Contents

Lecture 4

Lindahl Equilibrium

The Swedish economist Erik Lindahl [3] proposed an ingenious method for simultaneously resolving the allocation and distribution problems in an economy with public goods. This solution has come to be known as *Lindahl equilibrium*.

Lindahl Equilibrium in a Small Community

In a tiny hamlet on the shores of the Baltic, live two fishermen–Lars and Olaf. Each consumes only one private good, fish. Lars catches F_L fish per year and Olaf catches F_O fish per year. The only other good that they consume is one that they can consume jointly–"adult" video cassettes.[1] Olaf's utility function is, $U_O(X_O, Y)$ where X_O is the number of fish he consumes per year and Y is the number of video cassettes per year that he and Lars consume. Lars's utility function $U_L(X_L, Y)$ is defined similarly. They are able to rent video cassettes at a price of p fish per cassette. When they rent a video cassette, both can watch it together. The allocation problem that they have to solve is, how many video cassettes should they rent per year. The distribution problem is how should the costs of the rentals be divided.

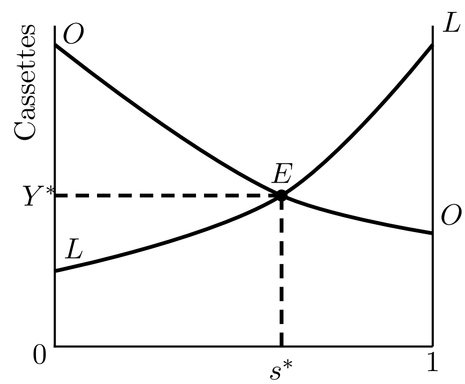
Though Lars and Olaf both like fish and video cassettes, their preferences are not identical. For many years, they argued both about how many video cassettes to rent and who should pay for them. One long winter night, they devised a clever way to solve these two problems simultaneously. Each of them would write down a "demand function" that states the number of cassettes he would want to rent as a function of the share of the cost that

[1] When Lindahl conceived this model, video cassettes were not yet invented. No doubt he had other public goods in mind.

he has to pay. They would then find a division of cost shares at which both agree about the number of cassettes to rent. They rent that number of cassettes and divide the costs in the shares that made them agree.

We can show this solution in the pretty little graph. On the horizontal axis we measure Olaf's share of the cost and on the vertical axis we measure cassettes demanded. The curve OO represents Olaf's demand as a function of his share and the curve LL represents Lars's demand as a function of Olaf's share. When Olaf's share is s (and Lars's share is $1-s$) Olaf's preferred number of video cassettes is the quantity Y that maximizes $U_O(X_O, Y)$ subject to the budget constraint $X_O + spY = F_O$ and Lars's preferred number of video cassettes is the quantity Y that maximizes $U_L(X_L, Y)$ subject to the budget constraint $X_L + (1-s)pY = F_L$

Figure 4.1: Lindahl Equilibrium



If $s = 0$, then Olaf has to pay nothing and Lars has to bear the full cost of the casettes. At these prices, Olaf's demand for casettes would be greater than Lars's demand. But if $s = 1$ so that Olaf has to pay the full cost and Lars pays nothing, then Lars would want more casettes than Olaf. Therefore the curve OO starts out higher than the curve LL on the left side of the graph and ends up lower than LL on the right side of the graph. Assuming that these are continuous curves, they must cross somewhere. If they both slope downwards, then they will cross exactly once.

The point E where the two curves cross determines both the amount of public goods and the way the cost is shared. This, together with the initial allocation of fish determines the consumption of private goods by each fisherman as well as the amount of public goods. The equilibrium cost share for Olaf is shown as s^* and the equilibrium number of casettes is Y^*. When Olaf's tax share is s^* and Lars' share is $1 - s^*$, we see from the graph that both fishermen agree in demanding Y^* video casettes. This outcome is the Lindahl equilibrium for Lars and Olaf.

The Lindahl equilibrium has the very satisfactory property that, not only do Lars and Olaf agree on the amount of public goods, but the quantity on which they agree satisfies the Samuelson conditions for Pareto optimality. In Lindahl equilibrium, Olaf enjoys $X_O^* = F_O - s^*pY^*$ units of fish and Y^* video casettes per year, while Lars consumes $X_L^* = F_L - (1 - s^*)pY^*$ units of fish and Y^* video casettes per year. Since the bundle (X_O^*, Y^*) maximizes Olaf's utility subject to the budget constraint $X_O + s^*pY = F_O$, Olaf's marginal rate of substitution between casettes and fish must be equal s^*p, which is the ratio of the price he pays for casettes relative to the price of fish. Similarly, since (X_L^*, Y^*) maximizes Lars's utility subject to the budget $X_L + (1 - s^*)pY = F_L$, Lars's marginal rate of substitution between casettes and fish must be equal to $(1 - s^*)p$. Therefore in Lindahl equilibrum, the sum of Lars's and Olaf's marginal rates of substitution must equal $s^*p + (1 - s^*)p = p$, which tells us that the Samuelson condition is satisfied.

Lindahl Equilibrium More Generally

Lindahl's solution concept can be generalized to large communities with many public goods and many private goods. Consider a community that has n consumers. There are ℓ *pure private goods* and m *pure public goods.* The vector of private goods consumed by person i is written x^i. Everyone must consume the same vector of public goods and that vector is denoted by y. Each consumer i has a utility function of the form $U^i(x^i, y)$.

In the general Lindahl model there are prices both for public goods and for private goods. The twist is this. With private goods, different people can consume different quantities, but in equilibrium they all must pay the same prices. With public goods, everyone must consume the same amount quantity, but in Lindahl equilibrium, they may pay different prices. Let us denote the (ℓ dimensional) price vector for public goods by p. Let us denote the (m dimensional) price vector paid by citizen i for public goods by q^i and let us define $q = \sum_i^n q^i$.

The vector of aggregate consumption of private goods is $x = \sum_i^n x^i$. An *aggregate output vector* (x, y) consists of a vector of an aggregate private good supply and a vector of public goods output. The *set of feasible ag-*

gregate outputs is asssumed to be a closed bounded set $\mathcal{F}$ in $\Re^{\ell+m}$. The *set of feasible allocations* is the set of allocations that can be achieved by distributing the private goods from a feasible aggregate output in a way that adds up to the aggregate amount. That is the set of feasible allocations consists of all allocations $(x^1, \dots, x^n, y)$ such that $(\sum_i^n x^i, y) \in \mathcal{F}$.

We assume that the wealth of each consumer i is determined by a wealth distribution function $W_i(p, q)$. Where the private goods are valued at the price vector p and public goods are valued at the price vector $q = \sum_{i=1}^n q^i$, the wealth distribution functions have the property that at any vector of private goods prices and Lindahl prices, the sum of individual wealths adds up to the value of the most valuable feasible aggregate output. Thus

$$\sum W_i(p, q) = \max\{px + qy | (c, y) \in \mathcal{F}\}. \tag{4.1}$$

The wealth distribution function is simply a generalization of the wealth distribution function that one finds in a competitive private goods economy. In a private goods economy each person has a specified initial endowment of goods and rights to some specified share of firms' profits, which are maximized over all feasible input-output combinations. For an economy with public goods.

A Lindahl equilibrium consists of a vector of private goods prices $\bar{p}$, individualized public goods prices (Lindahl prices), $(\bar{q}^1, \dots \bar{q}^n)$ and an allocation $(\bar{y}, \bar{x}^1, \dots \bar{x}^n)$ such that for each consumer i, $(\bar{x}^i, \bar{y})$ maximizes $U^i(x^i, y)$ subject to $\bar{p}x^i + \bar{q}^i y \le W_i(\bar{p}, \bar{q})$.

Having defined Lindahl equilibrium for such a general economy, we must ask whether a general Lindahl equilibrium has interesting properties. It turns out that with great generality, Lindahl equilibrium is Pareto optimal. The proof is similar to the beautiful Arrow-Debreu proof of the Pareto optimality of competitive equilibrium.

Proposition 1 (Pareto Optimality of Lindahl Equilibrium) *If all consumers have locally non-satiated preferences, then a Lindahl equilibrium is Pareto optimal.*

Proof: Suppose that allocation $(\bar{y}, \bar{x}^1, \dots \bar{x}^n)$ is a Lindahl equilibrium with private goods prices $\bar{p}$ and Lindahl prices, $(\bar{q}^1, \dots \bar{q}^n)$. Consider an alternative allocation $(y, x^1, \dots, x^n)$ that is Pareto superior to $(\bar{y}, \bar{x}^1, \dots \bar{x}^n)$. We will show that the alternative allocation is not feasible. If $(y, x^1, \dots x^n)$ is Pareto superior to $(\bar{y}, \bar{x}^1, \dots \bar{x}^n)$, then it must be that for all individuals i, $U^i(x^i, y) \ge U^i(\bar{x}^i, \bar{y})$ with strict inequality for some i. In Lindahl equilibrium, $(\bar{x}^i, \bar{y})$ maximizes $U^i(x^i, y)$ subject to $\bar{p}x^i + \bar{q}^i y \leq W_i(\bar{p}, \bar{q})$. Therefore when preferences are locally non-satiated, it must be that if $U^i(x^i, y) \geq U^i(\bar{x}^i, \bar{y})$, then $\bar{p}x^i + \bar{q}^i y \geq W_i(\bar{p}, \bar{q})$ with strict inequality for those consumers who strictly prefer $(y, x^1, \ldots, x^n)$ to $(\bar{y}, \bar{x}^1, \ldots \bar{x}^n)$ i. Adding these inequalities over all consumers i, we have

$$\sum_{i=1}^{n} \left(\bar{p}x^i + \bar{q}^i y \right) > \sum_{i=1}^{n} W_i(\bar{p}, \bar{q}) \tag{4.2}$$

From the definition of a wealth distribution function, we know that

$$\sum_{i=1}^{n} W_i(\bar{p}, \bar{q}) \geq \bar{p}x' + \left(\sum_{i=1}^{n} \bar{q}^i \right) y' \tag{4.3}$$

for all $(x', y') \in \mathcal{F}$.

Rearranging the sums on the leftside of Expression 4.2, it follows from 4.2 and 4.3 that

$$\bar{p} \left(\sum_{i=1}^{n} x^i \right) + \left(\sum_{i=1}^{n} \bar{q}_i \right) y > \bar{p}x' + \left(\sum_{i=1}^{n} \bar{q}^i \right) y' \tag{4.4}$$

for all $(x', y') \in \mathcal{F}$. But this means that $(\sum x^i, y) \notin \mathcal{F}$, and hence it follows that any allocation $(y, x^1, \ldots, x^n)$ that is Pareto superior to $(\bar{y}, \bar{x}^1, \ldots \bar{x}^n)$ is not feasible.

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However fascinating its properties might be, Lindahl equilibrium would not be very interesting if it turned out that Lindahl equilibrium doesn't exist. Although Lindahl showed diagrammatically that Lindahl equilibrium exists for a community with two consumers and one public good, the requirements for Lindahl equilibrium in a general model may seem pretty stringent. Even with separate prices for each individual and each public good, getting everybody to agree about the quantity of every single public good might be too much to ask, except in very special cases. As it turns out, however, Lindahl equilibrium exist for a rich class of models. And, moreover, we don't need much in the way of new mathematical apparatus to show that this is true. Kenneth Arrow [1] pointed out that the powerful apparatus developed by economists to prove the existence of competitive equilibrium can be carried over almost intact to show the existence of Lindahl equilibrium. Details of this argument are worked out in Bergstrom [2]. We sketch this line of reasoning below.

How are Public Goods Like Sheep?

Pure public goods are not very much like real sheep at all. But they are very like the abstract sheep that economists know and love. An economist's sheep produces wool and mutton in fixed proportions. With real sheep, there are many ways to alter the proportions of these outputs. (One method is simply to allow a sheep to live longer and be sheared more times before he is eaten. Sheep[2] can also be bred to be more or less meaty relative to how woolly they are. (Alfred Marshall, *Principles of Economics*, Book V, chapter VI) understood these matters well.) Pure public goods are like economists' sheep, in that a pure public good jointly produces public good services for all consumers in the community.

Let us return to the Baltic shores. The set of feasible allocations for Lars and Olaf were those allocations (X_O, X_L, Y) such that $X_O + X_L + pY = F_O + F_L$. Although in this example, Lars and Olaf must necessarily consume the same number of casettes as each other, let us define two separate variables to represent their consumptions. Let Y_L denote casettes for Lars and Y_O denote casettes for Olaf. Then we can write Lars's utility function as $U_L(X_L, Y_L)$ and Olaf's utility function as $U_O(X_O, Y_O)$. Written this way, the utility functions look just like ordinary utility functions of selfish consumers for two private goods, fish and casettes. Let us consider a competitive equilibrium for an artificially constructed private goods economy, in which these are the utility functions but where the production sector has the restriction that Y_L and Y_O must be produced in fixed proportions. In this constructed economy, the only available production technology is an activity that operates at constant returns to scale. For every p units of fish that this process uses as inputs, it produces outputs of 1 unit of the commodity "casettes for Lars" and one unit of the commodity "casettes for Olaf." The artificial economy that we have constructed in this way satisfies all of the standard conditions of an Arrow-Debreu competitive economy. In fact, this economy is isomorphic to the purely competitive economy studied by Alfred Marshal in which sheep produce wool and mutton in fixed proportions. Sheep correspond to video casettes. Mutton corresponds to "casettes for Lars." Wool corresponds to "casettes for Olaf." This interpretation of joint products for Lars and Olaf has a feature that is not usually present in the sheep-mutton example, and that is that "mutton" is useful to Lars but not to Olaf and "wool" is useful to Olaf but not to Lars. This extra feature, however, will cause no problems in applying the standard theorems on the existence or

[2] Much like economists.

optimality of equilibrium.

While it is a comfort to know that public goods are like sheep, we have a deeper reason for pointing out this similarity. A powerful technique in theoretical reasoning is to find an isomorphism between the theory you happen to be studying and some other theory that has been richly developed. Where such an isomorphism is found, the results of one theory can be lifted over and translated into theorems in the other. In this case, we have translated a public goods economy into a formally equivalent private goods economy. Furthermore, if the public goods economy has convex preferences and a convex production possibility set, the corresponding private goods economy will inherit these properties and it is straightforward to apply the standard general competitive equilibrium analysis to this economy. The Arrow-Debreu theory of the existence and optimality of competitive equilibrium goes through without a hitch when there are fixed coefficients in production.

If we map the Lars-Olaf economy into an artificial private goods economy in the way we have just proposed, we know that a competitive equilibrium exists and is Pareto optimal. You will find it easy to verify that this competitive equilibrium quantity of cassettes for the artificial private goods economy corresponds to the Lindahl equilibrium quantity of casettes and the competitive equilibrium prices of the artificial commodities "cassettes for Lars" and "casettes for Olaf" correspond respectively to Lars's Lindahl price, $(1-S_O^*)p$ and Olaf's Lindahl equilibrium price S_O^*p. The same correspondence applies in general to the case of many goods and many consumers.

Deception's Ugly Head

Lindahl equilibrium appears to be a very attractive concept. It provides a determinate solution to both the allocation and distribution problem. The Lindahl solution seems equitable, promotes harmony in public decisions, and is Pareto efficient and equilibrium exists for a large class of environments, much as is the case for competitive equilbrium.

But there is a serious problem that we have so far not faced. For straightforward implementation of a Lindahl equilibrium, even in a case as simple as that of Lars and Olaf, we would have to know the demand functions of each individual for public and private goods. But how are we to discover these curves? The simplest answer would be: Let's ask Lars and Olaf. But here is the rub. Can we expect them both to tell us the truth? The answer is pretty simple and pretty discouraging...No. (But later we will discuss interesting and clever ways that economists have found for extracting honest information.)

Let us simplify our problem by assuming that Lars and Olaf can describe their preference by reporting a single parameter, which at least before reporting is information private to its possessor. (For example it might be that each has a Cobb-Douglas utility function and only the owner of the utility function knows the exponent that is attached to public goods.) Let r_i be the value of this parameter reported by person i. Suppose that after the parameter values are reported, Lars and Olaf calculate and implement the resulting Lindahl equilibrium. Let us denote the Lindahl quantities of public goods and Lindahl cost shares for Olaf corresponding to these reports by $Y^*(r_L, r_O)$ and $S_O^*(r_L, r_O)$. When this Lindahl equilibrium is implemented, Olaf's consumption of private goods will be

$$X_O^*(r_L, r_O) = F_O - S_O^*(r_L, r_O) p Y^*(r_L, r_O). \tag{4.5}$$

We can thus write Olaf's utility as a function $\tilde{U}_O(r_O, r_L)$ of the parameters that he and Lars report. Thus we define

$$\tilde{U}_O(r_O, r_L) = U_O\left(X_O^*(r_L, r_O), Y^*(r_L, r_O)\right) \tag{4.6}$$

Suppose that we are initially in Lindahl equilibrium with both fishermen telling the truth. Would it pay Olaf to deviate from the truth?

Let's try differentiating both sides of Equation 4.6 with respect to r_O:

$$\begin{aligned} \frac{\partial \tilde{U}_O(r_O, r_L)}{\partial r_O} &= \frac{\partial U_O(X_O^*, Y^*)}{\partial X_O} \frac{\partial X_O^*(r_L, r_O)}{\partial r_O} \\ &+ \frac{\partial U_O\left(X_O^*, Y^*\right)}{\partial Y} \frac{\partial Y^*(r_L, r_O)}{\partial r_O} \end{aligned} \tag{4.7}$$

Differentiating Equation 4.5 we see that

$$\frac{\partial X_O^*(r_L, r_O)}{\partial r_O} = -\frac{\partial S_O^*(r_L, r_O)}{\partial r_O} p Y^*(r_L, r_O) - S_O^*(r_L, r_O) p \frac{\partial Y^*(r_L, r_O)}{\partial r_O} \tag{4.8}$$

If we substitute from Equation 4.8 into Equation 4.7 we find that

$$\begin{aligned} \frac{\partial \tilde{U}_O(r_O, r_L)}{\partial r_O} &= -\frac{\partial U_O\left(X_O^*, Y^*\right)}{\partial X_O} \frac{\partial S_O^*(r_L, r_O)}{\partial r_O} p Y^*(r_L, r_O) \\ &- \left(\frac{\partial U_O\left(X_O^*, Y^*\right)}{\partial X_O} S_O^*(r_L, r_O) p + \frac{\partial U_O\left(X_O^*, Y^*\right)}{\partial Y}\right) \frac{\partial Y^*(r_L, r_O)}{\partial r_O} \end{aligned} \tag{4.9}$$

Take a hard look at the second line of Equation 4.9. Since we have assumed that (X_O^*, Y^*) is Olaf's preferred bundle when he has to pay a price of $S_O^*(r_L, r_O)p$ per unit of private good, it must be that Olaf's marginal rate of substitution between private good and public good is equal to $S_O^*(r_L, r_O)p$. But this implies that

$$-\frac{\partial U_O\left(X_O^*, Y^*\right)}{\partial X_O} S_O^*(r_L, r_O)p + \frac{\partial U_O\left(X_O^*, Y^*\right)}{\partial Y} = 0 \tag{4.10}$$

Therefore the second line of Equation 4.9 is equal to zero and hence

$$\frac{\partial \tilde{U}_O(r_O, r_L)}{\partial r_O} = -\frac{\partial U_O\left(X_O^*, Y^*\right)}{\partial X_O} \frac{\partial S_O^*(r_L, r_O)}{\partial r_O} \, pY^*(r_L, r_O) \tag{4.11}$$

From Equation 4.11 we see that the partial derivative of $\tilde{U}_O(r_O, r_L)$ with respect to r_O must be of the opposite sign from the sign of the partial derivative of $S_O^*(r_L, r_O)$ with respect to r_O. This means that if there is any lie that Olaf can tell that will decrease his Lindahl price, it will be in his interest to tell this lie. Thus at a Lindahl equilibrium in which both individuals tell the truth about their preferences, either individual could gain by pretending to want less public goods than he really does.

Exercises

4.1 Las Venales, Nevada has N people, each of whom consumes a single private good, "bread". There is also a public good, "circus". Everybody in Las Venales has preferences representable by a utility function of the form $U^i(X_i, Y) = X_i^{1-\alpha} Y^\alpha$ where X_i is i's own consumption of bread per year and Y is the number of circus acts performed in Las Venales per year. (There is no congestion at the circus.) Bread is the *numeraire*. Circus acts can be purchased at a cost of p per unit. Although preferences don't differ, incomes do. Person i has an income of W_i.

a). Find the Pareto optimal amount of circus for Las Venales as a function of the parameters, N, p, α and $\sum W_i$.

b). Find the Lindahl equilibrium prices and quantities for Las Venales.

4.2 On the Isle of Glutton, there are two agricultural products, corn and pigs. There are N people. Each person likes a different cut of meat from the pig and has no use for any other part. Conveniently, on the isle of Glutton, the parts of a pig are named after the people who like them. Person i's utility function is $U^i(X_i, Y_i) = X_i^{1-\alpha} Y_i^\alpha$ where X_i is the amount of corn that i consumes and Y_i is the amount of the *ith* cut of pork that person i gets to consume. Person i initially owns W_i units of corn. There are constant returns to scale in raising pigs. To raise a pig, you need to feed him p units of corn. One pig yields one unit of each cut of pork. Ignore the labor cost of growing pigs, the cost of baby pigs, the cost of other foods, housing and entertainment for the pigs. Also ignore any costs of butchering and retailing. Find competitive equilibrium prices of each cut of the pig and the competitive equilibrium number of pigs as a function of the parameters of the problem.

4.3 Los Locos, California has N people, each of whom consumes a single private good, "grass", and a single public good, beach. A developer is interested in building condominiums along the beachfront. But he has offered to sell part of his holdings of beachfront land to the city of Los Locos at p per foot of beach front. There are three types of people in Los Locos, the α's, the β's and the γ's. The α's all have utility functions, $U(X_i, Y) = X^{1-\alpha} Y^\alpha$, and incomes, W_α, the β's all have utility functions, $U(X_i, Y) = X^{1-\beta} Y^\beta$ and incomes, W_β, and the γ's all have utility functions, $U(X_i, Y) = X^{1-\gamma} Y^\gamma$ and incomes W_γ.

a). Find the Lindahl equilibrium prices and quantities.

b). Find the Pareto optimal allocation in which all consumers in Los Locos get to consume the same amount of grass. How much beach would be purchased. Under what conditions would this amount of beach be greater than the amount provided in Lindahl equilibrium.

4.4 Brass Monkey, Ontario has 1000 citizens and each citizen i has a utility function $U^i(X_i, Y) = Y^\alpha(X_i + k_i)$ where Y is the size of the town skating rink, measured in square inches and X_i is then number of doughnuts that i consumes per year. Doughnuts are the numeraire in Brass Monkey, so the price of a doughnut is always one unit of the natural currency. The cost of building and maintaining one square inch of skating rink is also one doughnut. Different people have different incomes. Person i has income W_i. Find a Lindahl equilibrium for Brass Monkey. What quantity of public goods is supplied? In Lindahl equilibrium, how much money does the government collect from Person i?

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