Global Public Goods and Coalition Formation under Matching Mechanisms

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This paper links coalition theory with matching mechanisms in the presence of global public goods among heterogeneous players. This matching coalition may achieve Pareto-improving outcomes while avoiding side payments. The paper characterizes conditions of coalition profitability and stability at both interior and corner equilibria. A matching coalition is more profitable but less stable with a larger matching rate. Empirically there is no stable coalition but this can be overcome by introducing reputation mechanisms — there always exists a stable matching coalition if players value their reputation. The matching coalition faces a trade-off between matching depth and breadth.

Keywords*: Public goods; Matching mechanisms; Coalition formation*

JEL*: C78, H41, H77, Q54*

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As the world is increasingly moving towards economic globalization and integration, more and more transboundary externalities emerge such as transnational pollution, international trade, financial crises, development aid, contagious diseases, etc. (Sandler, 2004). In the presence of such global public goods, international cooperation is required to reach optimal outcomes, but this proves difficult in practice due to the absence of a supranational government with enforcement power. The most relevant example is arguably climate negotiations which have gained much attention but little progress. To overcome underprovision of such global public goods, a large literature examines cooperative games in coalitional form (see, e.g., Carraro and Siniscalco, 1993; Barrett, 1994; Hoel and Schneider, 1997; Finus, Altamirano-Cabrera and Van Ierland, 2005).[1] A typical coalitional game often comprises of four stages: (1) Players decide whether to sign an agreement or to behave as a singleton; (2) Signatories decide on the public good provision cooperatively so as to maximize aggregate welfare of the coalition when taking the reaction of the non-signatories into account; (3) Given the public good provision of signatories, every non-signatory chooses their public good provision so as to maximize their own welfare; (4) Signatories decide how the welfare gain is distributed.

There are two major issues of this coalitional game in practice particularly at the international level. First, even if players decide to form a coalition, it is difficult to maximize aggregate welfare of the coalition because of divergent interests and incomplete information. At the international level, divergent interests involve multiple dimensions including economic benefits, political relationships, cultural background, etc., and incomplete information include numerous uncertainties as to production costs of public goods, preferences of countries, etc. One solution to mitigate this issue is to form a coalition among subgroups of like-minded players (see, e.g., Buchholz, Haslbeck and Sandler, 1998; Buchholz, Cornes and Rübbelke, 2014). In climate protection, countries that have common culture, similar incomes and national tastes are more likely to make ambitious agreements. Moreover, like-minded countries tend to have similar information in incomes, costs and preferences, and thus reduce asymmetry of information to some extent. However, one major issue in climate negotiations is the conflict between developing countries and developed countries, and these two groups diverge in many dimensions including historical responsibility, mitigation capability and current welfare. Therefore, the first issue cannot be solved by forming a coalition among like-minded countries. Instead, another realistic perspective is that countries form a coalition to achieve a less

[1] Peleg and Sudholter (2003) provide a comprehensive coverage of cooperative games; Ray (2007) presents a general theory of coalition formation that combines cooperative and non-cooperative approaches.

ambitious solution - a Pareto-improving outcome - rather than maximize aggregate welfare in the coalition, i.e., each coalition member is better off than without the coalition.

The second issue involves side payments. In the conventional coalitional game, players have to decide how to share costs or distribute gains among the coalition. This is also very difficult without a central government because players have quite subjective perceptions of equity of sharing costs or distributing gains and players are subject to political constraints at the international level. For example, developed and developing countries have agreed to protect climate on the basis of equity but it is difficult to reach consensus on equity given their different responsibility, capability, costs and benefits, etc. In addition, Summers (2007) argues that due to political constraints governments are unlikely to write substantial checks to each other pursuant to international treaties in climate change. Therefore, it would be pragmatically useful to apply a mechanism that can avoid side payments among players.

This paper considers a coalitional setting under matching mechanisms and focuses on Pareto-improving outcomes while avoiding side payments. The matching mechanisms, first suggested by Guttman (1978, 1987), work as a two-stage game. At the first stage, each agent announces a matching rate indicating by how much the agent would subsidize public good contributions of all other agents. For example, should one announce a matching rate of 0.1, the agent would provide 0.1 units of the public good as a matching contribution if another agent provides one unit of the public good. At the second stage, all agents decide independently how much of the public good they would provide. The idea of matching mechanisms is still to stick to the non-cooperative mode of public good provision but meanwhile to subsidize individual public good contributions and hence to lower the effective price of the public good. In his seminal work Guttman shows that with quasi-linear preferences the sub-game perfect equilibrium in such a two-stage game of two identical players is fully efficient. This matching approach has been refined and applied in various ways,[2] but little literature considers coalition formation under matching mechanisms. Buchholz, Cornes and Rübbelke (2014) investigate matching coalitions among homogeneous players. They assume that all players are completely homogeneous in the preference and income, and then consider a two-stage game: The coalition members cooperatively determine the matching rate at the first stage and then play a non-cooperative game against the outsiders at the second stage. The situation of homogeneous players is a very special case with symmetric results across players.

[2] See, e.g., Boadway, Pestieau and Wildasin, 1989; Danziger and Schnytzer, 1991; Althammer and Buchholz, 1993; Varian, 1994a, 1994b; Andreoni and Bergstrom, 1996; Falkinger, 1996; Boadway, Song and Tremblay, 2007, 2011; Buchholz, Cornes and Rübbelke, 2011, 2012.

This paper relaxes the condition of homogeneity and assumes that players have the same preference but allows them to have different incomes. This relaxation is obviously important given the large income heterogeneity across countries in the current world, but gives rise to several theoretical issues. First, as the outcomes are asymmetric across heterogeneous players, not only the coalition size but also the member identity matters for coalition formation. Second, given income heterogeneity, it is a key issue to make poor players better off in a matching coalition without resort to side payments. Third, there are potentially corner solutions in which a subgroup of either coalition insiders or outsiders provides no public goods. Fourth, in general, there is no optimal matching rate for a coalition because players are heterogeneous and cannot maximize their own welfare simultaneously with a common matching rate. Therefore, this paper considers a different two-stage game with an exogenous matching rate: Given a common matching rate players decide whether to join the matching coalition or to behave as a singleton at the first stage; at the second stage all players decide on the public good provision independently so as to maximize their own welfare. Compared to the typical coalitional game, there are two distinct differences in this matching coalition: One is that players do not maximize aggregate welfare of the coalition but instead maximize their own welfare given the matching rate, and the other is that players do not distribute the coalition welfare gain and thus there are no side payments. The total public good provision of the matching coalition may not be optimal from the perspective of the coalition, but it is more realistic and desirable in practice if the coalition can make every member better off and, more importantly, can avoid side payments.

This paper follows the literature on coalitional games to characterize coalition equilibria (see, e.g., D'Aspremont et al., 1983; Donsimoni, 1985; Carraro and Siniscalco, 1993; Barrett, 1994). A coalition equilibrium must meet two conditions. The first is profitability, i.e., each coalition member must be better off than without the coalition. This is required to reach any international agreements in the absence of a supranational government. The second is stability, which involves two aspects: One is internal stability, i.e., no signatory is better off by leaving the coalition, and the other is external stability, i.e., the situation of a non-signatory cannot be improved by joining the coalition. To examine stability, the paper considers whether one player inside (outside) a coalition can attain higher utility by leaving (joining) the coalition, assuming that no other player changes their decisions of coalition memberships and that the player recognizes that his membership choice affects public good contributions of all other players. However, the literature above assumes homogeneous players so that it is relatively simple to compare utility levels of joining and leaving a coalition because it only needs to consider the coalition size. Given heterogenous players, this paper needs to consider not only the coalition size but also the member identity to compare the utility levels.

The key questions in this paper include: (1) Under what conditions is a matching coalition profitable? (2) Under what conditions is a matching coalition stable? (3) In a stable coalition which players stay in and which stay out? (4) How does the matching rate affect the profitability and stability of a matching coalition? (5) If a coalition is not stable, does a reputation mechanism make a remedy? To answer these questions, at the first step, this paper considers a simple case - a marginal matching rate, and characterizes the conditions of profitability and stability of a matching coalition at interior equilibria and corner equilibria respectively. The results show that coalition profitability is favored by a large coalition, a small group of outsiders and a strong preference for public goods. At interior equilibria, if the participation rate of players is above a certain threshold, the coalition is profitable. As an extreme case, if all players join a coalition, there always exists a profitable coalition at a marginal matching rate regardless of the preference. At corner equilibria, although some poor players do not provide flat contributions, they are also likely to be better off if the total public good is sufficiently increased to compensate their forgone private good consumption due to matching contributions. The most optimistic result in the corner case is that very poor players can be better off even if income heterogeneity is quite large. On the other hand, the results also show that coalition stability is favored by a small group of players in the economy and a strong preference for public goods. However, it is empirically plausible that the stability condition does not hold, which indicates that players have incentives to leave the coalition. At the second step, the paper considers a relatively large matching rate and investigates how the matching rate affects the coalition. It is shown that a coalition is more profitable but less stable with a larger matching rate. At the third step, the paper introduces reputation mechanisms and shows the existence of a stable matching coalition if players value their reputation in coalition formation. Players would stay in the coalition when the gain of free riding is lower than the reputation loss. At the last step, based on previous findings, the paper discusses the trade-off between cooperation depth and breadth in a matching coalition.

The remainder of this paper is organized as follows. Section I provides the framework and introduces the aggregative game approach to characterize interior equilibria. Section II examines coalition profitability and stability with a marginal matching rate at interior equilibria followed by corner equilibria in Section III. Section IV considers coalition formation with a large matching rate for comparative studies, and examines the effects of matching rates on coalition formation. Section V introduces reputation mechanisms and investigates coalition stability further. Section VI explores the trade-off between depth and breadth in a matching coalition. Section VII concludes.

I. The Framework

A. The model

Consider a pure public good economy with one private good, one pure public good and $n(n \geq 3)$ players who all have the same utility function $u_i(x_i, G)$ where x_i is the private good consumption of player i and G is the total public good provision. The utility function is strictly quasi-concave, strictly increasing and twice partially differentiable in both variables. Both goods are strictly normal and indifference curves asymptote to the two axes. Player i has an initial income of w_i units of the private good. Players are ranked by income as $w_1 \geq w_2 \geq \cdots \geq w_n$, and the total income is $W = \sum_{i=1}^{n} w_i$.

Assume that there is only one coalition with a common matching rate $\mu \geq 0$. The matching rate is exogenously given, which may be set by a super power or through negotiations among a few major players or in any other way. The player set $I = \{1, 2, ..., n\}$ is divided into two groups: One subgroup, the coalition $C = \{i_1, i_2, ..., i_m\}$, consists of $m(m \leq n)$ cooperating players which provide the public good in the common matching scheme. The other subgroup $\tilde{C} = \{i_{m+1}, i_{m+2}, ..., i_n\}$ consists of all remaining non-cooperating outsiders which behave individually. The public good contribution of player $i \in C$, denoted by z_i, consists of a flat contribution y_i and of a matching contribution that player i makes by matching flat contributions of all other players in the coalition, i.e., $z_i = y_i + \mu \sum_{j=i_1, j \neq i}^{i_m} y_j$. The public good contribution of player $i \in \tilde{C}$ only consists of a flat contribution y_i. The player does not match other players' flat contributions and is not matched by other players. The flat contribution of each player both in the coalition and out of the coalition is non-negative, i.e., $y_i \geq 0$. The total public good provision is

(1)
$$G = \sum_{j=i_1}^{i_m} z_j + \sum_{j=i_{m+1}}^{i_n} y_j = (1 + (m-1)\mu) \sum_{j=i_1}^{i_m} y_j + \sum_{j=i_{m+1}}^{i_n} y_j$$

The prices of the private good and the public good are both normalized to one. Therefore, the budget constraint of player $i \in C$ is $x_i + z_i = w_i$ and that of player $i \in \tilde{C}$ is $x_i + y_i = w_i$. In the coalition, the private marginal rate of transformation between the private good and the public good for player i is $\pi_i = 1 + (m-1)\mu$, and the effective public good price that player i has to pay for an additional unit of the public good is $p_i = 1/\pi_i$.

There are two states of the economy: One is the initial Nash equilibrium without a matching coalition (hereafter "the initial equilibrium") and the other is the Nash equilibrium in a matching coalition (hereafter "the matching equilibrium"). The initial equilibrium is a special case of the matching equilibrium when $\mu = 0$. Both equilibria may involve zero flat contributions of some players under certain conditions, so it is useful to distinguish between interior equilibria and corner equilibria.

Definition 1 (i) An interior equilibrium is an equilibrium where each player chooses a positive flat contribution; (ii) A corner equilibrium is an equilibrium where at least one player chooses a zero flat contribution. Players with positive flat contributions are referred to as interior players while those with zero flat contributions are referred to as corner players.

From the Samuelson rule, given all players in the coalition, an interior matching equilibrium is optimal if and only if $\sum_{i=1}^{n} p_i = 1$ which is solved as $\mu = 1$, so this paper looks at situations in which $0 \le \mu \le 1$. As the matching rate is important in coalition formation, this paper first considers a marginal matching rate and evaluates profitability and stability of a matching coalition at the margin of the initial equilibrium, and then considers a large matching rate which deviates from zero but is still less than one.

B. Profitability and Stability

A matching coalition must satisfy profitability and stability conditions. The profitability condition requires that each coalition member must be better off than without the matching coalition, i.e., the utility level at the matching equilibrium, $u_i(x_i, G)$, is higher than the utility level at the initial equilibrium, $\bar{u}_i(x_i, G)$.

Definition 2 A matching coalition C is profitable if $u_i(x_i, G) > \bar{u}_i(x_i, G)$ for all $i \in C$.

This paper will compare profitability between two coalitions differing either in the coalition size or in the matching rate, so it is necessary to introduce the following definition.

Definition 3 A matching coalition C^1 is more profitable than another coalition C^2 if $u_i^1(x_i,G) > u_i^2(x_i,G) > \bar{u}_i(x_i,G)$, i.e., player i is better off to join C^2 and is further better off to join C^1, for all $i \in C^1 \bigcap C^2$.

To define coalition stability, we consider a marginal player. Given m players in the coalition, player $i \in \tilde{C}$ has two strategies: (1) Stay out of the coalition. Denote the public good provision by $\breve{G}(m,\mu)$, the private good consumption of player i by $\breve{x}_i$ and the utility of player i by $\breve{u}_i(\breve{x}_i,\breve{G}(m,\mu))$; (2) Join the coalition. There are $m+1$ players in the coalition including player i. Denote the public good provision by $\widehat{G}(m+1,\mu)$, the private good consumption of player i by $\widehat{x}_i = e(\widehat{G}(m+1,\mu),\bar{\pi})$ and the utility of player i by $\widehat{u}_i(\widehat{x}_i,\widehat{G}(m+1,\mu))$. A stable coalition must be both internally and externally stable, which is formalized as follows.

Definition 4 A matching coalition C of size m at a matching rate μ is stable if (i) coalition insiders have no incentives to leave the coalition, i.e., $\widehat{u}_i(\widehat{x}_i,\widehat{G}(m,\mu)) > \breve{u}_i(\breve{x}_i,\breve{G}(m-1,\mu))$ for all $i \in C$; and (ii) coalition outsiders have no incentives to join the coalition, i.e., $\widehat{u}_i(\widehat{x}_i,\widehat{G}(m+1,\mu)) < \breve{u}_i(\breve{x}_i,\breve{G}(m,\mu))$ for all $i \in \tilde{C}$.

To measure the free-riding incentive of each player in a matching coalition, this paper defines a free-riding function as follows.

Definition 5 Given m players in a matching coalition at a matching rate μ, the free-riding incentive of player $i \in \tilde{C}$ is measured by the following function

$$f_i(m,\mu) = \frac{\breve{u}_i(\breve{x}_i,\breve{G}(m,\mu))}{\widehat{u}_i(\widehat{x}_i,\widehat{G}(m+1,\mu))}$$

This function compares the utility levels of player i with two strategies given the coalition of size m with a matching rate μ. Immediately, $f_i(m,0) = 1$, i.e., the player is indifferent to joining the coalition and leaving the coalition given $\mu = 0$ because there is no coalition. If $f_i(m,\mu) < 1$, the player is better off staying in the coalition; if $f_i(m,\mu) > 1$, the player is better off staying out of the coalition and has incentives to free ride. The larger the function value, the stronger the free-riding incentive and the less stable the coalition.

In general, both profitability and stability depend on the coalition size and thus it is important to take into consideration the process of coalition formation. This paper considers two types of processes: (1) Players join a coalition sequentially. For example, in climate protection poor countries wait for rich countries to take actions first and then consider their own strategies. In this process coalition formation must be evaluated step by step. If a coalition is not profitable or stable at a certain size, it cannot expand to a larger one; (2) Players join a coalition at once. For example, poor countries and rich countries negotiate a coalition and their participation is conditional on each other. In this process coalition formation is evaluated based on the situation in which a group of players join at once. Even if a coalition is not profitable or stable at a small size, it may be profitable or stable at a large size. Put differently, a coalition is not profitable or stable until the number of coalition members reaches a certain threshold or tipping point.

Besides the notations above, this paper denotes the initial public good provision by $\bar{G}$, the marginal rate of substitution of player i between the two goods by $MRS_i = \frac{\partial u_i / \partial x_i}{\partial u_i / \partial G}$, and the initial marginal rate of substitution by $\overline{MRS_i} = MRS_i\big|_{\mu=0}$.

C. Aggregative Game Approach

This paper applies the aggregative game approach developed by Cornes and Hartley (2003, 2007) to characterize interior equilibria. While the conventional approach – the reaction function method – struggles to deal with public good models which involve more than two players, the aggregative game approach exhibits its power to avoid high-dimensional problems when the number of players increases. Let $e_i(G, \pi_i)$ denote player i 's income expansion path which is a function of the total public good provision, on which player i 's marginal rate of transformation is π_i. At an interior matching equilibrium, the following two conditions must be satisfied:

(2) $$x_i = e_i(G, \pi_i)$$

(3) $$G + \sum_{i=1}^{n} e_i(G, \pi_i) = W$$

The first condition (hereafter "the interiority optimality condition") holds because, when any player chooses a positive flat contribution, the marginal rate of substitution between the private good and the public good must be equal to the private marginal rate of transformation between the two goods so that the choice is on the income expansion path. The second condition is the aggregate budget constraint.

Unfortunately, interior equilibria only emerge for specific income distributions. Interiority of equilibria is even much harder to achieve with matching than without (Buchholz, Cornes and Rübbelke, 2011). However, corner equilibria may also be important in some situations in a public good economy.[3] This paper will show that corner equilibria greatly lower the requirement on income distribution for coalition profitability.

II. Coalition with Interior Equilibria

A. Interior Equilibria

Given the same utility function, all players have an identical functional form representing their income expansion paths denoted by $e(G,\pi_i)$. At an interior matching equilibrium,

(4) $$x_i = x_C = e(G,\pi) \qquad i \in C, \pi = 1+(m-1)\mu$$

(5) $$x_i = x_{\tilde{C}} = e(G,1) \qquad i \in \tilde{C}$$

Combining the aggregate budget constraint yields

(6) $$G + m * e(G,\pi) + (n-m) * e(G,1) = W$$

If $\lim_{G\to 0} e(G,\pi_i) = 0$ and $\lim_{G\to\infty} e(G,\pi_i) = \infty$, the existence of a public good level $\hat{G}$ is implied by the Intermediate Value Theorem. Uniqueness is ensured by the strict monotonicity of the income expansion path. Given such $\hat{G}$, the private good consumption is obtained as

(7) $$x_C = e(\hat{G},\pi), x_{\tilde{C}} = e(\hat{G},1)$$

The following proposition provides some basic findings about the matching coalition.

Proposition 1 At interior matching equilibria, given the coalition size,

(i) Neutrality holds: Income heterogeneity or income redistribution does not affect the total public good provision and the private good consumption;

(ii) The total public good provision is increasing in the matching rate;

(iii) The private good consumption of each outsider is increasing in the matching rate while that of each coalition member is decreasing in the matching rate;

(iv) All outsiders have an identical utility level and so do all coalition members;

(v) The utility of each outsider is higher than that of each coalition member.

Proof: The total public good provision depends on the total income rather than the individual income, so income heterogeneity or income redistribution among players does not affect the total public good provision. Each player's private good consumption depends only on the total

[3] For example, Bergstrom, Blume and Varian (1986) show that large income redistributions may change the contributor set of public goods leaving some players at the corner and thereby break down Warr neutrality (Warr, 1982, 1983). Itaya, de Meza and Myles (1997) prove that social welfare can be increased by creating sufficient income inequality that only the rich provide public goods. Cornes and Sandler (2000) show that it is possible to achieve Pareto-improving redistribution from non-contributors to contributors if there are both a large number of non-contributors and a large response of aggregate contributions to a change in income of contributors.

public good provision and is thus unchanged. Differentiating the aggregate budget constraint with respect to the matching rate yields

$$\frac{\partial G}{\partial \mu} = -m * \frac{\partial e(G,\pi)}{\partial \pi}\frac{\partial \pi}{\partial \mu}\left(1 + m * \frac{\partial e(G,\pi)}{\partial G} + (n-m) * \frac{\partial e(G,1)}{\partial G}\right)^{-1} \tag{8}$$

As $\partial e(G,\pi)/\partial \pi < 0$ and $\partial \pi/\partial \mu > 0$, it follows that $\partial G/\partial \mu > 0$, i.e., the total public good provision is increasing in the matching rate. As $x_{\tilde{C}} = e(G,1)$ is increasing in G, the private good consumption of outsiders is increasing in the matching rate. The aggregate budget constraint $G + m * x_C + (n-m) * x_{\tilde{C}} = W$ implies that the private good consumption of coalition members x_C is decreasing in the matching rate. $x_C = e(\hat{G},\pi)$ and $x_{\tilde{C}} = e(\hat{G},1)$ indicate that all coalition members have an identical utility level and so do all outsiders. As $x_C = e(\hat{G},\pi)$ is decreasing in π, the utility of each outsider is higher than that of each insider. QED

Neutrality is a central topic in the theory of public goods (see, e.g., Warr, 1982, 1983; Bergstrom, Blume and Varian, 1986; Cornes and Sandler, 1996). The neutrality immediately leads to the following result in a matching coalition in this paper.

Corollary 1 At interior matching equilibria, the private good consumption and the total public good provision depend on the coalition size but are irrelevant to the member identity.

B. Coalition Profitability

This section investigates under what conditions coalition members are better off. Consider the utility change of player i with respect to a marginal matching rate. As all coalition members have an identical utility level at interior equilibria, the subscript i is dropped in notations for convenience.

$$\left.\frac{\partial u}{\partial \mu}\right|_{\mu=0} = \frac{\partial u}{\partial x}\left(\frac{\partial e(G,\pi)}{\partial G}\frac{\partial G}{\partial \mu} + \frac{\partial e(G,\pi)}{\partial \pi}\frac{\partial \pi}{\partial \mu}\right) + \frac{\partial u}{\partial G}\frac{\partial G}{\partial \mu} \tag{9}$$

The interiority optimality condition implies

$$\frac{\partial u / \partial x}{\partial u / \partial G} = \pi \Rightarrow \left.\frac{\partial u}{\partial x}\right|_{\mu=0} = \left.\frac{\partial u}{\partial G}\right|_{\mu=0} \tag{10}$$

The aggregate budget constraint $G + m * e(G,\pi) + (n-m) * e(G,1) = W$ implies

$$\left.\frac{\partial G}{\partial \mu}\right|_{\mu=0} = -\frac{m(m-1) * \partial e(G,\pi) / \partial \pi}{1 + n * \partial e(G,1) / \partial G} \tag{11}$$

Combining the above equations yields

(12) $$\left.\frac{\partial u}{\partial \mu}\right|_{\mu=0} = (m-1)\frac{\partial u}{\partial G}\frac{\partial e(G,\pi)}{\partial \pi}\left(1-m\left(\frac{\partial e(G,1)}{\partial G}+1\right)\left(1+n*\frac{\partial e(G,1)}{\partial G}\right)^{-1}\right)$$

This generates the profitability condition by setting $\partial u/\partial \mu|_{\mu=0} > 0$ as follows.

Proposition 2 At interior equilibria a matching coalition of size m at a marginal matching rate is profitable if the following condition holds

$$(n-m)\frac{\partial e(G,1)}{\partial G} < m-1$$

The term $\partial e(G,1)/\partial G$ is the slope of the income expansion path and captures the preference between the private good and the public good. Given a CES utility function $u_i(x_i,G) = \left(ax_i^{\rho} + (1-a)G^{\rho}\right)^{1/\rho}$ where $0 < a < 1$ and $\rho \le 1$, then $\frac{\partial e(G,1)}{\partial G} = \left(\frac{a}{1-a}\right)^{1/(1-\rho)}$ where a is the preference intensity between the two goods and $1/(1-\rho)$ is the elasticity of substitution. Therefore, the stronger the preference for the public good, the smaller the term. The effect of the elasticity of substitution is ambiguous depending on the preference intensity.

This proposition indicates that coalition profitability is favored by a large coalition, a small group of outsiders and a strong preference for the public good. Besides, if players value the public good more (less) than the private good, coalition profitability is favoured by a large (small) elasticity of substitution between the two goods. The proposition also suggests that profitability is independent of income distribution due to neutrality at interior equilibria.

Rearranging the profitability condition leads to

(13) $$\frac{m}{n} > 1 - \frac{1-1/n}{\partial e(G,1)/\partial G + 1}$$

The ratio m/n is the participation rate of the coalition in the whole economy and the right side of the condition is referred to as the participation threshold (or the tipping point). The participation rate must be above this threshold to make the coalition profitable. The threshold is determined by the preference and the number of players in the economy, and it has to be larger if there are fewer players in the economy and players have weaker preferences for the public good. Table 1 provides some numerical examples in the CES function. There are several implications. First, the threshold is sensitive to the preference intensity over the public good but not sensitive to the number of players in the economy. Second, the elasticity of substitution between the two goods have opposite effects on the threshold depending on the preference intensity over the public good. Third, empirically assuming $\alpha \ge 0.5$, at least half of players have to join the coalition for profitability. Fourth, as the threshold is always less than one, if all players join a coalition, they are always better off at a marginal matching rate regardless of the preference over the public good and of the number of players in the economy, which is summarized in Corollary 2.

TABLE 1 PARTICIPATION THRESHOLDS IN THE CES UTILITY FUNCTION

$1/(1-\rho)$		1/2				1				2			
n		10	20	50	100	10	20	50	100	10	20	50	100
$\frac{a}{1-a}$	0.1	0.32	0.28	0.26	0.25	0.18	0.17	0.11	0.1	0.11	0.06	0.03	0.02
	0.5	0.47	0.44	0.43	0.42	0.4	0.37	0.35	0.34	0.28	0.24	0.22	0.21
	1	0.55	0.53	0.51	0.51	0.55	0.53	0.51	0.51	0.55	0.53	0.51	0.51
	5	0.72	0.71	0.70	0.69	0.85	0.84	0.84	0.84	0.97	0.96	0.96	0.96
	10	0.78	0.77	0.76	0.76	0.92	0.91	0.91	0.91	0.99	0.99	0.99	0.99

Note: The CES function is degenerated to a Cobb-Douglas function when the elasticity of substitution is equal to one.

Corollary 2 At interior equilibria, a grand matching coalition at a marginal matching rate is always profitable regardless of the preference over the public good and of the number of players in the economy.

This conclusion shows the universal existence of a profitable matching coalition at interior equilibria. The interiority condition requires $y_i > 0$, i.e., $w_i > e(G(W),1)$. This condition implies that the individual income must be sufficiently large relative to the total income. Put differently, income heterogeneity cannot be too large to generate interior equilibria. If it is too large, rich players would provide relatively large contributions to the public good. As poor players have very small incomes, the marginal rate of substitution between the two goods would be very large. Given the public good provision of rich players, poor players would not provide any contribution.

Given a Cobb-Douglas utility function $u(x_i, G) = x_i^{\alpha} G$, the public good contribution at the initial equilibrium is solved as $y_i = w_i - \frac{\alpha}{1+n\alpha} W$. Interiority requires $y_i > 0$, i.e., $\frac{w_i}{W} > \frac{\alpha}{1+n\alpha}$. This is the minimum income share, and the maximum income share is obtained as $\frac{w_i}{W} < 1-(n-1)\frac{\alpha}{1+n\alpha} = \frac{1+\alpha}{1+n\alpha}$. Therefore, the maximum income ratio of two players is $\frac{1+\alpha}{\alpha}$.

Although Corollary 2 reveals an optimistic result for potential cooperation in this matching context, coalition profitability depends on participation of players. Even if players are better off staying in the coalition compared to the initial equilibrium, they may enjoy even higher utility if they leave the coalition given other players staying in the coalition. The following section investigates coalition stability at interior equilibria.

C. Coalition Stability

A stable coalition must be both internally and externally stable, i.e., no insider is better off by leaving the coalition and no outsider is better off by joining the coalition. Given m players in the coalition, player $i \in \tilde{C}$ has two strategies.

(i) The player stays out of the coalition. The public good provision, $\breve{G}(m,\mu)$, is uniquely characterized by the aggregate budget constraint as

(14) $$G + m * e(G,\breve{\pi}) + (n-m) * e(G,1) = W, \; \breve{\pi} = 1 + (m-1)\mu$$

At this interior equilibrium, the private good consumption of player i is $\breve{x}_i = e(\breve{G}(m,\mu),1)$.

(ii) The player joins the coalition. The public good provision, $\widehat{G}(m+1,\mu)$, is uniquely characterized by the aggregate budget constraint as

(15) $$G + (m+1) * e(G,\widehat{\pi}) + (n-m-1) * e(G,1) = W, \; \widehat{\pi} = 1 + m\mu$$

At this interior equilibrium, the private good consumption of player i is $\widehat{x}_i = e(\widehat{G}(m+1,\mu),\widehat{\pi})$.

The following proposition characterizes the stability condition at interior equilibria.

Proposition 3 At interior equilibria a matching coalition of size m at a marginal matching rate is stable if the following condition holds

$$(n-2)\frac{\partial e(G,1)}{\partial G} < 1$$

Proof: Consider the free-riding function $f_i(m,\mu)$ at a marginal matching rate.

(16) $$\left.\frac{\partial f(m,\mu)}{\partial \mu}\right|_{\mu=0} = \frac{1}{\bar{u}(x,G)}\left(\frac{\partial \breve{u}(\breve{x},\breve{G})}{\partial \mu} - \frac{\partial \widehat{u}(\widehat{x},\widehat{G})}{\partial \mu}\right)$$

$$= \frac{1}{\bar{u}(x,G)}\left(\frac{\partial \breve{u}(\breve{x},\breve{G})}{\partial \breve{x}}\frac{\partial e(\breve{G},1)}{\partial \mu} + \frac{\partial \breve{u}(\breve{x},\breve{G})}{\partial \breve{G}}\frac{\partial \breve{G}}{\partial \mu} - \frac{\partial \widehat{u}(\widehat{x},\widehat{G})}{\partial \widehat{x}}\left(\frac{\partial e(\widehat{G},\widehat{\pi})}{\partial \widehat{G}}\frac{\partial \widehat{G}}{\partial \mu} + \frac{\partial e(\widehat{G},\widehat{\pi})}{\partial \widehat{\pi}}\frac{\partial \widehat{\pi}}{\partial \mu}\right) - \frac{\partial \widehat{u}(\widehat{x},\widehat{G})}{\partial \widehat{G}}\frac{\partial \widehat{G}}{\partial \mu}\right)$$

The aggregate budget constraints imply

(17) $$\left.\frac{\partial \breve{G}}{\partial \mu}\right|_{\mu=0} = -\frac{m(m-1) * \partial e(G,\pi)/\partial \pi}{1 + n * \partial e(G,1)/\partial G}$$

(18) $$\left.\frac{\partial \widehat{G}}{\partial \mu}\right|_{\mu=0} = -\frac{(m+1)m * \partial e(G,\pi)/\partial \pi}{1 + n * \partial e(G,1)/\partial G}$$

Together with the interiority optimality condition,

(19) $$\left.\frac{\partial f(m,\mu)}{\partial \mu}\right|_{\mu=0} = \frac{1}{\bar{u}(x,G)}\frac{\partial u(x,G)}{\partial G}\frac{\partial e(G,\pi)}{\partial \pi}\left(1 + n\frac{\partial e(G,1)}{\partial G}\right)^{-1} * m\left(1 - (n-2)\frac{\partial e(G,1)}{\partial G}\right)$$

As $\frac{\partial e(G,\pi)}{\partial \pi} < 0$, this generates the stability condition as

(20) $$\left.\frac{\partial f(m,\mu)}{\partial \mu}\right|_{\mu=0} < 0 \Rightarrow (n-2)\frac{\partial e(G,1)}{\partial G} < 1$$

As $f(m,0)=1$, then $f(m,\mu)<1$ given small μ. Thus, $\breve{u}(\breve{x},\breve{G}) < \hat{u}(\hat{x},\hat{G})$. QED

This stability condition has several implications. First, coalition stability is favoured by a small group of players in the economy and a strong preference for the public good. Second, coalition stability is independent of income distribution due to neutrality at interior equilibria. Third, coalition stability is independent of the coalition size, which will be explained later. Fourth, coalition stability is also independent of the process of coalition formation. Whether players join the coalition sequentially or at once does not matter for stability. Therefore, at interior equilibria in a matching coalition at a marginal matching rate, all coalition members have the same free-riding incentive regardless of the income distribution, the coalition size and the formation process.

The reason that coalition stability is independent of the coalition size is that the stability is evaluated for one marginal player at a marginal matching rate. As one player is examined at once given all other players' memberships, the stability is determined by the utility difference given the marginal player's two strategies. At a marginal matching rate, whether or not to join the coalition, the marginal player has the same values in the marginal utility of the private good, the marginal utility of the public good and the marginal change in the private good with respect to the total public good. There are only two differences: One is the marginal change in the total public good with respect to the matching rate, and the other involves the income expansion path. When the player stays out of the coalition, the income expansion path is the same as without the coalition and the matching rate affects the private good consumption only through the total public good. However, while the player joins the coalition, the income expansion path is changed by the matching rate and the matching rate affects the private good consumption through both the total public good and the marginal rate of transformation. These two differences contribute to the utility difference given the marginal player's two strategies. As one marginal player is evaluated at once, the coalition size only scales up the difference. The sign of the difference, i.e., whether to gain or lose, is determined by the preference and the number of players in the economy.

We combine profitability and stability conditions and consider a matching coalition with the smallest size $m=2$. If the profitability condition holds, then $(n-2)\frac{\partial e(G,1)}{\partial G}<1$ holds which indicates that the stability condition also holds, so this coalition is internally stable but externally unstable. With more players joining the coalition, players are further better off and the coalition is always internally stable. This process leads to a grand coalition. On the other hand, if the profitability condition does not hold at $m=2$, then the existence of a stable coalition depends on the process of coalition formation. If players join sequentially, then there is no stable coalition. If players join at once and the number of joining players reaches the threshold, then the coalition is internally stable but externally unstable, which leads to a stable grand coalition. Therefore, at interior equilibria there is either no stable coalition or a stable grand coalition.

Proposition 4 At interior equilibria with a marginal matching rate,

(i) If $(n-2)\frac{\partial e(G,1)}{\partial G}<1$,

a) If players join sequentially, there is a stable grand coalition;
b) If players join at once, there is a stable grand coalition;

(ii) If $(n-2)\frac{\partial e(G,1)}{\partial G}>1$,

a) If players join sequentially, there is no stable coalition;
b) If players join at once, there is a stable grand coalition if the number of joining players is above the threshold, and there is no stable coalition if the number is below the threshold.

In the Cobb-Douglas example, the stability condition reads as $(n-2)\alpha<1$. Only very small α and n can sustain this condition. This indicates that players would not leave the coalition only when players value the public good much more than the private good and there are only a few players in the economy. However, it is empirically plausible that players value the private good much more than the public good, i.e., $\partial e(G,1)/\partial G>1$. Moreover, global public goods such as climate protection often involve hundreds of players. Therefore, the stability condition does not hold and players have incentives to leave the coalition.

Corollary 3 At interior equilibria, if there are $n\geq 3$ players in the economy and players value the private good more than the public good, there is no stable matching coalition at a marginal matching rate.

III. Coalition with corner equilibria

A. Corner equilibria

Interior equilibria only emerge for a narrow range of income distribution. In the Cobb-Douglas example, the maximum income ratio at interior equilibrium is $(1+\alpha)/\alpha$. Given $\alpha > 1$, the income distribution is quite narrow compared to the large income heterogeneity in the current world. If some poor players do not satisfy $y_i = w_i - e(G,1) \geq 0$, they would not provide public good contributions at the initial equilibrium. The following proposition identifies which are interior players and which are corner players.

Proposition 5 At the initial equilibrium,

(i) There exists such a unique number $k (1 \leq k \leq n)$ that the first k players provide positive public good contributions and the remaining $n-k$ players provide no contributions;

(ii) k is determined by the following conditions:

a) $w_k > e(G(k),1)$;

b) $w_{k+1} \leq e(G(k+1),1)$;

where $G(i)$ is determined by $G(i) + i * e(G(i),1) = \sum_{j=1}^{i} w_j \ (i = k, k+1)$.

Proof: If player j provides positive public good contributions, i.e., $y_j = w_j - e(G,1) > 0$, it follows that $y_i = w_i - e(G,1) > 0$ given $w_i \geq w_j$ for any player $i (i < j)$, i.e., all players with higher incomes also provide positive contributions. Suppose that the first k players provide positive contributions and the remaining $n-k$ players provide no contributions. The total public good provision depends on the number of interior players, denoted by $G(k)$, and it is characterized by the aggregate budget constraint as

$$G(k) + k * e(G(k),1) = \sum_{j=1}^{k} w_j$$

If an additional player $k+1$ also provides positive contributions, then

$$G(k+1) + (k+1) * e(G(k+1),1) = \sum_{j=1}^{k} w_j + w_{k+1}$$

As $e(G(k+1),1) < w_{k+1}$, it follows that

$$G(k+1) + k * e(G(k+1),1) > G(k) + k * e(G(k),1)$$

As $e(G,1)$ is increasing in G, it follows that $G(k+1) > G(k)$, i.e., the total public good provision is increasing in the number of interior players. Since w_i is decreasing in i and $G(i)$ is increasing in i, then $y_i = w_i - e(G(i),1)$ is decreasing in i. This monotonicity ensures uniqueness of k which satisfies, for any player $i(i \le k)$, $y_i = w_i - e(G(i),1) > 0$, and for any player $i(i > k)$, $y_i = w_i - e(G(i),1) < 0$. As $y_1 = w_1 - e(G(1),1) > 0$ always holds, then k always exists. For the two marginal players, player k provides positive contributions while player $k+1$ provides no contributions, i.e., $w_k > e(G(k),1)$ and $w_{k+1} \le e(G(k+1),1)$. QED

In the Cobb-Douglas example, the conditions in Proposition 5 read as $w_k \Big/ \sum_{j=1}^{k} w_j > \dfrac{\alpha}{1+k\alpha}$ and $w_{k+1} \Big/ \sum_{j=1}^{k+1} w_j \le \dfrac{\alpha}{1+(k+1)\alpha}$ respectively.

We now consider a coalition of size m at corner equilibria and examines a marginal matching rate so that the contributor set is unchanged, i.e., the first $k(k < m)$ players still choose positive flat contributions whether they join the coalition or stay out, and the remaining $m-k$ players in the coalition provide no flat contributions. The set of interior players is denoted by $II = \{1, 2, ..., k\}$ and is divided into two subgroups: One subgroup $CI = \{i_1, i_2, ..., i_j\}$ consists of $j(j \le k)$ interior players who join the coalition and provide positive flat contributions (hereafter "interior insiders"), and the other subgroup $\tilde{CI} = \{i_{j+1}, i_{j+2}, ..., i_k\}$ consists of the remaining interior players who stay out and provide positive flat contributions (hereafter "interior outsiders"). The set of corner players, $CC = \{i_{k+1}, i_{k+2}, ..., i_{k+m-j}\}$, consists of $s(s = m - j)$ corner players who joins the coalition and provide no flat contributions (hereafter "corner insiders"). These sets satisfy $CI \cup CC = C$, $CI \cup \tilde{CI} = II$ and $CI \cap \tilde{CI} = \varnothing$. For notation convenience, we denote $W_1 = \sum_{i=i_1}^{i_j} w_i$, $W_2 = \sum_{i=i_{j+1}}^{i_k} w_i$ and $W_3 = \sum_{i=i_{k+1}}^{i_{k+m-j}} w_i$.

B. Coalition Profitability

We first consider profitability for interior insiders. For player $i \in CI$, the utility change with respect to a marginal matching rate is

(21)
$$\left.\frac{\partial u}{\partial \mu}\right|_{\mu=0} = \frac{\partial u}{\partial x}\left(\frac{\partial e(G,\pi)}{\partial G}\frac{\partial G}{\partial \mu} + \frac{\partial e(G,\pi)}{\partial \pi}\frac{\partial \pi}{\partial \mu}\right) + \frac{\partial u}{\partial G}\frac{\partial G}{\partial \mu}$$

The aggregate budget constraint at the matching equilibrium is rearranged as

(22) $$\frac{1+(j-1)\mu}{1+(m-1)\mu}(G-(W_2-(k-j)*e(G,1)))+j*e(G,\pi)=W_1$$

The aggregate budget constraint at the initial equilibrium is rearranged as

(23) $$G-(W_2-(k-j)*e(G,1))=W_1-j*e(G,1))$$

Differentiating the first budget constraint at $\mu=0$ and then substituting the second budget constraint yields

(24) $$\left.\frac{\partial G}{\partial \mu}\right|_{\mu=0}=\frac{s(W_1-j*e(G,1))-j(m-1)*\partial e(G,\pi)/\partial \pi}{1+k*\partial e(G,1)/\partial G}$$

Combining the above equations together with the interiority optimality condition,

(25) $$\left.\frac{\partial u}{\partial \mu}\right|_{\mu=0}=\frac{\partial u}{\partial G}\left(\left(1+\frac{\partial e(G,1)}{\partial G}\right)\frac{s(W_1-j*e(G,1))-j(m-1)\partial e(G,\pi)/\partial \pi}{1+k*\partial e(G,1)/\partial G}+\frac{\partial e(G,\pi)}{\partial \pi}(m-1)\right)$$

This generates the profitability condition for interior insiders (PI) as

(26) $$s\left(1+\frac{\partial e(G,1)}{\partial G}\right)(W_1-j*e(G,1))+\left((k-j)\frac{\partial e(G,1)}{\partial G}-j+1\right)(s+j-1)\frac{\partial e(G,\pi)}{\partial \pi}>0$$

This general condition is very complicated and we first consider a special case in which $s=0$, i.e., there are no corner players. The profitability condition is reduced as

(27) $$(k-j)\frac{\partial e(G,1)}{\partial G}<j-1$$

It is not surprising that this is the profitability condition at interior equilibria, and the threshold is solved as $\frac{j}{k}>1-\frac{1-1/k}{1+\partial e/\partial G}$. The following proposition provides some intuitive insights on profitability of interior insiders at corner equilibria.

Proposition 6.1 At corner equilibria with a marginal matching rate,

(i) If the participation rate of interior players is above the threshold,

a) Interior insiders are better off at the same utility level;

b) Interior insiders are further better off if more interior players join the coalition;

c) Interior insiders are further better off if more corner players join the coalition;

(ii) If the participation rate of interior players is below the threshold,

a) Interior insiders are possible to be better off;

b) Interior insiders may be more or less likely to be better off if more interior players join the coalition;

c) Interior insiders may be more or less likely to be better off if more corner players join the coalition;

(iii)Profitability of interior insiders depends on income distribution, and interior insiders are more likely to be better off if they have larger incomes.

Proof: Denote the profitability condition of interior insiders as

$$g_{PI}(j,s)=s\left(1+\frac{\partial e(G,1)}{\partial G}\right)(W_1-j*e(G,1))+\left((k-j)\frac{\partial e(G,1)}{\partial G}-j+1\right)(s+j-1)\frac{\partial e(G,\pi)}{\partial \pi}$$

If the participation rate is above the threshold, i.e., $(k-j)\frac{\partial e(G,1)}{\partial G}-j+1<0$, then $g_{PI}(j,s)>0$ holds, so interior players are better off and, due to neutrality, they have the same utility level. It can also trivially be reached that $g_{PI}(j,s)$ is increasing in j and s, so interior insiders are further better off if more interior (corner) players join the coalition.

If the participation rate of interior players is below the threshold, $g_{PI}(j,s)>0$ is still possible to hold given $s>0$. We consider an additional interior player joining the coalition. Denote

$$\begin{aligned}\Delta g_{PI}(j)&=g_{PI}(j+1,s)-g_{PI}(j,s)\\&=s\left(1+\frac{\partial e(G,1)}{\partial G}\right)(w_{i_{j+1}}-e(G,1))-\frac{\partial e(G,\pi)}{\partial \pi}\left((s+2j-k)\frac{\partial e(G,1)}{\partial G}+s+2j-1\right)\end{aligned}$$

If this additional player has a small income so that $w_{i_{j+1}}$ is close to $e(G,1)$ and meanwhile there is a small group of interior and corner players in the coalition, then $\Delta g_{PI}(j)<0$, i.e., it is less likely to be better off with more interior players. Conversely, if this additional player has a large income and there is a large group of interior and corner players in the coalition, it is more likely to be better off with more interior players.

Then we consider an additional corner player joining the coalition. Denote

$$\begin{aligned}\Delta g_{PI}(s)&=g_{PI}(j,s+1)-g_{PI}(j,s)\\&=\left(1+\frac{\partial e(G,1)}{\partial G}\right)(W_1-j*e(G,1))+\left((k-j)\frac{\partial e(G,1)}{\partial G}-j+1\right)\frac{\partial e(G,\pi)}{\partial \pi}\end{aligned}$$

If players value the private good much more than the public good and only a small group of interior players with small incomes join the coalition, then $\Delta g_{PI}(s)<0$, i.e., it is less likely to be better off with more corner players. Conversely, if a large group of interior players join the coalition, it is more likely to be better off with more corner players. QED

At interior equilibria, if the participation rate is below the threshold, interior insiders are worse off. In contrast, at corner equilibria, even if the participation rate of interior players is below the threshold, interior insiders are still possible to be better off because a number of corner players join the coalition and provide public good contributions through matching.

Consider a special case in which $j = k$, i.e., all interior players join the coalition. The profitability condition is reduced as

$$g_{PI}(k,s) = s\left(1+\frac{\partial e(G,1)}{\partial G}\right)\bar{G} - (k-1)(k+s-1)\frac{\partial e(G,\pi)}{\partial \pi}$$

As $\partial e(G,\pi)/\partial \pi < 0$, it follows that $g_{PI}(k,s) > 0$ and hence $\partial u/\partial \mu|_{\mu=0} > 0$ always holds, which is not surprising. The previous section has shown that at interior equilibria if all players join a coalition they are always better off. Now at corner equilibria, given all interior players in a coalition, if corner players also join the coalition, they provide positive contributions through matching and hence further increase the total public good provision, so interior players are further better off.

We then consider profitability for corner insiders. For player $i \in CC$, the budget constraint is rearranged as

(28) $$x_i = w_i - \frac{\mu}{1+(m-1)\mu}(G-(W_2-(k-j)*e(G,1)))$$

Immediately, $\left.\frac{\partial x_i}{\partial \mu}\right|_{\mu=0} = -(W_1 - j*e(G,1))$ and $\left.\frac{\partial x_i}{\partial G}\right|_{\mu=0} = 0$.

The utility change of player i with respect to a marginal matching rate is

(29) $$\left.\frac{\partial u_i}{\partial \mu}\right|_{\mu=0} = \frac{\partial u_i}{\partial x_i}\left(\frac{\partial x_i}{\partial G}\frac{\partial G}{\partial \mu}+\frac{\partial x_i}{\partial \mu}\right)+\frac{\partial u_i}{\partial G}\frac{\partial G}{\partial \mu} = \frac{\partial u_i}{\partial x_i}\frac{\partial x_i}{\partial \mu}+\frac{\partial u_i}{\partial G}\frac{\partial G}{\partial \mu}$$
$$= \frac{\partial u_i}{\partial G}\left(\frac{s(W_1-j*e(G,1))-j(m-1)*\partial e(G,\pi)/\partial \pi}{1+k*\partial e(G,1)/\partial G} - \overline{MRS_i}*(W_1-j*e(G,1))\right)$$

This generates the profitability condition for corner insiders (PC) as

(30) $$\left(1+k*\frac{\partial e(G,1)}{\partial G}\right)^{-1}\left(s-\frac{\partial e(G,\pi)}{\partial \pi}*\frac{j(j+s-1)}{W_1-j*e(G,1)}\right) > \overline{MRS_i}$$

This condition leads to the following proposition.

Proposition 6.2 At corner equilibria with a marginal matching rate,

(i) Corner insiders are more likely to be better off if more corner players join the coalition, interior insiders have smaller incomes and corner insiders have larger incomes;

(ii) Corner insiders may be more or less likely to be better off if more interior players join the coalition.

Proof: Denote the profitability condition of corner insiders as

$$g_{PC}(j,s) = s - \frac{\partial e(G,\pi)}{\partial \pi} * \frac{j(j+s-1)}{W_1 - j*e(G,1)} - \overline{MRS_i} * \left(1 + k * \frac{\partial e(G,1)}{\partial G}\right)$$

$g_{PC}(j,s)$ is increasing in s and decreasing in W_1 and MRS_i, so part (i) of this proposition holds. We consider an additional interior player joining the coalition. Denote

$$\Delta g_{PC}(j) = g_{PC}(j+1,s) - g_{PC}(j,s)$$
$$= -\frac{\partial e(G,\pi)}{\partial \pi} \frac{(2j+1)(W_1 - j*e(G,1)) - j(j*w_{i_{j+1}} - j*e(G,1))}{(W_1 + w_{i_{j+1}} - (j+1)*e(G,1))(W_1 - j*e(G,1))}$$

If the existing insiders have small incomes so that $W_1 - j*e(G,1)$ is close to zero and meanwhile the additional player has a large income so that $w_{i_{j+1}} - j*e(G,1)$ is large, it follows that $\Delta g_{PC}(j) < 0$. Conversely, if the existing insiders have large incomes while the additional player has a small income, then $\Delta g_{PC}(j) > 0$. QED

Again consider the special case of $j = k$, and the profitability condition is reduced as

(31) $$\frac{(m-k) - k(m-1)*\partial e(G,\pi)/\partial \pi / \bar{G}}{1 + k*\partial e(G,1)/G} > \overline{MRS_i}$$

Given the Cobb-Douglas example, this condition leads to the following conclusion.

Corollary 4 Given $u(x_i,G) = x_i^{\alpha} G$ and k interior players at the initial equilibrium, a matching coalition consisting of the first $m(m>k)$ players at a marginal matching rate is profitable if the following conditions hold

(i) $w_m \Big/ \sum_{j=1}^{k} w_j > \frac{\alpha}{m-k+(m-1)k\alpha}$;

(ii) $w_{m+1} \Big/ \sum_{j=1}^{k} w_j < \frac{\alpha}{m+1-k+mk\alpha}$.

In such a profitable matching coalition at corner equilibria, the maximum income ratio between two players could be $\frac{(1+\alpha)(m-k+(m-1)k\alpha)}{\alpha(1+k\alpha)}$, which is much larger than the maximum income ratio at interior equilibria. The intuition is that, given all interior players in a coalition, if corner players also join the coalition and provide matching contributions, the total public good provision is sufficiently increased to compensate their forgone private consumption. Consider a Cobb-Douglas example $u(x_i,G) = x_i G$ and assume that there are 100 players with $w_i = 101 - i$. At the initial equilibrium, only the first 13 players provide contributions to the public good. However, if all players join a coalition at a marginal matching rate, they are all better off including the poorest player with $w_{100} = 1$.

What if players have such low incomes that they are worse off in such a matching coalition? They would certainly stay out of the coalition but this is not an issue at all in practice. First, the maximum income ratio is sufficiently large. For example, given $\alpha = 10$, $m = 100$ and $k = 10$, the maximum income ratio could be as large as 109. This indicates that even if income heterogeneity is very large, players can form a matching coalition to make them all better off. Second, given such large income heterogeneity in a matching coalition, the income of outsiders must be relatively small and their public good provision would also be very little if there is any. Third, international agreements must allow discrimination so that the most serious negotiating efforts can concentrate on the countries whose participation matters most. In climate change, poor countries are less industrialized and their contributions to carbon reduction do not matter much for global climate protection.

C. Coalition Stability

We first consider stability for interior insiders. Player $i \in CI$ has two strategies given other players' memberships.

(i) The player joins the coalition. The public good provision, $\widehat{G}(m,\mu)$, is uniquely characterized by the aggregate budget constraint below and the private good consumption of player i is $\widehat{x}_i = e(\widehat{G}(m,\mu),\widehat{\pi})$.

(32) $$\frac{1+(j-1)\mu}{1+(m-1)\mu}(G-(W_2-(k-j)*e(G,1)))+j*e(G,\widehat{\pi}) = W_1,\ \widehat{\pi} = 1+(m-1)\mu$$

(ii) The player stays out of the coalition. The public good provision, $\breve{G}(m-1,\mu)$, is uniquely characterized by the aggregate budget constraint below and the private good consumption of player i is $\breve{x}_i = e(\breve{G}(m-1,\mu),1)$.

(33) $$\frac{1+(j-2)\mu}{1+(m-2)\mu}(G-(W_2-(k-j)*e(G,1))-(w_i-e(G,1)))+(j-1)*e(G,\breve{\pi}) = W_1-w_i$$

where $\breve{\pi} = 1+(m-2)\mu$.

Consider $f_i(m,\mu)$ at a marginal matching rate.

(34) $$\left.\frac{\partial f_i(m,\mu)}{\partial \mu}\right|_{\mu=0} = \frac{1}{\overline{u}(x,G)}\left(\frac{\partial \breve{u}_i(\breve{x}_i,\breve{G})}{\partial \mu} - \frac{\partial \widehat{u}_i(\widehat{x}_i,\widehat{G})}{\partial \mu}\right)$$

$$= \frac{1}{\overline{u}_i(x,G)}\left(\frac{\partial \breve{u}_i}{\partial \breve{x}_i}\frac{\partial e(\breve{G},1)}{\partial \breve{G}}\frac{\partial \breve{G}}{\partial \mu} + \frac{\partial \breve{u}_i}{\partial \breve{G}}\frac{\partial \breve{G}}{\partial \mu} - \frac{\partial \widehat{u}_i}{\partial \widehat{x}_i}\left(\frac{\partial e(\widehat{G},\pi)}{\partial \widehat{G}}\frac{\partial \widehat{G}}{\partial \mu} + \frac{\partial e(\widehat{G},\pi)}{\partial \widehat{\pi}}\frac{\partial \pi}{\partial \mu}\right) - \frac{\partial \widehat{u}_i}{\partial \widehat{G}}\frac{\partial \widehat{G}}{\partial \mu}\right)$$

The aggregate budget constraints imply

(35) $$\left.\frac{\partial \breve{G}}{\partial \mu}\right|_{\mu=0} = \frac{s(W_1 - j*e(G,1) - (w_i - e(G,1))) - (j-1)(m-2)*\partial e(G,\pi)/\partial \pi}{1 + k*\partial e(G,1)/\partial G}$$

(36) $$\left.\frac{\partial \widehat{G}}{\partial \mu}\right|_{\mu=0} = \frac{s(W_1 - j*e(G,1)) - j(m-1)*\partial e(G,\pi)/\partial \pi}{1 + k*\partial e(G,1)/\partial G}$$

Combining the above equations together with the interiority optimality condition

(37) $$\left.\frac{\partial f_i(m,\mu)}{\partial \mu}\right|_{\mu=0} = \frac{1}{\overline{u}(x,G)}\frac{\partial u(x,G)}{\partial G}\left(1 + k\frac{\partial e(G,1)}{\partial G}\right)^{-1} *$$

$$\left(\frac{\partial e(G,\pi)}{\partial \pi}\left(j - 1 + (m + j + k - 2 - mk)\frac{\partial e(G,1)}{\partial G}\right) - s(w_i - e(G,1))\left(1 + \frac{\partial e(G,1)}{\partial G}\right)\right)$$

This generates the stability condition for interior insiders (SI) as

(38) $$s(w_i - e(G,1))\left(1 + \frac{\partial e(G,1)}{\partial G}\right) + \frac{\partial e(G,\pi)}{\partial \pi}\left(((k-1)s + (j-1)(k-2))\frac{\partial e(G,1)}{\partial G} - j + 1\right) > 0$$

The following proposition provides some intuitive insights of this stability condition.

Proposition 7.1 At corner equilibria with a marginal matching rate,

(i) Stability of interior insiders depends on their income distribution, and interior insiders with higher incomes are more likely to be stable;

(ii) Given $(k-2)\frac{\partial e(G,1)}{\partial G} > 1$, interior insiders are more likely to be stable if fewer interior players join the coalition; Given $(k-2)\frac{\partial e(G,1)}{\partial G} < 1$, interior insiders are more likely to be stable if more interior players join the coalition;

(iii)Interior insiders may be more or less likely to be stable if more corner players join the coalition.

Proof: Denote the stability condition of interior insiders as

$$g_{SI}(j,s) = s(w_i - e(G,1))\left(1 + \frac{\partial e(G,1)}{\partial G}\right) + \frac{\partial e(G,\pi)}{\partial \pi}\left((k-1)s\frac{\partial e(G,1)}{\partial G} + (j-1)\left((k-2)\frac{\partial e(G,1)}{\partial G} - 1\right)\right)$$

This term is more likely to be positive if interior insiders have higher incomes. We consider an additional interior player joining the coalition. Denote

$$\Delta g_{SI}(j) = g_{SI}(j+1,s) - g_{SI}(j,s) = \frac{\partial e(G,\pi)}{\partial \pi}\left((k-2)\frac{\partial e(G,1)}{\partial G} - 1\right)$$

This condition immediately leads to part (ii). We consider an additional corner player joining the coalition. Denote

$$\Delta g_{SI}(s) = g_{SI}(j,s+1) - g_{SI}(j,s) = w_i - e(G,1) + \frac{\partial e(G,1)}{\partial G}\left(w_i - e(G,1) + (k-1)\frac{\partial e(G,\pi)}{\partial \pi}\right)$$

The sign of this condition is ambiguous depending on the income and the preference. QED

We then consider stability for corner players. Player $i \in \tilde{C}$ at the corner has two strategies given other players' memberships.

(i) The player stays out of the coalition. The public good provision, $\breve{G}(m,\mu)$, is uniquely characterized by the aggregate budget constraint as

(39) $$\frac{1+(k-1)\mu}{1+(m-1)\mu}(G-(W_2-(k-j)*e(G,1))) + j*e(G,\breve{\pi}) = \sum_{i=} w_i \text{ , } \breve{\pi} = 1+(m-1)\mu$$

The private good consumption of player i is $\breve{x}_i = w_i$ and thus $\left.\frac{\partial \breve{x}_i}{\partial \mu}\right|_{\mu=0} = 0$.

(ii) The player joins the coalition. The public good provision, $\widehat{G}(m+1,\mu)$, is uniquely characterized by the aggregate budget constraint as

(40) $$\frac{1+(j-1)\mu}{1+m\mu}(G-(W_2-(k-j)*e(G,1))) + j*e(G,\widehat{\pi}) = \sum_{i=1}^{k} w_i \text{ , } \widehat{\pi} = 1+m\mu$$

The private good consumption of player i is $\widehat{x}_i = w_i - \frac{\mu}{1+m\mu}(G-(W_2-(k-j)*e(G,1)))$

and thus $\left.\frac{\partial \widehat{x}_i}{\partial \mu}\right|_{\mu=0} = -(W_1 - j*e(G,1))$.

Consider $f_i(m,\mu)$ at a marginal matching rate.

(41) $$\left.\frac{\partial f_i(m,\mu)}{\partial \mu}\right|_{\mu=0} = \frac{1}{\overline{u}_i(x,G)}\left(\frac{\partial \breve{u}_i(\breve{x}_i,\breve{G})}{\partial \breve{x}_i}\frac{\partial \breve{x}_i}{\partial \mu} + \frac{\partial \breve{u}_i(\breve{x}_i,\breve{G})}{\partial \breve{G}}\frac{\partial \breve{G}}{\partial \mu} - \frac{\partial \widehat{u}_i(\widehat{x}_i,\widehat{G})}{\partial \widehat{x}_i}\frac{\partial \widehat{x}_i}{\partial \mu} - \frac{\partial \widehat{u}_i(\widehat{x}_i,\widehat{G})}{\partial \widehat{G}}\frac{\partial \widehat{G}}{\partial \mu}\right)$$

The aggregate budget constraints imply

(42) $$\left.\frac{\partial \breve{G}}{\partial \mu}\right|_{\mu=0} = \frac{(m-j)(W_1 - j*e(G,1)) - j(m-1)*\partial e(G,\pi)/\partial \pi}{1+k*\partial e(G,1)/\partial G}$$

(43) $$\left.\frac{\partial \widehat{G}}{\partial \mu}\right|_{\mu=0} = \frac{(m+1-j)(W_1 - j*e(G,1)) - j*m*\partial e(G,\pi)/\partial \pi}{1+k*\partial e(G,1)/\partial G}$$

Combining the above equations yields

(44) $$\left.\frac{\partial f_i(m,\mu)}{\partial \mu}\right|_{\mu=0} = \frac{1}{\overline{u}_i(x_i,G)}\frac{\partial u_i(x_i,G)}{\partial G} * \left(\overline{MRS_i} *(W_1 - j*e(G,1)) - \frac{W_1 - j*e(G,1) - j*\partial e(G,\pi)/\partial \pi}{1+k*\partial e(G,1)/\partial G}\right)$$

This generates the stability condition for corner insiders (SC) as

(45) $$\left(1+k*\frac{\partial e(G,1)}{\partial G}\right)^{-1}\left(1-\frac{\partial e(G,\pi)}{\partial \pi}\frac{j}{W_1 - j*e(G,1)}\right) > \overline{MRS_i}$$

The following proposition provides some intuitive insights on stability of corner players.

Proposition 7.2 At corner equilibria with a marginal matching rate,

(i) Corner insiders may be more or less likely to be stable if more interior players join the coalition;

(ii) Stability of corner insiders depends on income distribution of interior players, and corner insiders are more likely to be stable if interior players with smaller incomes join the coalition;

(iii)Stability of any corner insider is independent of other corner players.

Proof: Denote the stability condition for corner insiders as

$$g_{SC}(j) = \left(1+k*\frac{\partial e(G,1)}{\partial G}\right)^{-1}\left(1-\frac{\partial e(G,\pi)}{\partial \pi}\frac{j}{W_1 - j*e(G,1)}\right) - \overline{MRS_i}$$

We consider an additional interior player joining the coalition. Denote

$$\begin{aligned}\Delta g_{SC}(j) &= g_{SC}(j+1) - g_{SC}(j) \\ &= \left(1+k*\frac{\partial e(G,1)}{\partial G}\right)^{-1}\frac{\partial e(G,\pi)}{\partial \pi}\frac{j(w_{i_{j+1}} - W_1/j)}{(W_1 - j*e(G,1))(W_1 + w_{i_{j+1}} - (j+1)*e(G,1))}\end{aligned}$$

If $w_{i_{j+1}} < W_1/j$, then $\Delta g_{SC}(j) > 0$. Therefore, only when the joining interior player has a smaller income than the average income of the existing interior insiders, corner insiders are more likely to be stable. Given the same number of interior insiders, the smaller their income W_1 , the more likely $g_{SC}(j)$ is positive. Besides, the stability condition is independent of other corner players. QED

We now consider the four conditions simultaneously (PI , PC , SI and SC). First, if SC holds, then PC holds and hence we only need to consider PI , SI and SC . Second, if the participation rate of interior players is above the threshold, then PI holds. We take the CES utility function as an example to examine SI and SC .

We first consider SC. Assuming the participation rate of interior players is above the threshold, we consider two cases: (1) $j = k$; (2) $j < k$ and j is above the threshold. In the first case, the stability condition for corner players reads as

$$\left(1+k*\left(\frac{a}{1-a}\right)^{1/(1-\rho)}\right)^{-1}\left(1+k*\frac{1}{1-\rho}*\left(\frac{a}{1-a}\right)^{1/(1-\rho)}\right) > \overline{MRS_i} \tag{46}$$

As $\overline{MRS_i} > 1$, it follows that $0 < \rho \le 1$. Consider an extreme situation in which ρ is close to one and w_i is close to w_k so that $\overline{MRS_i}$ is close to one, so the condition holds.

In the second case, if the income distribution of interior players is so polarized that a small fraction of interior players have high incomes while a large share of them have low incomes, and only the players with low incomes join the coalition so that $W_1 - j*e(G,1)$ is close to zero, then the stability condition holds regardless of ρ.

We then consider SI. Given the CES function, the stability condition for interior insiders is

$$s\frac{w_i - e(G,1)}{G}\left(1+\left(\frac{a}{1-a}\right)^{\frac{1}{1-\rho}}\right) - \frac{1}{1-\rho}*\left(\frac{a}{1-a}\right)^{\frac{1}{1-\rho}}\left(((k-1)s+(k-2)(j-1))\left(\frac{a}{1-a}\right)^{\frac{1}{1-\rho}} - j+1\right) > 0$$

This condition holds in two extreme cases: (1) a is close to zero so that $\left(\frac{a}{1-a}\right)^{1/(1-\rho)}$ is close to zero; (2) ρ goes to negative infinity so that $1/(1-\rho)$ is close to zero.

Combining the above analysis leads to the following qualitative proposition.

Proposition 7.3 At corner equilibria with a marginal matching rate, if players join a coalition at once, there is a stable matching coalition in one of the following three situations:

(i) Players value the public good sufficiently higher than the private good, and the elasticity of substitution between the two goods is sufficiently large, and the participation rate of interior players is above the threshold, and corner insiders have sufficiently high incomes;

(ii) The income distribution of interior players is so polarized that a small fraction of interior players have high incomes while a large share of them (above the threshold) have low incomes, and only the interior players with low incomes join the coalition, and corner insiders have sufficiently high incomes, and either of the following conditions:

   a) Players value the public good sufficiently higher than the private good;

   b) The elasticity of substitution between the two goods is sufficiently small.

We now turn to the process of joining sequentially. For simplicity, we assume that interior players make decisions first and then corner players take actions. From Proposition 5, only when $(k-2)\frac{\partial e(G,1)}{\partial G}<1$ do all interior players join the coalition sequentially. Given all interior players in the coalition, the stability of corner insiders requires (46) to hold, leading to the following proposition.

Proposition 7.4 At corner equilibria with a marginal matching rate, if players join a coalition sequentially, there is a stable matching coalition only when players value the public good sufficiently higher than the private good, and the elasticity of substitution between the two goods is sufficiently large and corner insiders have sufficiently high incomes.

We now put two restrictions on the preference of players. First, it is empirically plausible to assume that players value the private good more than the public good, i.e., $a>1/2$. Second, the elasticity of substitution is neither too large nor too small. For example, it is assumed to lie within a certain range $(1/10,10)$ centering around one which is the elasticity of substitution in the Cobb-Douglas function. Given these restrictions, none of the above three cases can be true.

Proposition 7.5 At corner equilibria with a marginal matching rate, if players value the private good more than the public good and the elasticity of substitution between the private good and the public good is within a reasonable range centering around one, there is no stable matching coalition.

IV. Coalition with Large Matching Rates

The previous sections have examined coalition formation with a marginal matching rate, and this section considers relatively large matching rates for comparative studies and investigates how matching rates affect profitability and stability of a matching coalition. Similarly, interior equilibria and corner equilibria are examined respectively.

A. Interior Equilibria

Coalition Profitability.—Consider the utility change with respect to a large matching rate. It is derived similarly to the case of a marginal matching rate but is evaluated at any positive matching rate rather than zero as follows.

(47)
$$\frac{\partial u}{\partial \mu}=(m-1)\frac{\partial u}{\partial G}\frac{\partial e(G,\pi)}{\partial \pi}*$$
$$\left(1+m*\frac{\partial e(G,\pi)}{\partial G}+(n-m)*\frac{\partial e(G,1)}{\partial G}\right)^{-1}\left((n-m)\frac{\partial e(G,1)}{\partial G}-\frac{(m-1)(1-\mu)}{1+(m-1)\mu}\right)$$

As $\frac{\partial e(G,\pi)}{\partial \pi} < 0$, it follows that

(48) $$\frac{\partial u}{\partial \mu} > 0 \Rightarrow (n-m)\frac{\partial e(G,1)}{\partial G} < \frac{(m-1)(1-\mu)}{1+(m-1)\mu}$$

If the utility change is evaluated at $\mu = 0$, it is immediately reduced to the profitability condition at the marginal matching rate as

(49) $$\left.\frac{\partial u}{\partial \mu}\right|_{\mu=0} > 0 \Rightarrow (n-m)\frac{\partial e(G,1)}{\partial G} < m-1$$

If the participation rate is above the threshold, i.e., $\left.\partial u/\partial \mu\right|_{\mu=0} > 0$, due to continuity, it follows that $\partial u/\partial \mu > 0$ holds given relatively large but sufficiently small matching rates, i.e., the utility level is increasing in the matching rate. Put differently, the larger the matching rate, the more profitable the coalition.

Optimal Matching Rates.—In general, there is no optimal matching rate for a matching coalition of heterogeneous players. However, at interior matching equilibria, coalition members have an identical level of utility. This indicates that there may exist an optimal matching rate maximizing the utility of all coalition members at the same time.

Consider the utility change with respect to the matching rate. Given $m < n$, as $\frac{\partial e(G(\mu),1)}{\partial G} > 0$ and $\frac{\partial e(G,\pi)}{\partial \pi} < 0$, it follows that $\left.\frac{\partial u}{\partial \mu}\right|_{\mu=1} < 0$. The Intermediate Value Theorem implies that there exists $\mu^*(0 < \mu^* < 1)$ satisfying $\partial u/\partial \mu = 0$, so the optimal matching rate is determined by the following condition

(50) $$(n-m)\frac{\partial e(G(\mu),1)}{\partial G} = \frac{(m-1)(1-\mu)}{1+(m-1)\mu}$$

There are two interesting special cases of this condition.

(i) Grand coalition

If all players form a grand coalition, i.e., $m = n$, then $\mu^* = 1$. This is the optimal matching rate from the social perspective.

(ii) Cobb-Douglas example

In the Cobb-Douglas example, the optimal matching rate can be solved as

(51) $$\mu^* = 1 - \frac{(n-m)\alpha}{m-1}$$

The optimal matching rate of coalition members is positively related to the coalition size while negatively related to the size of outsiders. The intuition is that if there are fewer outsiders (free riders), the coalition as a whole has incentives to provide a larger public good contribution so that the total public good provision is closer to the socially optimal level, and this requires a larger matching rate.

Coalition Stability.—Coalition stability also depends on matching rates. From Proposition 7.5, it is empirically plausible to assume that $\left.\frac{\partial f(m,\mu)}{\partial \mu}\right|_{\mu=0} > 0$. Due to continuity, $\frac{\partial f(m,\mu)}{\partial \mu} > 0$ holds given relatively large but sufficiently small matching rates. This indicates that the larger the matching rate, the stronger the free-riding incentive. Table 2 provides some numerical examples given a Cobb-Douglas utility function with $\alpha = 2$ and $n = 20,\ m = 15$.

TABLE 2 FREE-RIDING INCENTIVES IN THE COBB-DOUGLAS UTILITY FUNCTION

μ	0	0.01	0.02	0.04	0.06	0.08	0.10	0.15	0.20
$f(m,\mu)$	1.00	1.27	1.57	2.26	3.06	3.97	4.99	8.00	11.65

The following proposition summarizes the effects of matching rates on coalition formation at interior equilibria.

Proposition 8 At interior equilibria, in a matching coalition of size m with a matching rate μ, given the participation rate is above the threshold,

(1) All coalition members have the same optimal matching rate and it is determined by $(n-m)\frac{\partial e(G(\mu),1)}{\partial G} = \frac{(m-1)(1-\mu)}{1+(m-1)\mu}$;

(2) As long as the matching rate is smaller than the optimal matching rate, the larger the matching rate, the more profitable the coalition;

(3) Assuming that players have incentives to leave the coalition at a marginal matching rate, the larger the matching rate, the less stable the coalition.

B. Corner Equilibria

For simplicity, this section examines the effects of matching rates on coalition formation at corner equilibria in a special situation in which all interior players join the coalition. In such a coalition, at a marginal matching rate, all interior insiders and those corner insiders are better off, but empirically they all have incentives to leave the coalition. Now given a relatively large but sufficiently small matching rate, due to the same continuity argument as in the case of interior equilibria, the coalition is more profitable but less stable.

At interior equilibria, all coalition members have the same optimal matching rate. In contrast, at corner equilibria, the optimal matching rate is different across corner insiders. For player

$i \in CC$, the budget constraint is rearranged as $x_i = w_i - \frac{\mu}{1+(m-1)\mu} G$. It immediately follows that $\frac{\partial x_i}{\partial \mu} = -\frac{G}{(1+(m-1)\mu)^2}$ and $\frac{\partial x_i}{\partial G} = -\frac{\mu}{1+(m-1)\mu}$. Therefore,

(52)
$$\begin{aligned}\frac{\partial u_i}{\partial \mu} &= \frac{\partial u_i}{\partial x_i}\left(\frac{\partial x_i}{\partial G}\frac{\partial G}{\partial \mu} + \frac{\partial x_i}{\partial \mu}\right) + \frac{\partial u_i}{\partial G}\frac{\partial G}{\partial \mu} \\ &= \frac{\partial u_i}{\partial G}\left(\frac{\partial G}{\partial \mu} - MRS_i\left(\frac{\mu}{1+(m-1)\mu}\frac{\partial G}{\partial \mu} + \frac{G}{(1+(m-1)\mu)^2}\right)\right)\end{aligned}$$

The private good consumption of player i is decreasing in μ. As $\lim_{x_i \to 0} MRS_i = \infty$, when μ is sufficiently large so that x_i is close to zero, the following condition holds

(53)
$$\frac{\partial u_i}{\partial \mu} = \frac{\partial u_i}{\partial G}\left(\frac{\partial G}{\partial \mu} - MRS_i\left(\frac{\mu}{1+(m-1)\mu}\frac{\partial G}{\partial \mu} + \frac{G}{(1+(m-1)\mu)^2}\right)\right) < 0$$

The Intermediate Value Theorem implies that there exists $\mu = \hat{\mu}_i$ satisfying $\partial u_i / \partial \mu = 0$, so the optimal matching rate of player i is determined by the following condition

(54)
$$\frac{\partial G}{\partial \mu} = MRS_i\left(\frac{\hat{\mu}_i}{1+(m-1)\hat{\mu}_i}\frac{\partial G}{\partial \mu} + \frac{G}{(1+(m-1)\hat{\mu}_i)^2}\right)$$

As MRS_i is different across players at the corner, the optimal matching rate is also different for corner players. This shows there is no optimal matching rate for a coalition at the corner.

V. Free-riding and Reputation

It is empirically plausible that the larger the matching rate, the stronger the free-riding incentive. To mitigate this incentive, this section introduces reputation mechanisms. Assume that each player values their reputation in coalition formation, denoted by $R_i > 0$, which is heterogeneous and exogenous. More specifically, if one player joins the coalition, the player gains reputation and benefits from this reputation in some way (for example, through repeated games or issue linkage, see, e.g., Finus 2001), thereby improving the utility. The utility of player i is adjusted for reputation as

$$\hat{u}_i(x_i, G(m+\delta, \mu)) = \begin{cases} \breve{u}_i(\breve{x}_i, \breve{G}(m, \mu)) & \delta = 0 \ if \ i \in \tilde{C} \\ \widehat{u}_i(\widehat{x}_i, \widehat{G}(m+1, \mu)) + R_i & \delta = 1 \ if \ i \in C \end{cases}$$

The following proposition shows the universal existence of a stable grand coalition if players value their reputation in coalition formation.

Proposition 9 If players value their reputation in coalition formation, there always exists a stable grand matching coalition at a small matching rate.

Proof: Given $\mu = 0$, a matching coalition of size m is reduced to the initial equilibrium, so

$$\breve{u}_i(\breve{x}_i, \breve{G}(m,0)) = \widehat{u}_i(\widehat{x}_i, \widehat{G}(m+1,0)) = \overline{u}_i(x_i, G)$$

(i) If $f_i(m,\mu)$ is decreasing in μ, then $\breve{u}_i(\breve{x}_i, \breve{G}(m,\mu)) < \widehat{u}_i(\widehat{x}_i, \widehat{G}(m+1,\mu))$. Immediately, $\hat{u}_i(x_i, G(m+1,\mu)) > \hat{u}_i(x_i, G(m,\mu))$;

(ii) If $f_i(m,\mu)$ is increasing in μ, then $\breve{u}_i(\breve{x}_i, \breve{G}(m,\mu)) > \widehat{u}_i(\widehat{x}_i, \widehat{G}(m+1,\mu))$. As $f_i(m,0) = 1$, due to continuity, there exist small matching rates satisfying

$\widehat{u}_i(\widehat{x}_i, \widehat{G}(m+1,\mu)) + R_i > \breve{u}_i(\breve{x}_i, \breve{G}(m,\mu))$.

Therefore, given sufficiently small μ, $\hat{u}_i(x_i, G(m+1,\mu)) > \hat{u}_i(x_i, G(m,\mu))$ holds for any $m(2 \le m \le n-1)$, i.e., the coalition is internally stable. By setting $m = n-1$, it follows that $\hat{u}_i(x_i, G(n,\mu)) > \hat{u}_i(x_i, G(n-1,\mu))$, so the grand coalition at a small matching rate is stable. QED

Assuming $f_i(m,\mu)$ is increasing in μ, together with the above proposition, the following relationships hold given sufficiently large m and sufficiently small μ

(55) $$\widehat{u}_i(\widehat{x}_i, \widehat{G}(m+1,\mu)) + R_i > \breve{u}_i(\breve{x}_i, \breve{G}(m,\mu)) > \widehat{u}_i(\widehat{x}_i, \widehat{G}(m+1,\mu)) > \overline{u}_i(x_i, G)$$

This indicates that the player would be better off in the coalition than without the coalition but given other players' memberships the player has incentives to leave the coalition. However, if taking reputation into consideration, the player would stay in the coalition when the gain of free riding is lower than the reputation loss.

VI. Coalition Depth and Breadth

The previous section has shown that a stable grand matching coalition always exists when taking reputation into consideration and the matching rate plays an important role in coalition formation. This section examines the trade-off between coalition depth and breadth with different matching rates. Due to heterogeneity, international cooperation often faces a trade-off between depth and breadth and a coalition is either "narrow but deep" or "broad but shallow". Bottom-up approaches or decentralized cooperation is likely to reach "narrow but deep" agreements while top-down approaches or centralized cooperation tends to generate "broad but shallow" treaties (Barrett, 2003). This trade-off is also present in a matching coalition. Consider an experiment in which the matching rate in a coalition gradually increases from zero.

Assume $f_i(m,\mu)$ is increasing in μ. Given a small matching rate $\mu_1(\mu_1 > 0)$, for any player i in a stable coalition, the following condition holds

(56) $$\hat{u}_i(\hat{x}_i, \hat{G}(m+1,\mu_1)) + R_i > \breve{u}_i(\breve{x}_i, \breve{G}(m,\mu_1))$$

Now consider a larger matching rate. $\breve{u}_i(\breve{x}_i, \breve{G}(m,\mu))$ and $\hat{u}_i(\hat{x}_i, \hat{G}(m+1,\mu))$ both increase, but $\breve{u}_i(\breve{x}_i, \breve{G}(m,\mu))$ increases to a larger extent because $f_i(m,\mu)$ is increasing in μ. Therefore, with a sufficiently large matching rate $\mu_2(\mu_2 > \mu_1)$, for those players with lower reputation values, the following condition holds

(57) $$\hat{u}_i(\hat{x}_i, \hat{G}(m+1,\mu_2)) + R_i < \breve{u}_i(\breve{x}_i, \breve{G}(m,\mu_2))$$

This indicates that the players with lower reputation values would leave the coalition. As the matching rate continues to increase, the coalition gradually shrinks.

One caveat here as well as in Section IV is that the set of interior players may be changed with the matching rate increasing given specific income distributions, but this paper does not discuss this situation for simplicity.

VII. Conclusions

Matching mechanisms have been proposed to mitigate underprovision of public goods in voluntary contribution models. This paper has investigated coalition formation under matching mechanisms among multiple players with different incomes. First, it has characterized the conditions of profitability and stability in a matching coalition with a marginal matching rate. The results show that coalition profitability is favored by a large coalition, a small group of outsiders and a strong preference for public goods. At interior equilibria, if the participation rate of players is above the threshold, the coalition is profitable. As an extreme case, if all players join a coalition, the grand coalition at a marginal matching rate is always profitable regardless of the preference. At corner equilibria, although some poor players do not provide flat contributions, they are also likely to be better off if the total public good is sufficiently increased to compensate their forgone private good consumption. The most optimistic message in the corner case is that very poor players can be better off even if income heterogeneity is quite large. However, it is empirically plausible that the stability condition does not hold and players have incentives to free ride until the coalition collapses. Second, it has also examined coalition formation with different matching rates for comparative studies, and found that a coalition is more profitable but less stable with a larger matching rate. Third, it has introduced reputation mechanisms to mitigate the incentive of free riding, and shown the existence of a stable matching coalition if players value their reputation in coalition formation. They would stay in the coalition when the gain of free riding is lower than the reputation loss. Due to heterogeneity, the matching coalition faces a trade-off between matching depth and breadth. The policy implication is that the matching rate can be flexibly set to compromise between cooperation depth and breadth and it may achieve Pareto-improving outcomes while avoiding side payments.

REFERENCES

Althammer, Wilhelm and Wolfgang Buchholz (1993). "Lindahl-Equilibria as the Outcome of a Non-Cooperative Game: A Reconsideration." *European Journal of Political Economy* 9(3): 399-405.

Andreoni, James and Ted Bergstrom (1996). "Do Government Subsidies Increase the Private Supply of Public Goods?" *Public Choice* 88(3/4): 295-308.

Barrett, Scott (1994). "Self-Enforcing International Environmental Agreements." *Oxford Economic Papers* 46: 878-894.

Barrett, Scott (2003). *Environment and Statecraft: The Strategy of Environmental Treaty-Making*. Oxford University Press, Oxford.

Bergstrom, Theodore, Lawrence Blume and Hal Varian (1986). "On the Private Provision of Public Goods." *Journal of Public Economics* 29(1): 25-49.

Boadway, Robin, Pierre Pestieau and David Wildasin (1989). "Tax-Transfer Policies and the Voluntary Provision of Public Goods." *Journal of Public Economics* 39(2): 157-176.

Boadway, Robin, Zhen Song and Jean-Francois Tremblay (2007). "Commitment and Matching Contributions to Public Goods." *Journal of Public Economics* 91(9): 1664-1683.

Boadway, Robin, Zhen Song and Jean-Francois Tremblay (2011). "The Efficiency of Voluntary Pollution Abatement When Countries Can Commit." *European Journal of Political Economy* 27(2): 352-368.

Buchholz, Wolfgang, Richard Cornes and Dirk Rübbelke (2011). "Interior Matching Equilibria in a Public Good Economy: An Aggregative Game Approach." *Journal of Public Economics* 95(7-8): 639-645.

Buchholz, Wolfgang, Richard Cornes and Dirk Rübbelke (2012). "Matching as a Cure for Underprovision of Voluntary Public Good Supply." *Economics Letters* 117(3): 727-729.

Buchholz, Wolfgang, Richard Cornes and Dirk Rübbelke (2014). "Potentially Harmful Cooperation on Global Public Good Provision." *Economica* 81(322): 205-223.

Buchholz, Wolfgang, Christian Haslbeck and Todd Sandler (1998). "When Does Partial Co-Operation Pay?" *FinanzArchiv/Public Finance Analysis* (55): 1－20.

Carraro, Carlo and Domenico Siniscalco (1993). "Strategies for the International Protection of the Environment." *Journal of Public Economics* 52(3): 309-328.

Cornes, Richard and Roger Hartley (2003). "Risk Aversion, Heterogeneity and Contests." *Public Choice* 117(1/2): 1-25.

Cornes, Richard and Roger Hartley (2007). "Aggregative Public Good Games." *Journal of Public Economic Theory* 9(2): 201-219.

Cornes, Richard and Todd Sandler (2000). "Pareto-Improving Redistribution and Pure Public Goods." *German Economic Review* 1(2): 169-186.

Cornes, Richard and Todd Sandler (1996). *The Theory of Externalities, Public Goods and Club Goods*. Cambridge University Press, Cambridge.

D'Aspremont, Claude, Alexis Jacquemin, Jean Jaskold Gabszewicz and John A. Weymark (1983). "On the Stability of Collusive Price Leadership." *Canadian Journal of Economics* 16(1): 17-25.

Danziger, Leif and Adi Schnytzer (1991). "Implementing the Lindahl Voluntary-Exchange Mechanism." *European Journal of Political Economy* 7(1): 55-64.

Donsimoni, Marie-Paule (1985). "Stable Heterogeneous Cartels." *International Journal of Industrial Organization* 3(4): 451-467.

Falkinger, Josef (1996). "Efficient Private Provision of Public Goods by Rewarding Deviations from Average." *Journal of Public Economics* 62(3): 413-422.

Finus, Michael (2001). *Game Theory and International Environmental Cooperation*. Edward Elgar, Cheltenham and Northampton.

Finus, Michael, Juan-Carlos Altamirano-Cabrera and Ekko C. Van Ierland (2005). "The Effect of Membership Rules and Voting Schemes on the Success of International Climate Agreements." *Public Choice* 125(1/2): 95-127.

Guttman, Joel M. (1978). "Understanding Collective Action: Matching Behavior." *The American Economic Review* 68(2): 251-255.

Guttman, Joel M. (1987). "A Non-Cournot Model of Voluntary Collective Action." *Economica* 54(213): 1-19.

Hoel, Michael and Kerstin Schneider (1997). "Incentives to Participate in an International

Environmental Agreement." *Environmental and Resource Economics* 9(2): 153-170.
Itaya, Jun-ichi, David de Meza and Gareth D. Myles (1997). "In Praise of Inequality: Public Good Provision and Income Distribution." *Economics Letters* 57(3): 289-296.
Peleg, Bezalel and Peter Sudholter (2003). *Introduction to the Theory of Cooperative Games*. Kluwer Academic, Boston.
Ray, Debraj (2007). *A Game-Theoretic Perspective on Coalition Formation*. Oxford University Press, Oxford.
Sandler, Todd (2004). *Global Collective Action*. Cambridge University Press, Cambridge.
Summers, Lawrence (2007). Foreword, in J. E. Aldy and R. N. Stavins (eds.), *Architectures for Agreement: Addressing Global Climate Change in the Post-Kyoto World*. Cambridge University Press, Cambridge.
Varian, Hal R. (1994a). "Sequential Contributions to Public Goods." *Journal of Public Economics* 53(2): 165-186.
Varian, Hal R. (1994b). "A Solution to the Problem of Externalities When Agents Are Well-Informed." *The American Economic Review* 84(5): 1278-1293.
Warr, Peter G. (1982). "Pareto Optimal Redistribution and Private Charity." *Journal of Public Economics* 19(1): 131-138.
Warr, Peter G. (1983). "The Private Provision of a Public Good Is Independent of the Distribution of Income." *Economics Letters* 13(2-3): 207-211.