the convex set Y.

We next show that any feasible interior allocation $(x_1, \ldots, x_n, y)$ where $y \neq y^*$, is Pareto dominated by a feasible allocation with public choice y^*. If $(x_1, \ldots, x_n, y)$ is feasible, it must be that $\sum x_i \leq W - c(y)$. Since y^* is the unique maximizer of $F(y)$ on Y, it follows that $F(y^*) - F(y) = \delta > 0$. For each i, let

$$x_i^* = \frac{A(y)x_i + B^i(y) + \frac{\delta}{n} - B^i(y^*)}{A(y^*)}. \tag{6}$$

Then it must be that for all i,

$$A(y^*)x_i^* + B^i(y^*) - \big(A(y)x_i + B^i(y)\big) = \frac{\delta}{n} > 0. \tag{7}$$

For $\lambda \in (0,1)$, define $x_i(\lambda) = x_i + \lambda(x_i^* - x_i)$, and $y(\lambda) = y + \lambda(y^* - y)$. Since the allocation $(x_1, \ldots, x_n, y)$ is feasible, it must be that $\sum_i x_i = W - c(y)$. Since $\sum_i x_i^* = W - c(y^*)$, and since the function $c(\cdot)$ is assumed to be convex, it follows that $\sum_i x_i(\lambda) \leq W - c\left(y(\lambda)\right)$. Since $x_i > 0$ for all i, it follows that for λ positive but sufficiently small, $x_i(\lambda) > 0$ for all i. Therefore for sufficiently small positive values of λ, the allocation $(x_1(\lambda), \ldots, x_n(\lambda), y(\lambda))$ is feasible.

Since preferences are assumed to be strictly convex, it must be that for all i, $u_i\left(x_i(\lambda), y(\lambda)\right) > u_i\left(x_i, y\right)$. Therefore the allocation $(x_1, \ldots, x_n, y)$ cannot be Pareto optimal. It follows that the only Pareto optimal allocations have public choice y^*.

□

5 Necessary Conditions for Coase Independence

5.1 Money metric utility

We have shown that a necessary and sufficient condition for parallel y-contingent utility possibility sets is that preferences are uniformly affine in private goods. Of course, since utility representations are invariant to monotonic transformations, not every utility function that represents such preferences is of the appropriate affine form.

A useful diagnostic tool for determining whether preferences are uniformly affine in private goods is the "money metric utility function,"[4] defined as follows. Choose a reference public choice vector $\bar{y}$. This reference public choice might represent a *status quo* public choice, but could be chosen quite arbitrarily. The $\bar{y}$-based money metric utility function $\bar{u}(\cdot,\cdot)$ is defined so that $\bar{u}(x_i, y)$, is the amount of private good that consumer i would need so as to be indifferent between (x_i, y) and $(\bar{u}_i(x_i, y), \bar{y})$.

Definition 10 (Money metric utility function)**.** *Where preferences of each consumer i are represented by a utility function $u_i(x_i, y)$, the $\bar{y}$=based money metric utility function $\bar{u}_i(x_i, y)$ is defined implicitly by the equation*

$$u_i(\bar{u}_i(x_i, y), \bar{y}) = u_i(x_i, y).$$

To ensure that the money metric utility function $\bar{u}(x_i, y)$ is well-defined, we restrict its domain to a set of admissible outcomes.

Definition 11 (Admissible outcomes)**.** *The set of $\bar{y}$-admissible outcomes for consumer i with preferences $\succeq_i$ consists of all (x_i, y) such that $(x_i, y) \succ (0, \bar{y})$ and such that for some z, $(z, \bar{y}) \succ (x, y)$.*

The set of $\bar{y}$-admissible outcomes excludes outcomes that are worse for i than having no private goods, and having public choice $\bar{y}$. This set also excludes public outcomes y that are so good for i that no possible amount of private wealth would compensate i for having $\bar{y}$ rather than y.

Lemma 2. *Let consumer i have preferences that are strictly increasing in private goods. Preferences of i on the set of $\bar{y}$ admissible outcomes can be represented by a $\bar{y}$-based money metric utility function $\bar{u}_i$, defined by the condition*

$$(\bar{u}_i(x_i, y), \bar{y}) \sim_i (x_i, y)$$

where $\sim_i$ denotes indifference.

Proof. Where (x_i, y) is a $\bar{y}$ admissible outcome for i, it must be that $(x_i, y) \succeq_i (0, \bar{y})$, and for some $z > 0$, $(z, \bar{y}) \succeq_i (x_i, y)$. Since preferences are continuous and monotone increasing in private goods, there is exactly one real number $\bar{u}_i(x_i, y)$ such that $(\bar{u}_i(x_i, y), \bar{y}) \sim_i (x_i, y)$. Since preferences are strictly increasing in private goods, it follows that $\bar{u}_i(x_i, y)$ represents preferences of i over the set of admissible outcomes. □

The following proposition shows us a way to check whether a set of utility functions can be transformed by monotonic transformations to a set of functions that are uniformly affine in private goods.

Proposition 5. *Preferences over the set of $\bar{y}$ admissible outcomes are uniformly affine in private goods if and only if for each i, the $\bar{y}$-based money metric utility function over this set takes the functional form*

$$\bar{u}_i(x_i, y) = \alpha(y)x_i + \beta_i(y)$$

where $\alpha(\bar{y}) = 1$ and $\beta_i(\bar{y}) = 0$.

[4]The term "money metric" utility function was coined by Samuelson, in a classic paper.[15])

Proof. If preferences are uniformly affine in private goods, then preferences of each i can be represented by some utility function of the form $A(y)x_i + B_i(y)$. The definition of the money metric utility function $\bar{u}(\cdot,\cdot)$ implies that

$$A(\bar{y})\bar{u}_i(x_i, y) + B_i(\bar{y}) = A(y)x_i + B_i(y) \tag{8}$$

Rearranging terms of Equation 8, we have

$$\bar{u}_i(x_i, y) = \alpha(y)x_i + \beta_i(y)$$

where

$$\alpha(y) = \frac{A(y)}{A(\bar{y})}$$

and

$$\beta_i(y) = \frac{B_i(y) - B_i(\bar{y})}{A(\bar{y})}.$$

It follows immediately that $\alpha(\bar{y}) = 1$ and $\beta(\bar{y}) = 0$.

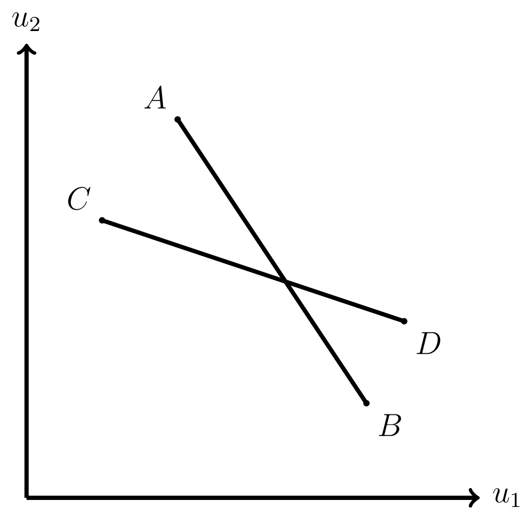
The converse is also true since the money metric utility function represents preferences. □

5.2 Examples with and without Coase independence

Example 3–A case without Coase independence

There are two consumers and the set Y has two elements, y and y'. The line segment AB shows the y-contingent utility possibility set and CD shows the y'-contingent utility possibility set. Since the two lines cross, some of the allocations possible with public choice y are Pareto dominated by allocations possible public choice y', and some of the allocations possible with y' are Pareto dominated by allocations possible with public choice y. Thus, neither y nor y' is always Pareto optimal. Therefore this economy does not satisfy either weak or strong Coase independence.

Figure 4: Crossing Contingent UPF's



Example 4– Coase independence without uniformly affine preferences

In this example there is strong Coase independence, but preferences are *not* affine in private goods. There are two consumers and one public good. The public good can be produced costlessly in any quantity between 0 and 2. Thus, the set of feasible allocations is

$$\mathcal{F} = \{(x_1, x_2, y) | x_1 + x_2 = W \text{ and } y \in [0, 2]\}.$$

Let preferences of each consumer i be represented by a utility function of the form

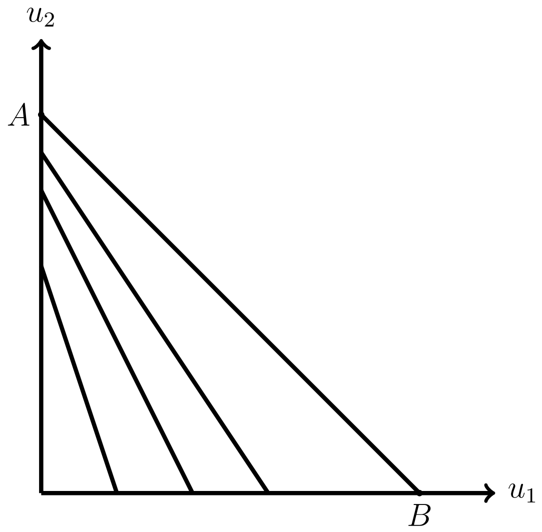
$$u_i(x_i, y) = x_i e^{-\alpha_i (y-1)^2}, \tag{9}$$

where $\alpha_1 > \alpha_2 > 0$. From Equation 9, it follows that $x_i = u_i(x_i, y)e^{-\alpha_i(y-1)^2}$ and hence the for $y \in [0, 2]$, the y-contingent utility possibility frontier is the line segment $(u_1, u_2) \geq 0$ such that

$$u_1 e^{\alpha_1 (y-1)^2} + u_2 e^{\alpha_1 (y-1)^2} = W. \tag{10}$$

The y-contingent utility possibility frontiers are illustrated in Figure 5. The y-contingent utility possibility frontier for $y = 0$ satisfies the equation $u_1 + u_2 = W$ and is shown by the line AB. The y-contingent utility possibility frontiers for $y \neq 0$ are straight lines lying below AB with their slopes becoming steeper as $|y - 1|$ increases.

Figure 5: Coase independence without uniform linearity



It follows that the public choice $y = 1$ is always Pareto optimal and therefore the economy satisfies strong Coase independence. Although there is strong Coase independence, the y-contingent utility possibility frontiers are not parallel and preferences are not uniformly affine in private goods.

Although the economy in Example 4 displays Coase independence when producing the public good is costless, it turns out that if the unit cost of the public good is $c > 0$, there is neither weak nor strong Coase independence. To show this, we note that any allocation (x_1, x_2, y) that is an interior Pareto optimum for this economy must satisfy the "Samuelson condition", equating the sum of marginal rates of substitution to the marginal cost of the public good. For this example the Samuelson condition is

$$-2(y - 1)(\alpha_1 x_1 + \alpha_2 x_2) = c. \tag{11}$$

The allocation $(x_1, x_2, y) > 0$ is Pareto optimal only if it satisfies Equation 11. If $c > 0$, then Equation 11 is satisfied only if $0 < y < 1$. If $0 < y < 1$, then since $\alpha_1 > \alpha_2$, the sum of marginal rates of substitution changes when income is redistributed. Therefore any public choice that is optimal for one income distribution is not optimal after income is redistributed. It follows that in this case, there is no always Pareto optimal public choice. Thus the economy does not satisfy weak Coase independence.

In this example with zero production cost there is Coase independence, although contingent utility possibility sets are not parallel. However, if we alter the technology without changing preferences, Coase independence fails. This example reflects a more general result, namely that if an economy is to have Coase independence regardless of technology and costs, then it must be that contingent utility possibility sets are parallel and hence that preferences can be represented by utility functions that are uniformly affine in private goods.

5.3 General Coase independence

Example 4 shows that for a given set of preferences and technological possibilities, there can be Coase independence without parallel y-contingent utility possibility sets. Proposition 2 and Corollary 1 inform us that if preferences are uniformly affine in private goods, then there will be weak Coase independence for *every* economy in which feasible technology is specified by some continuous cost function $c(y)$. As it turns out, for this to be the case for every such technology, y-contingent utility possibility sets must be parallel and thus preferences must necessarily be uniformly affine in private goods.

Definition 12 (General Coase Independence)**.** *Let $\bar{y} \in Y$ be a reference public choice, and let $\bar{Y}$ be the set of bary-admissible public choices. A set of n consumers, with preferences $\succeq_1, \ldots, \succeq_n$ satisfies general Coase independence with respect to $\bar{y}$ if there is weak Coase independence for every economy in which consumers have these preferences and where the set of feasible outcomes is*

$$\mathcal{F} = \{(x_1, \ldots, x_n, y) \geq 0 | \sum_i x_i = W - c(y) \text{ and } y \in \bar{Y}\}$$

where $c(\cdot)$ is a continuous function.

Proposition 6. *Suppose that for all consumers, preferences are continuous and monotonic in private goods, and suppose that there is general Coase independence with respect to the reference vector $\bar{y}$. Then it must be that for all $y \in \bar{Y}$, the y-contingent utility possibility sets are parallel.*

Proof. For each i, let $\bar{u}_i(x_i, y)$ be the money metric utility function defined with reference public choice vector $\bar{y}$. It is immediate from the definition of $u_i(\cdot, \cdot)$ that $\bar{u}_i(x_i, \bar{y}) = x_i$ for all $x_i \geq 0$. It then follows that for any W and any function $c(y)$, the $\bar{y}$-contingent utility possibility set is the set $\{(u_1, \ldots, u_n) | \sum u_i = W - c(\bar{y})\}$.

Suppose that y-contingent utility possibility sets are not parallel on $\bar{\mathcal{F}}^+$. Then for some $\hat{y}$ there exist two allocations $(x_1, \ldots, x_n, \hat{y})$ and $(x'_1, \ldots, x'_n, \hat{y})$ in $\bar{\mathcal{F}}^+$ such that $\sum x'_i = \sum x_i$ and $\sum_i \bar{u}_i(x'_i, \hat{y}) > \sum_i \bar{u}_i(x_i, \hat{y})$.

Consider the economy with only two possible public choices $\hat{y}$ and $\bar{y}$. General Coase independence requires that for any cost function, one of the two public choices $\hat{y}$ or $\bar{y}$ is always Pareto optimal. Let us choose the function $G(\cdot)$

$$\sum_i \bar{u}_i(x'_i, \hat{y}) > G(\bar{y}) > \sum_i \bar{u}_i(x_i, \hat{y}) \tag{12}$$

Now the $\bar{y}$-contingent utility possibility set is $\{(u_1, \ldots, u_n) | \sum u_i = G(\bar{y})\}$. Since according to Expression 12, $\sum_i \bar{u}_i(x'_i, \hat{y}) > G(\bar{y})$, it must be that the public choice $\bar{y}$ is not always Pareto optimal. But since we also have $G(\bar{y}) > \sum_i \bar{u}_i(x_i, \hat{y})$, it follows that the public choice $\hat{y}$ is also not always Pareto optimal. Therefore this economy has no always Pareto optimal allocation.

It follows that if y-contingent utility possibility frontiers are not parallel for all $y \in Y$, then there is not general Coase independence. Hence if there is general Coase independence, y-contingent utility possibility frontiers must be parallel on the set of feasible allocations that are the $\bar{y}$-extremes.

$\square$

We have shown that y-contingent utility possibility frontiers are parallel if and only if preferences can be represented by utility functions that are uniformly affine in private goods. Thus we have the following corollary.

Corollary 2. *If preferences of all consumers are continuous and y-contingent utility possibility sets are parallel for all $y \in Y$ and $W > c(y)$, then it must be that preferences of all consumers i can be represented by utility functions of the form*

$$A(y)x_i + B^i(y).$$

6 Coase independence with many private goods

So far we have considered Coase independence for economies in which, though there may be many public goods, there is only one private good.

In a paper called "When do Market Games Have Transferable Utility?", Bergstrom and Hal Varian [6] found necessary and sufficient conditions for "transferable utiity" in a pure exchange economy with many private goods and no public goods.

They defined transferable utility for an exchange economy so as to require that utility representations could be found such that for any aggregate initial endowment ω of private goods, the set of interior Pareto optimal utility distributions consist of all $(u_1, \ldots, u_n)$ such that $\sum_{i=1}^{n} u_i = F(\omega)$ for some function F. In the case of two consumers, transferable utility would mean that the utility possibility frontiers conditional on aggregate endowment vectors ω would be parallel straight line segments all with slope -1. Bergstrom and Varian show that when preferences are continuous, there will be transferable utility for ane exchange economy if and only if indirect utility of each consumer i can be represented in the Gorman polar form, $v(p, m_i) = \alpha(p)m_i + \beta_i(p)$.

In a later paper [3], Bergstrom remarked that with many private goods and many public goods there will be transferable utility if and only if there exist functions $\alpha(p, y)$ and $\beta_i(p, y)$ such that when the vector of public good is y, indirect utility of each consumer i with price vector p and income m_i can be represented by a function

$$V(y, p, m_i) = \alpha(p, y)m_i + \beta_i(p, y). \tag{13}$$

Elisabeth Gugl [12], in a paper titled "Transferable utility in the case of many private and many public goods", noted that Bergstrom [3] does not formally prove this assertion, but "claims that the result is obvious by combining the Bergstrom-Varian (many private goods) and the Bergstrom-Cornes (many public goods, one private goods) results." After further thought, I agree with Gugl that this result is not so obvious and that both its statement and proof deserve clarification. Gugl and Chiappori [7] wrote a follow-up paper in which they prove that for a price-taking household in a competitive economy, there is transferable utility within the household if and only if all household members have indirect utility functions of the form shown in Equation 7.

To relate the Chiappori-Gugl results to the current paper, we need to define "transferable utility" in such a way that it coincides with the property of parallel contingent utility possibility sets as in Definition 6. In this discussion, we will follow Chiappori and Gugle in confining our discussion to the "small country" case of a set of consumers who are price-takers in a competitive economy.[5]

We will assume that technological possibilities satisfy the following:[6]

Assumption 3 (Technology with many goods)**.** *A community has k consumers, m public goods and n private goods. There is a compact set $Y \subset \Re^m_+$ of possible public choices. There is a competitive price vector $p \in \Re^n_+$ for private goods. There is a function $c(y,p)$ which is convex in y and which represents the cost of purchasing the vector y of public goods when the price vector for private goods is p. Each individual i has initial wealth $W_i > 0$, and total wealth of all community members is $W = \sum_i W_i$.*

Definition 13 (Feasible allocations contingent on p)**.** *The set of feasible allocations contingent on price vector p is*

$$\mathcal{F}(p) = \left\{ (x_1, \ldots, x_n, y) | (x_1, \ldots, x_n) \geq 0, y \in Y, \text{ and} p \sum_i x_i = W - c(y) \right\}.$$

We define the (y,p)-contingent utility possibility set to be the set of utility distributions that can be achieved by possible distributions of income if the price vector is p and the vector of public goods is y.

Definition 14. *The (y,p)-contingent utility possibility frontier consists of all utility distributions $(u_1, \ldots, u_k)$ such that for each i, $u_i = u_i(x_i, y)$ such that $(x_1, \ldots, x_n, y)$ belongs to the set $\mathcal{F}(p)$ contingent on price vector p*

Definition 15 (Transferable utility with many private and many public goods)**.** *There is transferable utility with many private and many public goods if there exist continuous utility functions $u_i(x_i, y)$ representing preferences of each consumer i such that for all $y \in Y$ and for all price vectors $p > 0$, the (y,p)-contingent utility possibility frontier is contained in a set of the form*

$$\{(u_1, \ldots, u_k) | \sum_i u_i = F(y,p)\}.$$

Definition 16 (Extended indirect utility)**.** *The extended indirect utility function of Consumer i is defined to be $v_i(p, m_i, y)$ where $v_i(p,y) = \max_{px_i \leq m_i} u(x_i, y)$,*

Proposition 7. *There is transferable utility with many private and many public goods if and only if there exist continuous functions $\alpha(p,y)$ and $\beta_i(p,y)$ such that extended indirect utility function of each consumer i can be represented by a utility function of the form*

$$v_i(p, m_i, y) = \alpha(p,y) m_i + \beta_i(p,y).$$

Proof. If indirect utility is representable by a utility function of the form 7, it follows that

$$\sum v_i(p, m_i, y) = \alpha(p,y) \sum_i m_i + \sum_i \beta_i(p,y). \tag{14}$$

□

[5]In contrast, Bergstrom and Varian [6] treat a closed exchange economy, where competitive equilibrium prices are determined by the aggregate endowment vector. This approach could be extended to an economy with public goods, but to do so we would need to add a more detailed theory of production than is needed under price-taking.

[6]Chiappori and Gugle assume that public goods are purchased at constant unit prices and hence additively separable and linear in quantities. Our assumption generalizes this by allowing the possibility that the cost of a bundle of public goods to be any function that is convex in y, and hence allowing the possibility of complementarity or substitutability in the production of public goods.

7 Applications with externalities

Here we consider two examples with Coase independence despite the presence of income effects. In the first example, there is an externality that is offensive to the wealthy, but agreeable to the poor. In the second example, individual citizens gain from producing an externality like congestion, noise, or air pollution, but all suffer damages that depend on the sum of individual actions.

7.1 Coase's Fish and Chips

Coase [10] (p 21) described a court case in which a fried fish shop in a "predominantly working class district was set up near houses 'of a much better character' ". Occupants of these houses sought to close the shop on grounds of the "odour and fog or mist" emitted. The judge ruled that the shop must be moved, but could be allowed to locate near houses of less high character, whose occupants would be likely to find that the convenience of proximity would more than compensate for any adverse aromatic effects.

Coase's fried fish story is clearly not consistent with quasi-linear utility, since aversion to the smell of fish and chips is assumed to increase with income. Nevertheless, we can construct an economy that is qualitatively similar to Coase's fish and chips case and also exhibits strong Coase independence.

A community has n people and a fish and chips store. Let y be the number of hours per year that the store is open. The quantity of private goods consumed by consumer i is x_i. Let $W = \sum x_i$ be total income and let $\bar{W} = W/n$ be average per capita income of members the community. Consumers have identical utility functions of the form

$$u_i(x_i, y) = A(y)x_i + B(y), \tag{15}$$

where $u_i(\cdot,\cdot)$ is strictly quasi-concave and where $A(y) > 0$, $A'(y) < 0$, $B'(y) > 0$, and with $A''(y) < 0$ and $B''(y) < 0$.

Remark 1. *For the community described in the previous paragraph, there is strong Coase independence. The greater is total income in the community, the shorter will be the Pareto optimal number of hours for the fish shop to be open. If the number of opening hours is set at the Pareto optimal level, then all citizens with income less average will favor longer hours and those with income greater than average will prefer shorter hours.*

Proof. Since preferences are uniformly affine in private goods and strictly convex, it follows from Proposition 4 that there is strong Coase independence. Let W be total community income and $\bar{W} = W/n$ be mean income. income. The number of hours y^* that is always Pareto optimal is the value of y that maximizes

$$\begin{aligned}\sum_{i=1}^{n} u_i(x_i, y) &= A(y)\sum_{i=1}^{n} x_i + nB(y) \\ &= A(y)W + nB(y), \end{aligned} \tag{16}$$

Taking a derivative, we see that this expression is maximized when

$$-\frac{B'(y^*)}{A'(y^*))} = \bar{W} \tag{17}$$

The assumptions that $B''(y) < 0$ and $A''(y) < 0$ imply that the right side of Equation 17 is decreasing in y. It then follows that the Pareto optimal number of shop hours, y^* is a decreasing function of mean income.

The derivative of Citizen i's utility with respect to number of hours is

$$\frac{\partial u_i(x_i, y^*)}{\partial y} = A'(y^*)x_i + B'(y^*). \tag{18}$$

Since, by $A(y^*)$ is assumed to be positive, it must be that this derivative is positive if $x_i > \bar{W}$ and negative if $x_i < barW$. It follows that if the number of hours that the shop is open is Pareto optimal, then those whose income is above the community mean wish that it would be open fewer hours and those with income below the mean would wish it to be open for more hours.

□

7.2 Congestion, Noise, and Pollution

Negative externalities often result from the actions of many individuals, each of whom benefits from taking actions that adversely affect all others in the community. Where a community has n residents, we model the public choice as a vector $y = (y_1, \ldots, y_n)$, where y_i is the level of activity by resident i.

According to Proposition 4, there is weak Coase independence if preferences of each resident i can be represented by the functional form

$$u_i(x_i, y) = A(y)x_i + B_i(y). \tag{19}$$

For examples such as congested highways, cacophony in a busy restaurant, or exhaust fumes in the air, the adverse effects imposed by individual activities can be modeled as additive, with total damage determined by the sum, $Y = \sum_{j=1}^n y_j$. Let us assume that the utility functions in 19 take the special form

$$u_i(x_i, y) = A\left(\sum_{j=1}^n y_j\right) x_i + B_i(y_i).,\tag{20}$$

where where the function $A(\cdot)$ and $B_i(\cdot)$ are concave and continously differentiable, with $A(Y) > 0$, $A'(Y) < 0$, and such that $B_i(y_i)$ is maximized at some $\bar{y}_i > 0$.

Assume that the community has total wealth W which can be divided in any way such that $x_i > 0$ for all residents i and $\sum_i x_i = W$. It follows from Proposition 4 that this community has strong Coase independence, and has a unique Pareto optimal solution $y^* = (y_1^*, \ldots, y_n^*)$ such that y^* maximizes

$$\sum_{i=1}^n u_i(x_i, y) = A\left(\sum_{j=1}^n y_j\right) W + \sum_{i=1}^n B_i(y_i). \tag{21}$$

Where $Y^* = \sum_{i=1}^n y_i^*$, expression 21 has an interior maximum at y^* if for all i,

$$B_i'(y_i^*) = -A'(Y^*)W \tag{22}$$

Differentiating both sides of Equation 22, and rearranging terms, we find

$$\frac{dy^*}{dW} = \frac{-A'(y^*)}{A''(y^*)W + B_i''(y_i^*)}. \tag{23}$$

Since, we have assumed that $A'(\cdot) < 0$ and that $A(\cdot)$ and $B_i(\cdot)$ are concave functions it follows that $\frac{dy_i^*}{dW} < 0$. Thus, for this community, although the Pareto optimal level of activity for each resident is independent of income distribution, an increase in total wealth implies a lower Pareto optimal level of activity for each resident.

Equation 22 shows that at an efficient outcome, the marginal benefit gained from externality-generating activity must be positive and the same for all residents. As we will see, this outcome can be approximately achieved by means of a uniform tax rate on the activity, with tax revenue rebated in equal shares to all residents. Much as in a large competitive economy, consumers can reasonably neglect the effect of their own purchases on equilibrium prices, residents of a large community, acting in their own self-interest, can neglect the effect of their own externality-generating activity on the aggregate level. Suppose that resident i has initial wealth w_i and the externality-generating activity is taxed at the rate t, with tax revenues divided equally among residents. Then the utility of resident i who chooses activity level y_i will be

$$u_i(x_i, y_i, Y^*) = A(Y^*)\left(w_i - ty_i + \frac{\sum_j ty_j}{n}\right) + B_i(y_i). \tag{24}$$

The first order condition for maximization of i's utility with respect to y_i is then

$$A(Y^*)t(\frac{n-1}{n}) = B_i'(y_i^*). \tag{25}$$

It follows from Equation 22, that resident i will choose the Pareto optimal level of activity y_i^* if there is a uniform tax rate t where

$$t = \frac{-A'(Y^*)}{A(Y^*)}\left(\frac{n}{n-1}\right). \tag{26}$$

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