# Lindahl's Cost-sharing Game

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## 1 Introduction

Most economists are familiar with "Lindahl equilibrium" in public goods models. In a Lindahl equilibrium, each consumer pays the unit costs of each public good are distributed in such a way that all desire the same level of total expenditure on each public good.[1]

This allocation is described in Lindahl's celebrated work, "Just Taxation–A Positive Solution," and was introduced to a broad audience in Musgrave's classic public finance text.[2]. But this is not the only equilibrium notion that Lindahl discussed in this paper. Lindahl presents another concept of "equilibrium" which he motivates by a discussion of bargaining between two "parties", A and B. This equilibrium is the output of what today would be thought of as a simple sequential two-player game with complete information. Player A decides the fraction of the cost of the public good that will be paid by each player. Player B then decides on the quantity of the public good that will be supplied, and the costs are divided as specified by Player A.

### 1.1 Lindahl's Game

In Lindahl's discussion, there is a single public good and a single private good. The production cost per unit of public good is one unit of private good. Both players have quasi-linear utility functions with diminishing marginal utility of the public good. Player $A$ and $B$ have initial endowments of $W_A$ and $W_B$ units of private good, respectively. Where $Y$ is the amount of public good supplied and $X_i$ is the amount of public goods consumed by $i$, players $A$ and $B$ have utility functions

$$U_A(Y, X_A) = f(Y) + X_A \tag{1}$$

and

$$U_B(Y, X_B) = \phi(Y) + X_B \tag{2}$$

If Player $A$ chooses to pay the share $s$ of the cost of the public good, then for Player B, the cost per unit of public good is $1 - s$. Given that Player $A$ has chosen $s$, if Player $B$ chooses $Y$, her utility will be

$$\phi(Y) + W_B - (1 - s)Y \tag{3}$$

Player $B$ will therefore choose $Y(s)$ so that

$$\phi'(Y(s)) = 1 - s. \tag{4}$$

Then if Player $A$ selects the cost share $s$ for himself, the amount of public good will be $Y(s)$ and the total amount he will have to pay for this public good will be $sY(s)$. Then his utility will be

$$\tilde{U}(s) = f(Y(s)) + W_A - sY(s) \tag{5}$$

and his utility will be maximized at $s \in [0,1]$ if

$$\tilde{U}'(s) = f'(Y(s))\,Y'(s) - sY'(s) - Y(s) = 0. \tag{6}$$

and $\tilde{U}''(s) < 0$.

To find the equilibrium $Y$ and $s$, we note that $Y$ and $s$ must satisfy Equation 4 since Party $B$ chooses $Y$ and must satisfy Equation 6 since Party $A$ chooses the $s$ that results in the best outcome for him.

### Lindahl's Linear Symmetric Example

Lindahl discusses the special case in which there is quasi-linear utility, marginal utility is linear in $Y$ and the two parties have identical utility functions. In this case, for $Y \geq 0$,

$$f'(Y) = \phi'(Y) = \alpha - \beta Y. \tag{7}$$

From Equation 4 it follows that if $\alpha + s - 1 \geq 0$, then

$$Y(s) = \frac{\alpha + s - 1}{\beta} \text{ and } Y'(s) = \frac{1}{\beta} \tag{8}$$

Substituting from Equations 7 and 8 into Equation 6, we find that

$$\begin{aligned} \tilde{U}'(s) &= \left(\alpha - \beta\left(\frac{\alpha + s - 1}{\beta}\right)\right)\frac{1}{\beta} - \frac{s}{\beta} - \left(\frac{\alpha + s - 1}{\beta}\right) && (9) \\ &= \frac{2 - 3s - \alpha}{\beta} && (10) \end{aligned}$$

and

$$\tilde{U}''(s) = -\frac{3}{\beta} < 0. \tag{11}$$

Therefore $\tilde{U}(s)$ is maximized on $[0,1]$ if and only if

$$s = \frac{2 - \alpha}{3} \tag{12}$$

We see from Equation 12 that $0 \leq s \leq 1$ if and only if $\alpha < 2$. If Party $A$ must choose $s \geq 0$, then Party $A$ maximizes his payoff by choosing $s = 0$. That is, Party $A$ would commit to making no contribution to the supply of the public good. In fact, when $\alpha > 2$, if Party $A$ is able to "steal" some fraction of the public good provided by Party $B$, he would choose to do so.[1]

Lindahl did not complete his example by displaying the amount of public goods $Y(s)$ that results from this process. When $s$ satisfies Equation 12, it must be that the amount of public goods provided is

$$Y(s) = \frac{\alpha + s - 1}{\beta} = \frac{2\alpha - 1}{3\beta} \tag{13}$$

Examination of Equation 13 reminds us that we have one more non-negativity constraint to deal with. This equation gives us the equilibrium solution for $Y$ if and only if $Y(s)$ is nonnegative, which requires that $\alpha \geq 1/2$. In fact, we notice that if $\alpha < 1/2$, then for any positive amount of public good $Y$, the sum of the two parties' marginal rates of substitution between public and private goods is less than 1. Thus there is no positive quantity of public goods that both would agree to, no matter how the costs are divided.

Thus we find that for $\alpha \in [1/2, 1]$, Lindahl's bargaining solution is given by $s$ and $Y(s)$ satisfying Equations 12 and 13. For $\alpha < 1/2$, a there is a solution with an arbitrary choice of $s$ and with $Y = 0$. For $\alpha > 2$, the solution is $s = 0$ and $Y = \alpha/\beta$.

The traditional "Lindahl equilibrium" for this economy is one in which costs are shared in such a way that both choose the same amount of public good. From the symmetry of this problem, it is apparent that this occurs when $s = 1/2$. With $s = 1/2$, both consumers maximize their payoffs when $\alpha - \beta Y = 1/2$, which implies that

$$Y = \frac{2\alpha - 1}{2\beta} \tag{14}$$

Thus we see from Equations 13 and 14 that the quantity supplied in equilibrium for Lindahl's cost-sharing game is exactly 2/3 as large as that supplied in ordinary Lindahl equilibrium.

# References


[1] Erik Lindahl. Just taxation-a positive solution. In Richard Musgrave and Alan Peacock, editors, *Classics in the Theory of Public Finance*, pages 98–123. Macmillan, London, 1958.

[2] Richard Musgrave. *The Theory of Public Finance*. McGraw Hill, New York, 1959.


[1] Lindahl observed that if $\alpha > 2$, the solution for $s$ in Equation 12 would be negative. He says that in this case "it would be most advangageous for party $A$ if it could make its agreement to the public expenditures in question contingent upon $B$ making a certain contribution to them in proportion to the amount of expenditures." I have not been able to figure out what Lindahl meant by this.