Solutions for Public Finance problem sets

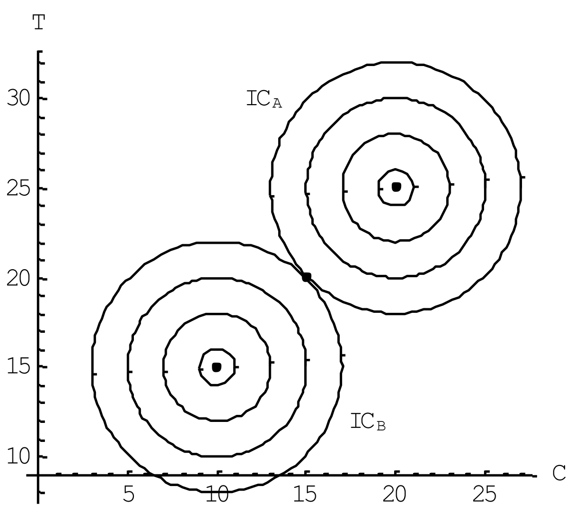
By Anita Gantner and Sofia Frankel, UCSB

1. Chapter 1: A Primitive Public Economy

Exercise 1.1

(a) As we can see, Anne's preferences and Bruce's preferences are represented by utility functions that are the equations for circles. In the case of Anne's preferences we have circles we center (20,25) and radius 10 and for Bruce we have circles with center (10,15) and radius 10. The centers of the circles represent their bliss points.

So their indifference curves look like:



Where IC_A stands for Anne's indifference curve and IC_B stands for Bruce's indifference curve.

(b) The point (10,15) is Pareto optimal since it represents Bruce's *bliss point*: the point where he has the maximum of utility. Any movement away from this point would make him worse off, that is, it is not possible to improve Anne's well being without making Bruce worse off.

To find the set of all Pareto optimal situations we can solve the following problem:

$$\max_{C,T} -(C-20)^2 - (T-25)^2$$

$$\text{s.t.} -(C-10)^2 - (T-15)^2 \geq \overline{U}^B$$

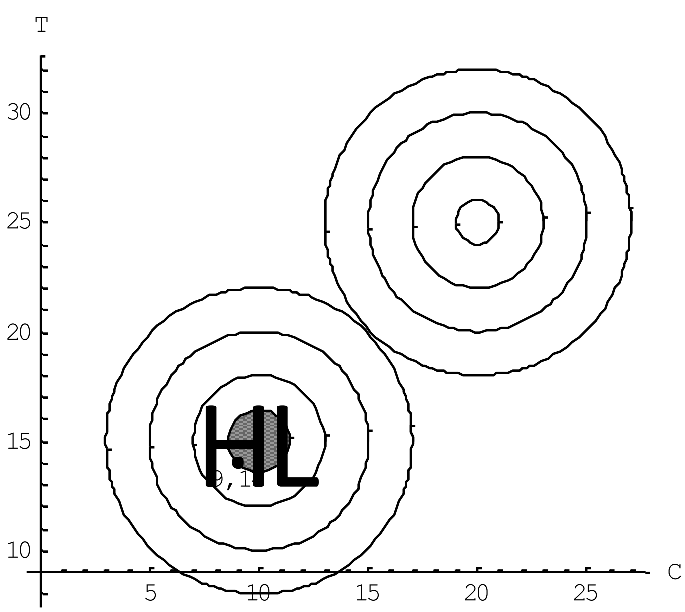
and we would get the P.O set $\{(C,T), C\in[10,20]: T = C+5\}$. This is just a straight line between the two bliss points, where the two indifference curves have tangencies.

(c) First notice that both indifference curves are tangent in the point (9,14). For that just evaluate Anne's MRS and Bruce's MRS at this point.

$$MRS^A = MRS^B = 1$$

but this is not a P.O since it is possible to find situations that are Pareto superior to it. Take for instance Bruce's bliss point (10,15). At this point $U^B = 0$ and $U^A = -200$, while at point (9,14) $U^B = -2$ and $U^A = -242$. So, at Bruce's bliss point they are both better. Take also, for instance, $(10, 15+\sqrt{2})$. At this point $U^B = -2$ and $U^A = -187.858$. As we see at this point Bruce has the same utility but Anne is better!

So, the set of situations that is Pareto superior to (9,14) is the set $\{(C,T)\in R^2 / (9,14): -(C-10)^2 - (T-15)^2 \geq -2\}$,because are the points where it is possible to improve the well being of at least one person without worse off the other. This is just a circle around Bruce's bliss point of radius $\sqrt{2}$. Graphically, it corresponds to the shaded area including Bruce's indifference curve frontier.

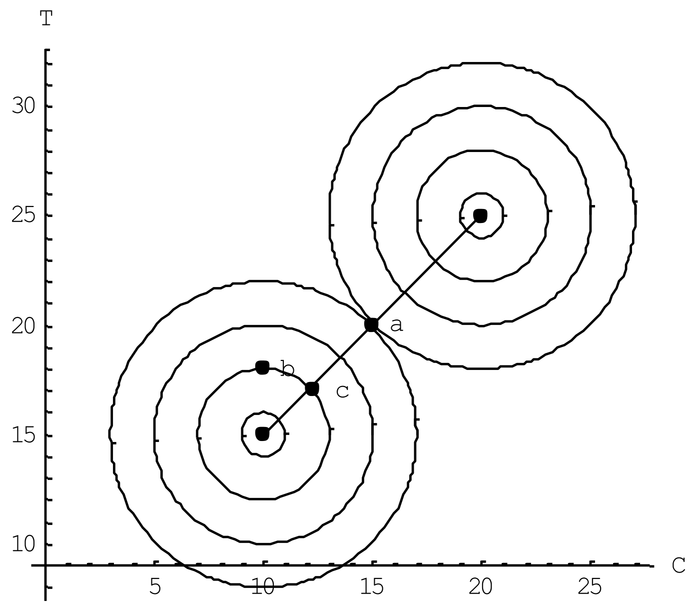


(d) For the indifference curves to be tangent, the condition $MRS^A = MRS^B$ must hold.

In our exercise it means that: $MRS^A = \dfrac{C-20}{T-25} = \dfrac{C-10}{T-15} = MRS^B$.

As we noticed before the point (9,14) is an example where we have tangency of both indifferences curves but it is not a Pareto optimal point. At this point we have U^B= -2 and U^A= -242. However, Anne and Bruce prefer to have more cribbage and more temperature. So they would be better at point (10,15), where we get U^B= 0 and U^A= -200. A necessary condition for P.O. is that preferred directions of change be opposite. At (9,14), the indifference curves are tangent on the "wrong" side.

(e) To answer this question, consider the following graph:



Remember that for a point to be *Pareto optimal* it must not have other possible situations that are Pareto superior to it. And if point is *not Pareto optimal* is because it is possible to obtain unanimous consent for a beneficial change.

The straight line, that was obtained in (b), is just the set of all Pareto optimals.

Consider the point *b*, which is not Pareto optimal since there is *c*, which is Pareto superior to it.

Both *a* and *c* are Pareto optimal points. But while *c* is Pareto superior to *b*, *a* is not Pareto superior to *b*, since moving from *b* to *a* would make Bruce worse off.

So, the fact that a point is Pareto optimal does not mean that is Pareto superior to a non Pareto Optimal point.

Exercise 1.2.

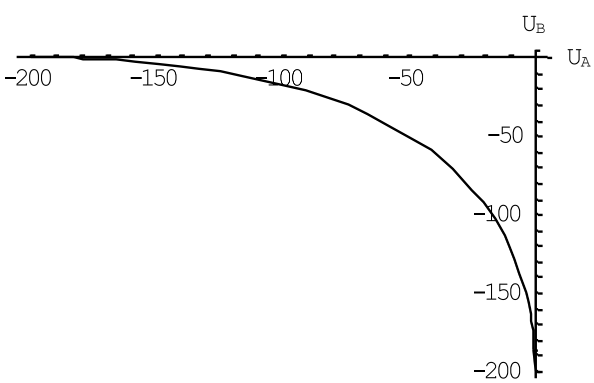
The Utility Possibility Frontier (UPF) for Anne and Bruce is represented in the space (U_A, U_B) and corresponds to Pareto optimal allocations.

To find the UPF we just solve the following system in order to find an expression that just depends on U_A and U_B:

$$\begin{cases} U_A = -(C-20)^2 - (T-25)^2 \\ U_B = -(C-10)^2 - (T-15)^2 \\ T = C+5 \end{cases}$$

and we get : $U_A = -2\left(\sqrt{\frac{U_B}{-2}} - 10\right)^2$.

Graphically the UPF looks like:



Exercise 1.3

(a) Since the utility functions proposed in exercise 1.1 are the Von-Neumann Morgenstern representations of both people's preferences, the UPS of exercise 1.2 is a UPS corresponding to a Von-Neumann Morgenstern representation of utility (sure thing utility possibility set).

Since the *sure thing utility possibility set* is *convex* then for any gamble there is always some certainty situation, which is at least as good for everybody. Thus they cannot improve on any of the sure thing Pareto optimal allocations by gambling.

(b) The idea here is to find Von-Neumann Morgenstern utility functions that would represent the same sure thing preferences as that in exercise 1. However, the UPS cannot be a convex set. So if Anne and Bruce play a game they will have an expected utility higher than the utility they get with a bundle in the security line.

$$U_A = \frac{1}{(C-20)^2 + (T-25)^2}$$

$$U_B = \frac{1}{(C-10)^2 + (T-15)^2}$$

Exercise 1.4

To have "elliptical" indifference curves instead of "circle" indifference curves means that the indifference curves are "sloped". That is, we have a negative slope when we have "too little" or "too much" of both cribbage games and temperature, and a positive slope when we have "too much" of one of the goods. When we have too much of one of the goods, it become a bad. So reducing the consumption of the "bad" good moves us closer to our "*bliss point*". If we have too much of both goods, they both are bads, so reducing the consumption of each moves us closer to the "*bliss point*".

So, less cribbage games can be compensated by more temperature and vice-versa. Anne and Bruce prefer much more playing in colder temperature to playing just a little more with a little higher temperature. The more "sloped" the indifference curves, the more extreme are the trade-offs described.

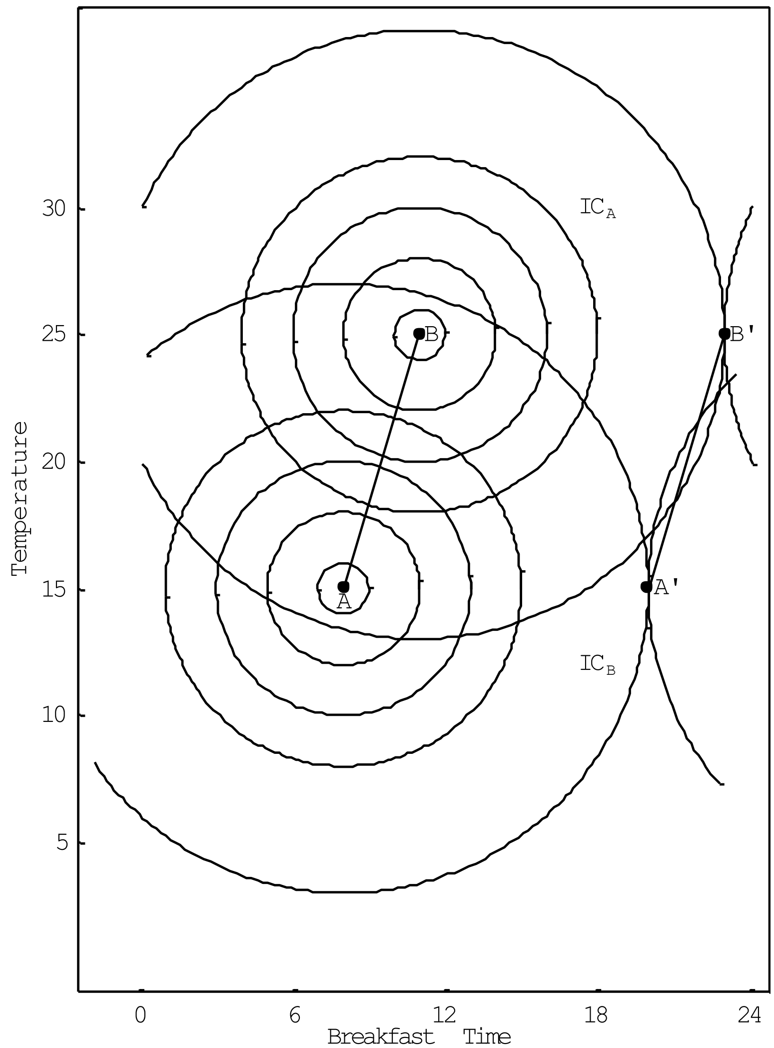
Exercise 1.5

Since it is impossible to make Bruce any better off than he is at his "*bliss point*" (point B^* in fig. 1.2) then the highest point that the *utility possibility frontier* (UPF) attains is precisely point B^*. If Anne is to be made better off than she is at Bruce's bliss point (take for example any point between B^* and A^*), then Bruce will have to be made worse off.
So, the points on the left of B^*, are points where Anne is worse than she would be at Bruce's "*bliss point*". Notice, however, that since Anne and Bruce share the same environment, if Anne is to be worse off than she is at Bruce's "*bliss point*", Bruce must also be worse off.
Thus, for the points on the west of point B^*, the boundary of UPF must slope upwards, which means that those points are not Pareto optimal and are representations of the "bad" tangency points. Moving away from these points we can improve both Anne's and Bruce's utility.

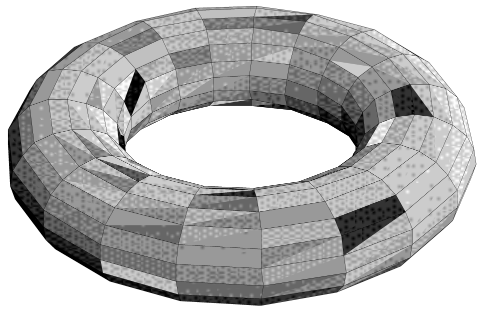
Exercise 1.6

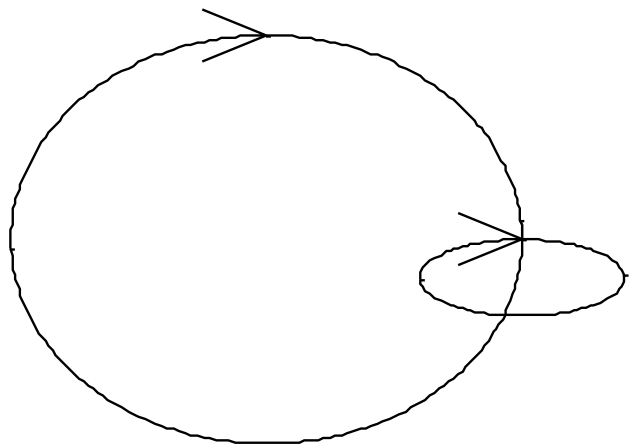
(a) Observe the following figure. One could glue this onto a tube in such a way that the left edge of the figure is laid on top of the right edge. The point A is Bruce's *"bliss point"* and B is Anne's *"bliss point"*. The Pareto optimal points all lie on the straight line between these two points.

Note: On the "back side of the roll," the indifference curves through the points B' and A' are tangent to themselves at these two points. Given the temperature, this is these points represent worst times for each person to eat breakfast. If breakfast is at this time, the person would prefer to eat either earlier or later.



(b) The donut looks something like this with breakfast time measured on a loop going around the hole and lunch time measured on a loop passing through the hole. Each person will have a bliss point somewhere on the bagel. Run a string between the two bliss points in the shortest possible way. Points along the string will be Pareto optimal.





Exercise 1.7 The set of Pareto optimal allocations is the entire triangular region bounded by the Bliss points of Anne, Bruce, and Charles.

According to the Kuhn Tucker method:

$$\underset{C,T}{Max} -(\text{C-20})^2 - (\text{T-25})^2$$

$$\text{s.t.} \begin{cases} -(\text{C}-10)^2 - (\text{T}-15)^2 \geq \overline{U}^{\text{B}} \\ -(\text{C}-20)^2 - (\text{T}-15)^2 \geq \overline{U}^{\text{C}} \end{cases}$$

The Lagrangean is: L = – (C-20)^2 – (T-25)^2 + λ (– (C-10)^2 – (T-15)^2 - $\overline{U}^{B}$) + μ (– (C-20)^2 – (T-15)^2 - $\overline{U}^{C}$)

$$\frac{\partial L}{\partial C} = -2(C-20) - 2\lambda(C-10) - 2\mu(C-20) = 0 \qquad and \qquad C > 0. \quad \text{F.O.C.}$$

$$\frac{\partial L}{\partial T} = -2(T-25) - 2\lambda(T-15) - 2\mu(T-15) = 0.$$

$$\frac{\partial L}{\partial \lambda} = -(C-10)^2 - (T-15)^2 = \overline{U}^{B} \; if \quad \lambda > 0.$$

$$\frac{\partial L}{\partial \mu} = -(C-20)^2 - (T-15)^2 = \overline{U}^{C} \; if \quad \mu > 0.$$

$$\begin{cases} -2(C-20) = 2\lambda(C-10) + 2\mu(C-20). \\ -2(T-25) = 2\lambda(T-15) + 2\mu(T-15) \\ -(C-10)^2 - (T-15)^2 > \overline{U}^{B} \; if \quad \lambda = 0. \\ -(C-20)^2 - (T-15)^2 > \overline{U}^{B} \; if \quad \mu = 0. \end{cases}$$

So there are multipliers λ , μ ε R_0^+ that satisfy the above system.

Take for instance the point (15, 15). Then we have λ=1 and μ= 0.
With the point (20,20) we have λ=0 and μ= 1.
With point (15,20) we have λ=1 and μ= 0.
With point (20,25) we have λ= μ= 0.

So the Lagrangean multiplier method applies here and it gives us a perfectly good necessary conditions. The only "oddity" is that there is a two-dimensional set of solutions rather than just a one-dimensional set as in the case of Anne and Bruce.