

The lazy housekeepers' problem*

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1 The House-Cleaning Problem

Alice and Bob share an apartment. Both like the apartment to be clean, but each hates to spend time cleaning. Nothing else matters to either of them. Their utility functions are

$$U^A(h_A, c) = c - f_A(h_A) \tag{1}$$

and

$$U^B(h_B, c) = c - f_B(h_B) \tag{2}$$

where c is the cleanliness of the apartment and h_i is the number of hours that i spends cleaning the house. Let us assume that $f'_i(x) \geq 0$ and $f''(x) > 0$ for all $x \geq 0$, that the level of cleanliness is determined by a “production function”

$$c = h_A + h_B. \tag{3}$$

Alice and Bob are each endowed with a sufficiently large amount of time H , so that there will be Pareto optima in which neither spends all of his or her time cleaning house.

Remarks:

1. The setup is similar to the standard Samuelsonian public goods problem with two consumers and one public and one private good, but not exactly the same. In the standard problem, the private good can be given to either person or used to make the public good. In this problem, there is no private good that can be exchanged.

*This problem was posed by Cheng-Zhong Qin of UCSB.

- Another thing that is “unusual” in this problem is that utilities are linear in the public good, but not in the private good. As you know, in the Samuelsonian public goods problem, if preferences are linear in private goods, then the efficient amount of public goods is independent of the distribution of private goods, so long as both consumers consume positive amounts of both goods.

A Samuelson-like necessary condition

Let’s try to find a necessary condition for an interior Pareto optimum for Alice and Bob. A Pareto optimum in which Bob’s utility is \bar{U}^B is a solution to the constrained maximum problem: Maximize $U^A(h_A, c)$ subject to the constraints $U^B(h_B, c) = \bar{U}^B$ and $h_A + h_B = c$. As we show in the Appendix to these notes, if we write down the Lagrangean for this problem, set its partial derivatives equal to zero, and make appropriate substitutions, we find that:

$$-\frac{\frac{\partial U^A(h_A, c)}{\partial c}}{\frac{\partial U^A(h_A, c)}{\partial h_A}} - \frac{\frac{\partial U^B(h_B, c)}{\partial c}}{\frac{\partial U^B(h_B, c)}{\partial h_B}} = 1 \quad (4)$$

To interpret Equation 4, it is useful to notice that the expression

$$-\frac{\frac{\partial U^i(h_i, c)}{\partial c}}{\frac{\partial U^i(h_i, c)}{\partial h_i}}$$

shows person i ’s marginal rate of substitution between cleanliness of the house and effort spent cleaning. This is the limiting ratio of the additional amount of time one would be willing to spend cleaning the house to gain an additional unit of cleanliness as the changes are made small. If the sum of the two persons’ marginal rates of substitution is greater than one, a Pareto superior arrangement could be found in which an extra unit of cleanliness is produced and the extra effort required of each person is less than the amount would make him indifferent about the additional cleanliness. Similarly if the sum of marginal rates of substitution is smaller than one, a Pareto improving decrease in the amount of cleanliness could be found. It would reduce cleanliness by a little bit, while reducing each person’s cleaning efforts by more than the amount needed to compensate for the loss of cleanliness.

In the special case where the utilities are linear in c as in Expressions 1 and 2, Equation 4 simplifies to

$$\frac{1}{f'_A(h_A)} + \frac{1}{f'_B(h_B)} = 1 \quad (5)$$

2 An Equivalent Formulation with Externalities

We could model this problem as a simple problem with externalities, where each player i has only one choice variable h_i and no mention is made of a public good c . Instead, each person's cleaning efforts shows up directly in the other person's utility function as an externality.

If we write the model in this way, the utility of person A is $U_A(h_A + h_B, h_A)$ and that of person B is $U_B(h_A + h_B, h_B)$. Pareto optimality implies that the outcome (h_A, h_B) maximizes $U_A(h_A, h_B)$ subject to the constraint that $U_B(h_A, h_B) = \bar{U}_B$. The Lagrangean for this problem is

$$L(h_A, h_B) = U_A(h_A, h_B) + \lambda (U_B(h_A, h_B) - \bar{U}_B). \quad (6)$$

If we set the partial derivatives of the Lagrangean with respect to h_A and h_B equal to zero and eliminate λ from these two equations, we have the condition:

$$\frac{\frac{\partial U_A(h_A, h_B)}{\partial h_A}}{\frac{\partial U_B(h_A, h_B)}{\partial h_A}} = \frac{\frac{\partial U_A(h_A, h_B)}{\partial h_B}}{\frac{\partial U_B(h_A, h_B)}{\partial h_B}} \quad (7)$$

Some Useful Special Cases

If the utilities are linear in c , as in Expressions 1 and 2, Equation 7 simplifies to

$$\frac{1 - f'_A(h_A)}{1} = \frac{1}{1 - f'_B(h_B)} \quad (8)$$

Cross-multiplying this expression and simplifying, we find that this equation is equivalent to Equation 5.

Let us consider the class of special cases in which for some parameters, a , B , and n , $f_A(h_A) = ah_A^n$ and $f_B(h_B) = bh_B^n$. Then with simple manipulations, we find that if x and y satisfy Equation 8, it must be that

$$h_B^{n-1} = \frac{a}{b} \left(\frac{h_A^{n-1}}{nah_A^{n-1} - 1} \right) \quad (9)$$

A particularly manageable class of special cases are those where $n = 2$ and $a = 1$. Then at any interior Pareto optimum, we must have

$$h_B = \frac{1}{b} \left(\frac{h_A}{2h_A - 1} \right) \quad (10)$$

Figure 1: Edgeworth Box with symmetric preferences

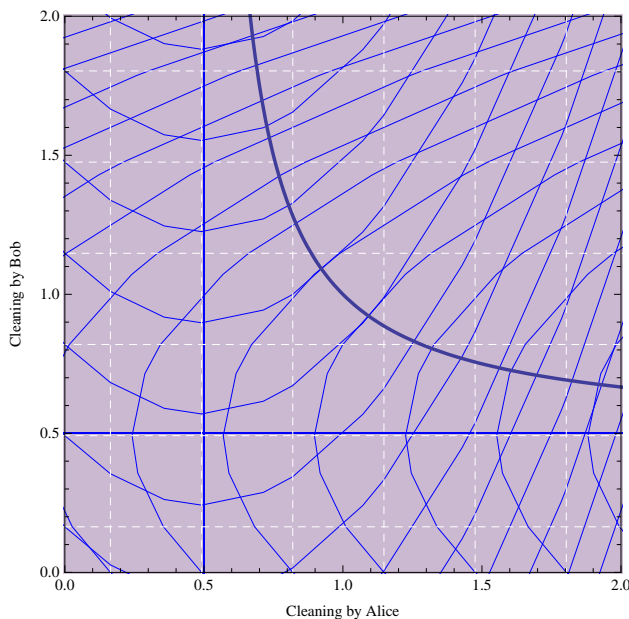


Figure 1 is an “Edgeworth box” for the case where Alice and Bob are equally averse to housekeeping with utility functions $U_A = h_A + h_B - h_B^2$ and $U_B = h_A + h_B - h_B^2$. Alice’s indifference curves are U -shaped with the bottom of each U appearing on the line $h_A = 1/2$. Bob’s are C -shaped, with the leftmost portion of each C appearing on the line $h_B = 1/2$. This happens because with these utility functions if the other person’s housecleaning effort is fixed at any level, each of the two persons maximizes his or her own utility by doing $1/2$ unit of housework. The thicker downward-sloping curve in Figure 1 is the locus of tangencies between indifference curves of Alice and of Bob. These represent the Pareto optimal assignments of housecleaning effort for Alice and Bob. Notice that at any of the Pareto optimal assignments depicted, both persons do more than $1/2$ unit of housework.

Figure 2 is an Edgeworth box for Alice and Bob in a case where Bob is more averse to housework than Alice. This figure is drawn for utility functions $U_A = h_A + h_B - h_B^2$ and $U_B = h_A + h_B - 2h_B^2$. In this case, the bottoms of Alice’s U -shaped indifference curves lie on the line $h_A = 1/2$, while the leftmost portions of Bob’s indifference curves lie on the line $h_B = 1/4$. Holding constant the cleaning effort or the other person, Alice would

prefer to do $1/2$ unit of housekeeping effort and Bob would prefer to do $1/4$ unit. As we see from Figure 2, although Bob hates housework more than Alice does, there are Pareto optima in which Bob does more housework than Alice.¹

Figure 2: Edgeworth Box when Bob hates housework more

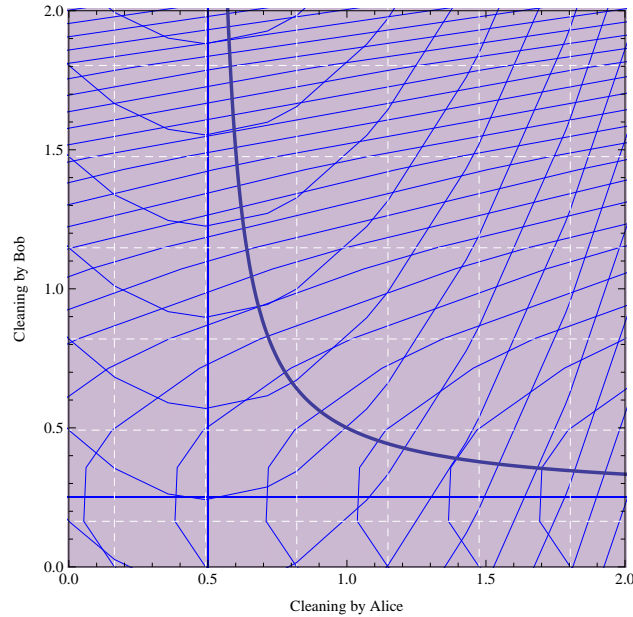
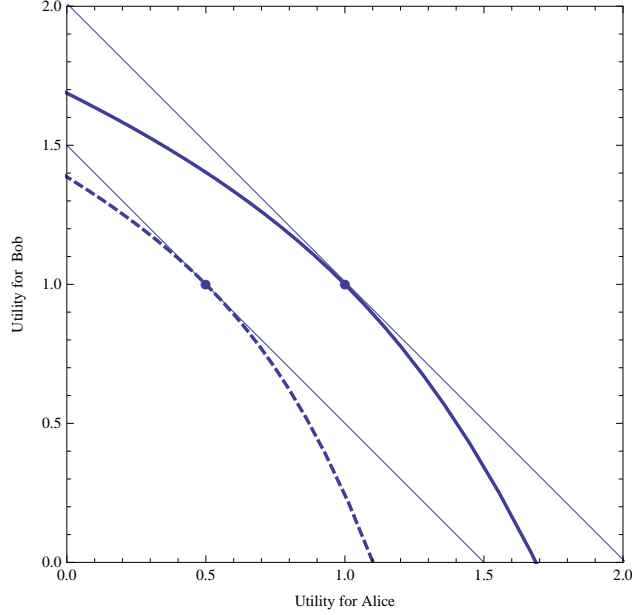


Figure 3 shows utility possibility frontiers for Alice and Bob in the two examples shown in Figures 1 and 2. The continuous curved line shows the frontier for the symmetric case and the dashed line shows the frontier for the case where Bob hates housework more. The downward-sloping straight lines are level curves for the sum of utilities $u_A + u_B$. In the symmetric case, we see that the maximum achievable sum of utilities is found at the point where $h_A = h_B = 1$ and where $u_A = u_B = 1$. In the case, where Bob hates homework more, we find that the maximum achievable utility happens where $h_A = 1$, $h_B = 1/2$, and where $u_A = 1/2$ and $u_B = 1$.

¹However, it is true that while there are no Pareto optima in which Alice does less than $1/2$ unit of housework, there are Pareto optima in which Bob does less than $1/2$ (but more than $1/4$) units of housework.

Figure 3: Utility Possibility Frontiers



3 Appendix

Proof that Equation 4 is a necessary condition for an interior Pareto optimum

The Lagrangean for the constrained maximization problem used to find necessary conditions for a Pareto optimum is

$$L(h_A, h_B, c) = U^A(h_A, c) + \lambda_1 (U^B(h_B, c) - \bar{U}^B) + \lambda_2 (h_A + h_B - c). \quad (11)$$

Setting partial derivatives of the Lagrangean equal to zero yields:

$$\frac{\partial U^A(h_A, c)}{\partial h_A} + \lambda_2 = 0 \quad (12)$$

$$\lambda_1 \frac{\partial U^B(h_B, c)}{\partial h_B} + \lambda_2 = 0 \quad (13)$$

$$\frac{\partial U^A(h_A, c)}{\partial c} + \lambda_1 \frac{\partial U^B(h_B, c)}{\partial c} - \lambda_2 = 0 \quad (14)$$

Divide both sides of Equation 14 by λ_2 to obtain

$$\frac{1}{\lambda_2} \frac{\partial U^A(h_A, c)}{\partial c} + \frac{\lambda_1}{\lambda_2} \frac{\partial U^B(h_B, c)}{\partial c} = 1 \quad (15)$$

Then using Equations 12 and 13 to eliminate the λ 's, we have Equation 4.