Contents

Lecture 3

Allocation and Distribution

The undisputed standard graduate public finance textbook when I was in graduate school in the 1960's was Richard Musgrave's *The Theory of Public Finance* [2]. In this book, Musgrave proposes that the main economic functions of government could be divided among three branches, the Allocation, the Distribution, and the Stabilization Branches of government. The job of the Allocation Branch is to "secure adjustments in the allocation of resources". The job of the Distribution Branch is to "secure adjustments in the distribution of income and wealth", and the job of the Stabilization Branch is to secure "economic stabilization".

Musgrave suggests that we think of each branch as run by a "manager" who is instructed to "plan his job on the assumption that the other two branches will perform their functions properly." Thus the Allocation Branch proceeds on the "assumption of full employment of resources and that the proper distribution of income has been secured." The distribution branch assumes that "a full-employment income is available for distribution and that the satisfaction of public wants is taken care of." Similarly for the Stabilization Branch–(but I'm afraid my class never got as far as the stabilization part of Musgrave's book.)

Musgrave's proposed division of labor was and is an attractive one. A good part of the appeal of this separation is that it approximately coincides with lines of specialization in the academic world. The Stabilization Branch could be staffed by macroeconomists, the Allocation Branch by microeconomists and the Distribution Branch by welfare economists, ethical philosophers, and perhaps a few stray theologians and political scientists. The macroeconomists and microeconomists would never have to communicate directly and the microeconomists would rarely have to communicate with the Distribution mélange.

In this lecture we consider the relation between the Allocation Branch and the Distribution Branch. In case utility is quasi-linear, this relation is especially simple. In fact, the Allocation Branch can get its job done while paying almost no attention to the actions of the Distribution Branch. As you recall from our discussion in the last chapter, if there is quasilinear utility, then so long as the Allocation Branch knows that the Distribution Branch is not going to be so cold-hearted as to leave some consumers with zero private goods, there is a unique Pareto optimal amount of public goods. All the Allocation Branch needs to do is to solve for the Pareto optimal quantity of public goods and provide it.[1]

But in general, the Allocation Branch will not be able to determine the right amount of public goods to supply unless it knows what the Distribution Branch is doing. This makes life more complicated, but does not necessarily mean that we must abandon Musgrave's program of divisional separation. Recall that Musgrave's suggestion was not that each branch should ignore the actions of the others, but rather that each branch should assume that the other branches "will perform their functions properly". Let's see how this goes in an explicit example.

A Case where The Allocation Branch Needs to Know What the Distribution Branch is Doing

We return to Cecil and Dorothy, but now suppose that they both have Cobb–Douglas utility functions. In particular:

$$U_C(X_C, Y) = X_C Y^2 \tag{3.1}$$

$$U_D(X_D, Y) = X_D Y \tag{3.2}$$

Suppose that $p_x = p_y = 1$. A little calculation shows the Samuelson condition to be

$$2\frac{X_C}{Y} + \frac{X_D}{Y} = 1 \tag{3.3}$$

[1] At the time when Musgrave's book was written, public finance economists paid little attention to the problem of how the Allocation Branch was to find out the utility functions of consumers who would be willing to tell the truth about their preferences only if it was in their interest. Perhaps if Musgrave were writing this book today, he would add an Investigative branch, or to make it sound a little less sinister, an Econometric Survey Research branch.

or equivalently:

$$2X_C + X_D = Y. \tag{3.4}$$

This equation together with the family budget equation,

$$X_C + X_D + Y = W \tag{3.5}$$

gives us two equations in the three unknowns, X_C, X_D, and Y. There is not enough information in equations 3.4 and 3.5 to solve uniquely for Y, without postulating something about the distribution (X_C, X_D) of income between Cecil and Dorothy. Indeed if we use equation 3.5 to eliminate X_D from equation 3.4, we find that the efficiency conditions are satisfied for any choice of Y and X_C such that $Y = X_C/2 + W/2$. This means that the optimal amount of Y depends on how the private goods are divided between Cecil and Dorothy. The more generously the Distribution Branch chooses to treat Cecil relative to Dorothy, the more public good the Allocation Branch should supply.

If the Allocation Branch knows the rule according to which the Distribution Branch is going to operate, then in typical cases it can solve uniquely for the right amount of public good. For our example, the rule used by the Distributive branch adds one more equation to the two equations with which the Allocation Branch has to work. Suppose, for instance, that the Distribution Branch decides that Cecil and Dorothy should always have equal incomes. Then in addition to equations A and B, we have

$$X_C = X_D. \tag{3.6}$$

Solving the system of equations 3.4, 3.5, and 3.6, we find that $Y = \frac{3}{5}W$ and $X_C = X_D = W/5$.

When can the Allocation Branch Ignore Distribution?

We showed that when Cecil and Dorothy have quasi-linear utility functions, there is a unique amount of public goods that satisfies the Samuelson condition. This fact generalizes to the case of many consumers, all of whom have quasilinear utility functions. While quasilinear utility functions are very easy to work with, the assumption of quasilinear utility in public goods is not very realistic. If preferences are quasi-linear, then a consumer's marginal rate of substitution between public and private goods must be independent of wealth. If this assumption held in the real world, we would expect to find that rich communities would choose the same menu of public goods as poor communities. We would also expect to see that within a given community, if the rich are taxed at a higher rate than the poor, then the rich would always favor less public goods than the poor. As we will see later, both of these conclusions are strongly refuted by available empirical evidence.

It turns out, however, that there is an interesting class of preferences, broader than the class of quasilinear preferences, for which the Pareto optimal amount of public goods does not change when income is redistributed among consumers. Before examining a more general class of utility functions that have this property, we will look at one specific example where preferences are not quasilinear, but where there is a unique Y that satisfies the Samuelson conditions.

The Case of Identical Cobb-Douglas Utilities

Suppose that there are n consumers, each of whom has a utility function of the form:

$$U(X_i, Y) = X_i^{\alpha} Y^{\beta}. \tag{3.7}$$

where $\alpha > 0$ and $\beta > 0$. Suppose also that this economy begins with a total endowment of W units of private good and no public goods, but it is possible to produce public goods at a constant cost of c units of private good per unit of public good. Then an allocation $(X_1, \ldots, X_n, Y) \geq 0$ is feasible if and only if

$$X + cY = W \tag{3.8}$$

where $X = \sum_{i=1}^{n} X_i$. The Samuelson necessary condition for a Pareto optimal allocation requires that the sum of the marginal rates of substitution between public and private goods equals the marginal cost of public goods. Consumer i's marginal rate of substitution between public and private goods is

$$\frac{\beta}{\alpha}\frac{X_i}{Y}.$$

The sum of the marginal rates of substitution over all consumers is:

$$\frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{X_i}{Y} = \frac{\beta}{\alpha}\frac{X}{Y}. \tag{3.9}$$

Therefore the Samuelson condition can be written:

$$\frac{\beta}{\alpha}\frac{X}{Y} = c. \tag{3.10}$$

We see that in this case, the sum of marginal rates of substitution depends only on the total amount X of private consumption. Thus any redistribution of income that leaves total private consumption unchanged will have no effect on the sum of marginal rates of substitution. Although individual marginal rates of substitution depend on individual private consumption, we see that the sum of these marginal rates of substitution is not changed if private consumption is redistributed while X remains constant.

Solving the simultaneous equations 3.8 and 3.10, we find that the unique value of Ys that satisfies the Samuelson necessary condition for an efficient allocation is

$$\bar{Y} = \frac{\beta}{\alpha + \beta}\left(\frac{W}{c}\right) \tag{3.11}$$

Since the sum of the marginal rates of substitution depends only on the total amount of private consumption and not on who gets it, any allocation $(X_1, \ldots, X_n, \bar{Y}) \geq 0$ such that

$$\sum X_i = W - c\bar{Y} = \frac{\alpha}{\alpha + \beta}W \tag{3.12}$$

will be feasible and satisfy the Samuelson condition.

More General Results

Bergstrom and Cornes [1] found a more general class of utility functions for which the Pareto efficient amount of public goods is independent of the distribution of private goods. Suppose that there is a single private good and k public goods. Let X_i denote the amount of private goods consumed by individual i and let $Y = (Y_1, \ldots, Y_k)$ be the vector of public goods. The Bergstrom-Cornes family of utility functions take the following form for each consumer i:

$$U_i(X_i, Y) = A(Y)X_i + B_i(Y). \tag{3.13}$$

Notice that each individual has the same function $A(\cdot)$, but that the functions $B_i(\cdot)$ can differ from person to person.

Samuelson Conditions

If there is just one public good and utility functions take this form, then the marginal rate of substitution of each consumer i between the public good and the private good is seen to be:

$$MRS_i(X_i, Y) = \frac{A'(Y)}{A(Y)} X_i + \frac{B_i'(Y)}{A(Y)}. \tag{3.14}$$

Summing Equation 3.13 over all i and rearranging terms slightly, we find that the Samuelson condition can be written as

$$\frac{A'(Y)}{A(Y)} X + \sum_{i=1}^{n} \frac{B_i'(Y)}{A(Y)} = c. \tag{3.15}$$

where $X = \sum_{i=1}^{n} X_i$.

Thus we see that the sum of the marginal rates of substitution depends only on the aggregate amount of consumption X and on the amount of public goods Y and does not depend on the distribution of the private goods among individuals. We can use the feasibility condition $X + cY = W$ to eliminate the variable X from Equation 3.15. Then we have:

$$\frac{A'(Y)}{A(Y)} (W - cY) + \sum_{i=1}^{n} \frac{B_i'(Y)}{A(Y)} = c. \tag{3.16}$$

The only variable in Equation 3.16 is Y. From the previous lecture we know that (if utility is continuously differentiable) the Samuelson condition is necessary for an interior Pareto optimum and that if utility functions are also quasiconcave, then the Samuelson condition together with the feasibility equation is sufficient for an allocation to be Pareto efficient. Therefore we can conclude that at any interior Pareto optima, the amount of public goods must solve Equation 3.16. We also know that if Equation 3.16 is satisfied for $Y = \bar{Y}$, then every allocation $(X_1, \ldots, X_n, \bar{Y})$ such that $\sum X_i = W - cY$ and such that $X_i > 0$ for all i is Pareto optimal.

A Non-calculus Treatment

We haven't yet answered the question of when there is a *unique* value of Y that satisfies Equation 3.16. Nor have we worked out the story of what happens at boundary solutions. We could approach the uniqueness questions with calculus arguments and the boundary solutions with Kuhn-Tucker methods, but I think it is more instructive to take a different approach. We can use simpler arguments based on addition, multiplication, some inequalities and some simple geometry of convex sets.

Let us begin by extending our discussion to a more general set of feasible allocation than we have considered previously. Specifically, let us assume that there is some closed bounded subset $\mathcal{F}$ of the Euclidean plane such that the set of feasible allocations consists of all allocations $(X_1, \ldots, X_n, Y)$ such that $(\sum X_i, Y) \in \mathcal{F}$.[2]

Preferences that can be represented by utility functions of the Bergstrom-Cornes form as in Equation 3.13 all have the same linear coefficient for private consumption. Therefore, the sum of individual utilities is determined by the amount Y of public goods and the total amount X of private goods consumed and does not depend on how the private goods are divided among individuals. Specifically, we have

$$\sum_{i=1}^{n} U_i(X_i, Y) = A(Y)X + \sum_{i=1}^{n} B_i(Y) \tag{3.17}$$

where $X = \sum_{i=1}^{n} X_i$.

Now consider the combination $(\bar{X}, \bar{Y})$ of public goods and aggregate private good output that maximizes the sum of utilities subject to the feasibility constraint, $(X, Y) \in \mathcal{F}$.[3] Why should we be interested in the feasible outcome that maximizes the sum of utilities? Because any allocation that maximizes the sum of utilities must be Pareto optimal. (I leave this as an exercise for you to prove.)

The sum of utilities will be the same at all allocations in which the amount of public goods is $\bar{Y}$ and the total amount of the private good is $\bar{X}$ although, of course, different allocations of the same total amount of private goods will lead to different distributions of utility. But since $(\bar{X}, \bar{Y})$ maximizes the sum of utilities over all feasible allocations, it must be that every allocation in which the amount of public goods is $\bar{Y}$ and the total amount of a private goods is $\bar{X}$ is Pareto optimal. We can state this result more formally.

Proposition 1 *Suppose that preferences of all consumers can be represented by utility functions of the form $U_i(X_i, Y) = A(Y)X_i + B_i(Y)$ and suppose that $(\bar{X}, \bar{Y})$ maximizes $A(Y)X + \sum_i B_i(Y)$ over the set of all feasible combinations of X and Y, where $X = \sum_i X_i$. Then every allocation $(X'_1, \ldots, X'_n, \bar{Y})$ such that $\sum X'_i = \bar{X}$ is Pareto optimal.*

[2] In the special case where there is a fixed initial endowment of private goods and public goods are produced from private goods at constant cost of c per unit, the set $\mathcal{F}$ is $\{(X, Y) | X + cY = W\}$.

[3] A standard mathematical result (known as the Weierstrass theorem) tells us that if the feasible set is a non-empty closed bounded set in a finite-dimensional space and if the function to be maximized is continuous then there is at least one point in the feasible set that maximizes the function over the feasible set. Therefore so there always is at least one $(\bar{X}, \bar{Y})$ that solves this constrained maximization problem.

Proposition 1 tells us that we can find a whole lot of Pareto optima by choosing $\bar{X}$ and $\bar{Y}$ to maximize $A(Y)X + \sum B_i(Y)$ subject to $(X, Y) \in \mathcal{F}$ and then distributing the total amount $\bar{X}$ of private goods in any way that adds up. This theorem does not, however, tell us whether $\bar{Y}$ is the only possible amount of public goods in a Pareto optimum, or even if $\bar{Y}$ is the only possible amount of public goods at an *interior* Pareto optimum.

Figure 3.1: **Maximizing the Sum of Utilities**

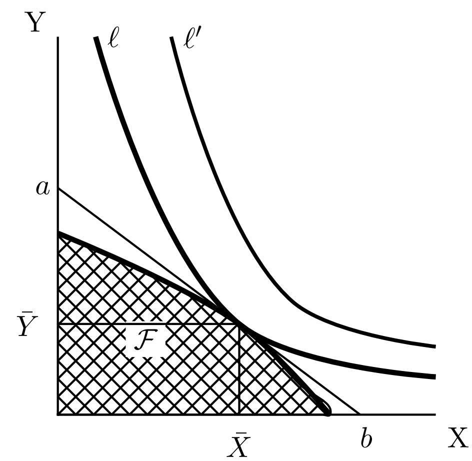


Figure 3, which will look familiar to you from consumer theory will give you a good idea of how to answer these questions. The crosshatched region in Figure 3 shows the set of feasible allocations $\mathcal{F}$. We have drawn two level curves (indifference curves) ℓ and ℓ' for the function $A(Y)X + \sum B_i(Y)$. Notice that we have drawn the set $\mathcal{F}$ as a convex set and we have also drawn the level curves to be convex toward the origin, in such a way that the set of points above each level curve is a convex set with no flat edges. A standard result in consumer theory is that this is appropriate if and only if the function $A(Y)X + \sum B_i(Y)$ is a strictly quasi-concave function. The picture shows the indifference curve ℓ to be the highest indifference curve that touches the feasible set $\mathcal{F}$. The point of tangency is the point $(\bar{X}, \bar{Y})$. The line shown as ab is tangent both to the set $\mathcal{F}$ and to the set $\{(X, Y)|A(Y)X + \sum B_i(Y) \geq A(\bar{Y})\bar{X} + \sum B_i(\bar{Y}\}$ at the one and only point $(\bar{X}, \bar{Y})$ belonging to both sets.

Looking at Figure 3, we see that when the feasible set $\mathcal{F}$ is closed, bounded and convex and the function $A(Y)X + \sum B_i(Y)$ is a strictly quasi-concave function, that there is exactly one quantity $\bar{Y}$ of public goods that corresponds to an outcome which maximizes the sum of utilities. Bergstrom and Cornes are able to show that when this is true, in any Pareto optimal allocation that gives a positive amount of private goods to each consumer, the amount of public goods must be the unique quantity $\bar{Y}$ that maximizes the sum of utilities.

Proposition 2 *Suppose that preferences of all consumers can be represented by strictly quasiconcave utility functions of the form* $U_i(X_i, Y) = A(Y)X_i + B_i(Y)$ *and that the set* $\mathcal{F}$ *of feasible combinations of aggregate output and public good supply is closed, convex and bounded. Then there is a unique quantity of public goods* $\bar{Y}$ *such that in every Pareto optimal allocation in which each consumer has a positive amount of private goods, the amount of public goods must be* $\bar{Y}$.

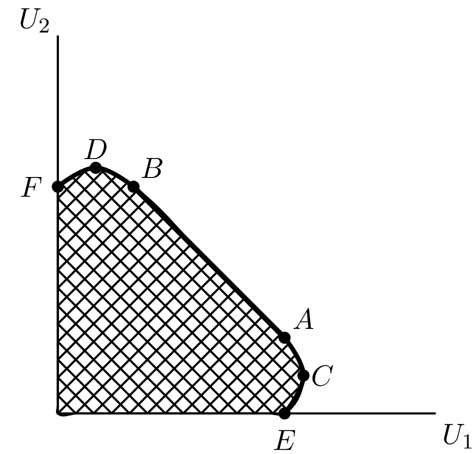
Although any allocation that maximizes the sum of utilities must be Pareto optimal, (You will be asked to prove this in an exercise), it is not in general true that every Pareto optimum maximizes the sum of utilities. Example 3.1, shows why this is the case. Example 3.2 shows why we need the convexity assumption for Proposition 2.

Example 3.1

Suppose that there are two persons, 1 and 2, and that each consumer i has utility function $U_i(X_, Y) = X_i + \sqrt{Y}$. Public goods can be produced from private goods at a cost of 1 unit of private goods per unit of public goods and there are initially 3 units of private goods which can either be used to produce public goods or can be distributed between persons 1 and 2. Thus the set of feasible allocations is $\{(X_1, X_2, Y) \geq 0 | X_1 + X_2 + Y \leq 3\}$. The sum of utilities is $U_1(X_1, Y) + U_2(X_2, Y) = X_1 + X_2 + 2\sqrt{Y}$ which is equal to $X + 2\sqrt{Y}$ where $X = X_1 + X_2$. Therefore we would maximize the sum of utilities by maximizing $X + 2\sqrt{Y}$ subject to $X + Y \leq 3$. The solution to this constrained maximization problem is $Y = 1$ and $X = 2$. Any allocation $(X_1, X_2, 1) \geq 0$ such that $X_1 + X_2 = 2$ is a Pareto optimum.

In Figure 3, we draw the utility possibility set. We start by finding the utility distributions that maximize the sum of utilities and in which $Y = 1$ and $X_1 + X_2 = 2$. At the allocation $(2, 0, 1)$, where Person 1 gets all of the private goods, we have $U_1 = 2 + 1 = 3$ and $U_2 = 0 + 1 = 1$. This is the point A. If Person 2 gets all of the private goods, then $U_1 = 0 + 1 = 1$ and $U_2 = 2 + 1 = 3$. This is the point B. Any point on the line AB can be achieved by supplying 1 unit of public goods and dividing 2 units of private goods between Persons 1 and 2 in some proportions.

Figure 3.2: **A Utility Possibility Set**



Now let's find the Pareto optimal points that do not maximize the sum of utilities. Consider, for example, the point that maximizes Person 1's utility subject to the feasibility constraint $X_1 + X_2 + Y = 3$. Since Person 1 has no interest in Person 2's consumption, we will find this point by maximizing $U_1(X_1, Y) = X_1 + \sqrt{Y}$ subject to $X_1 + Y = 3$. This is a standard consumer theory problem. If you set the marginal rate of substitution equal to the relative prices, you will find that the solution is $Y = 1/4$ and $X_1 = 2\frac{3}{4}$. With this allocation, $U_1 = 3\frac{1}{4}$ and $U_2 = 1/2$. This is the point C on Figure 3. Notice that when Person 1 controls all of the resources and maximizes his own utility, he still leaves some crumbs for Person 2, by providing public goods though he provides them from purely selfish motives. The curved line segment CA comprises the utility distributions that result from allocations $(W - Y, 0, Y)$ where Y is varied over the interval $[1/4, 1]$. An exactly symmetric argument will find the segment DB of the utility possibility frontier that corresponds to allocations in which Person 2 gets no private goods.

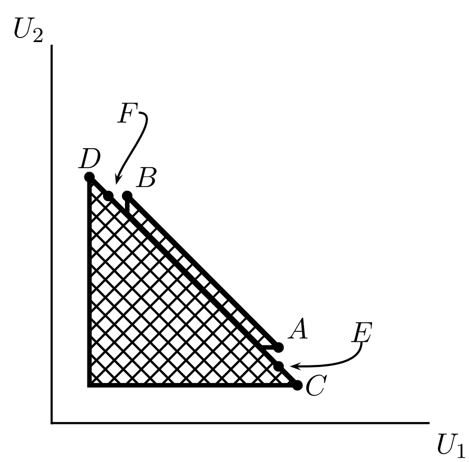
The utility possibility frontier, which is the northeast boundary of the utility possibility set, is the curve $CABD$. There are also some boundary points of the utility possibility set that are not Pareto optimal. The curve segment CE consists of the distributions of utility corresponding to allocations $(W - Y, 0, Y)$ where Y is varied over the interval $[0, 1/4]$. At these allocations, Person 1 has all of the private goods and the amount of public goods is less than the amount that Person 1 would prefer to supply for himself. Symmetrically, there is the curve segment DF in which Person 2 has all of the private goods and the amount of public goods is less than 1/2. Finally, every utility distribution in the interior of the region could be achieved by means of an allocation in which $X_1 + X_2 + Y < 3$.

In this example, we see that at every Pareto optimal allocation in which each consumer gets a positive amount of private goods the amount of public goods must be $Y = 1$, which is the amount that maximizes the sum of utilities.

Example 3.2

Suppose that as in Example 3.1, there are two persons, 1 and 2, and each person i has utility function $U_i(X_{,}Y) = X_i + \sqrt{Y}$. As in the previous example, public goods can be produced from private goods at a cost of 1 unit of private goods per unit of public goods and there are initially 3 units of private goods. But in this example, the amount of public goods supplied must be either $Y = 0$ or $Y = 1/4$. In this case the set of possible allocations is not a convex set.

Figure 3.3: **A Non-convex Utility Possibility Set**



The utility possibility set includes the two lines AB and CD. Points on the line AB show all of the utility distributions that are possible when $Y = 1$ and 2 units of private goods are divided between the persons 1 and 2. Points on the line CD show the utility distributions that are possible when $Y = 1/4$ and $2\frac{3}{4}$ units of private goods are divided between the persons 1 and 2. The cross-hatched area shows the entire utility possibility set. (Points that are not on AB or CD are obtained by wasting some of the private goods.) The utility possibility frontier consists of the three line segments AB, CE, and DF. The Pareto optimal allocations on the line segments CE and DF are reached with $Y = 1/4$ rather than $Y = 1$, even though $Y = 1$ maximizes the sum of utilities. Moreover, except for the endpoints C and D, points on these lines correspond to allocations in which both persons get a positive amount of public goods.

Exercises

3.1 Suppose in the example where Cecil and Dorothy have utility functions $X_C Y^2$ and $X_D Y$ respectively, the Distribution Branch has the rule that $X_C = 2X_D$. Solve for the Pareto optimal choice of Y by the Allocation Branch.

3.2 Where $\alpha > 0$ and $\beta > 0$, show that if all consumers have identical Cobb-Douglas utility functions $X_i^\alpha Y^\beta$ then these same preferences can also be represented by a utility function of the form $A(Y)X_i + B_i(Y)$. What are the functions $A(Y)$ and $B_i(Y)$?

> **Hint:** *What monotonic transformation of the Cobb-Douglas functions will give a utility function of the Bergstrom-Cornes form?*

3.3 Consider an economy with two individuals. Person i has utility function $Y(X_i + k_i)$ where $k_i > 0$. Public goods can be produced from private goods at a cost of one unit of private goods per unit of public goods, and there is an initial allocation of W units of private goods.

a). Find the unique amount of public goods that satisfies the Samuelson condition.

b). Show that there are some Pareto optima that do not satisfy the Samuelson condition and that have a different amount of public goods.

c). Describe the utility possibility set and the utility possibility frontier. Sketch the way it would look, qualitatively.

d). Suppose that one or both of the k_i's are negative. Compare the quantity of public goods at Pareto optimal outcomes that do not satisfy the Samuelson conditions with those at Pareto optimal outcomes that do. Interpret your result.

3.4 Consider an economy with n individuals where individual i has utility function $U_i(X_i, Y) = Y^\alpha(X_i + \beta_i Y + \gamma_i)$, where $0 < \alpha < 1$, $\sum_i \beta_i = 0$, and $\gamma_i > 0$ for all i. Assume that public goods can be produced from private goods at a cost of one unit of private goods per unit of public goods, and that there is an initial allocation of W units of private goods. Find the unique quantity of Y that satisfies the Samuelson conditions.

3.5 Prove the following results which are claimed in the text of the lecture:

a). An allocation that maximizes the sum of individual utilities over all feasible allocations must be Pareto optimal.

b). Where $a_i > 0$ for all $i = 1, \ldots, n$, any allocation that maximizes the sum

$$\sum_{i=1}^{n} a_i U_i(X_i, Y)$$

of individual utilities over all feasible allocations must be Pareto optimal.

Hint: *Consider an allocation that is feasible and Pareto superior to the allocation that solves your maximization problem. What is true of the sum or weighted sum of the utilities in this allocation? Can this allocation be feasible? Why not?*

3.6 There are two consumers and one public good. Person 1 always prefers more of the public good to less. Person 2's preferences are more subtle. Their utility functions are given by

$$U_1(X_1, Y) = (1 + X_1)Y$$

$$U_2(X_2, Y) = X_2 Y - \frac{1}{2}Y^2.$$

The feasible allocations are those such that $X_1 + X_2 + Y = W$.

a). Are these utility functions of the Bergstrom-Cornes form?

b). Draw some sample indifference curves for Person 2 between private and public goods. How would you describe Person 2's attitude toward public goods?

c). Find the allocations that maximize the sum of utilities. Take care to distinguish the case where W is large enough for there to be an interior solution from the case where it is not.

d). In the case where $W = 4$, find all of the Pareto optimal allocations and draw the utility possibility set and show the utility possibility frontier.

e). In the case where $W = 1/2$, find all of the Pareto optimal allocations and draw the utility possibility frontier. What distributions of private consumption are consistent with a Pareto optimal allocation?

f). For what values, if any, of W are there Pareto optimal allocations in which both consumers consume some private goods and where the sum of utilities over the set of feasible allocations is not maximized.

3.7 Bergstrom and Cornes prove that under fairly weak assumptions representability of preferences in the functional form $A(Y)X_i + B_i(y)$ is both necessary and sufficient for it to be true that *regardless of the level of aggregate income* starting from a Pareto optimal allocations in which both consumers have some private goods, if one leaves the amount of public goods unchanged and redistributes private goods to reach another allocation in which all consumers have some private goods, the resulting outcome will also be Pareto optimal. This exercise shows that the "necessity" part of this proposition depends critically on the qualification *regardless of the level of aggregate income.*

Consider an economy with one public good, one private good, and two consumers. Consumer 1 likes the public good and Consumer 2 hates it. The amount of public good provided must be either 0 or 1. The public good is costless to produce. There is one unit of private good which can be divided between Consumers 1 and 2 in any way such that private goods consumption adds to 1. Thus the set of feasible allocations is $\{(x_1, x_2, y)|x_1 \geq 0, x_2 \geq 0, x_1 + x_2 = 1, y \in \{0, 1\}, \}$. The utility functions of Consumers 1 and 2 respectively are:

$$U_1(x_1, y) = (x_1 + 1)(1 + y) = (1 + y)x_1 + y + 1$$
$$U_2(x_2, y) = (x_2 + 1)(2 - y) = (2 - y)x_2 - y + 2.$$

a). Show that no monotonic transformations of these utility functions will make it possible to write them both in the Bergstrom-Cornes form $A(Y)X_i + B_i(y)$.

> **Hint:** *If this were possible, the sum of marginal rates of substitution would not change after a redistribution of private goods.*

b). Draw the utility possibility frontier for this economy.

c). Assuming that lotteries are not possible, show that every possible allocation is Pareto optimal.

d). Suppose that the functions U_1 and U_2 are the von Neuman Morgenstern representations. Starting from allocations in which both consumers have some private goods, would it be possible to find a lottery that would make both consumers better off?

e). Suppose that instead of 1 unit of private goods, there were two units of private goods to be allocated, so that the set of feasible allocations is $\{(x_1, x_2, y) | x_1 \geq 0, x_2 \geq 0, x_1 + x_2 = 2, y \in \{0, 1\},\}$. Draw and label two separate lines; one showing the utility distributions possible if $y = 0$ and one showing the utility distributions possible if $y = 1$. Do these lines cross? Identify the utility possibility frontier and show that the Pareto optimal amount of public goods depends on the distribution of income.

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