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Lecture 3

Congestion

Trouble on the Highways

Residents of Hot Rod, Indiana drive cars and eat Big Macs. That's it. Resident i has initial wealth, W_i and Big Macs cost one dollar each.[1] Each hour of driving uses fuel that costs p_F. Driving also causes congestion. Let H be the size of the highway network in Hot Rod and let D be the total amount of driving by citizens of Hot Rod. The level of congestion is given by a function $C(D, H)$. Denote the partial derivatives of $C(D, H)$ with respect to D and H respectively by $C_D(D, H)$ and $C_H(D, H)$. We assume that $C_D(D, H) > 0$ and $C_H(D, H) < 0$ for all $D > 0$ and $H > 0$.

Let M_i and D_i denote the number of Big Macs consumed and D_i the number of hours of driving by resident i. Preferences of Hot Rodder i are represented by a utility function:

$$U^i\left(M_i, D_i, C(\sum_{i=1}^{n} D_i, H)\right) \tag{3.1}$$

We assume that U^i is an increasing function of its first two arguments and a decreasing function of its third argument. The set of feasible allocations in Hot Rod is described as the set of vectors $(M_1, \cdots, M_n, D_1, \cdots, D_n, H)$ such that total expenditures on Big Macs, fuel, and highways in Hot Rod add to the total wealth of its citizens. If the price of a Big Mac is \$1, a unit of fuel costs p_F, and a unit of highway costs the community p_H, this

[1]At the time this lecture was written, the U S. Federal Reserve Bank was committed to a monetary policy which maintained the price of Big Macs at this level.

feasibility constraint is:

$$\sum_{i=1}^{n} M_i + p_F \sum_{i=1}^{n} D_i + p_H H \leq \sum_{i-1}^{n} W_i. \tag{3.2}$$

Samuelson Conditions for Efficiency

We could use Lagrange multipliers to solve for the necessary conditions for Pareto optimality. Alternatively we could notice that this model is formally equivalent to one with $n+1$ public goods and one private good. Since everyone dislikes congestion, and i's driving increases congestion, driving by i makes each everyone other than i worse off. Therefore driving by any individual must be treated as a public good. Of course this public good is one that everyone except the driver hates, but the Samuelson conditions apply just as well when some people like and some people hate a public good as when all people like it. Highway expenditures are also a public good, since increased highway expenditures reduces congestion for everyone. But Big Macs remain private goods. Nobody except i cares about how many Big Macs i eats.

Before we state the Samuelson conditions, it is useful to make note of a subtle point about the marginal effect of one's own driving on one's utility. Let us denote the partial derivatives of $U^i(M_i, D_i, C)$ with respect to its three arguments by $U^i_M(M_i, D_i, C)$, $U^i_D(M_i, D_i, C)$, and $U^i_C(M_i, D_i, C)$, respectively. Then $U^i_D(M_i, D_i, C)$ is the derivative of i's utility with respect to i's own driving *holding constant the level of congestion.* This is not the same thing as the derivative of i's utility with respect to her own driving *holding constant the amount of driving by others*, because if i drives more and others continue to drive the same amount as before, then congestion will increase. The partial derivative of i's utility with respect to her own driving, holding constant the amount of driving by others is given by

$$\frac{\partial U^i(M_i, D_i, C)}{\partial D_i} = U^i_D(M_i, D_i, C) + U^i_C(M_i, D_i, C) C_D(D, H) \tag{3.3}$$

For all i and j, let us denote resident j's marginal rate of substitution between driving by resident i and Big Macs by

$$MRS^D_{ij}(M_i, D_i, C). \tag{3.4}$$

Using Equation 3.3, we find that

$$MRS^D_{ii}(M_i, D_i, C) = \frac{U^i_D(M_i, D_i, C)}{U^i_M(M_i, D_i, C)} + \left(\frac{U^i_C(M_i, D_i, C)}{U^i_M(M_i, D_i, C)} \right) C_D(D, H) \tag{3.5}$$

The part of Equation 3.5 to the right of the plus sign is the difference between i's marginal willingness to pay for a unit of driving when the amount of driving by others is held constant and i's marginal willingness to pay for a unit of driving when the congestion level is the same after his entry as before. The difference between these two measures is the amount that i would be willing to pay to have one less car on the road. Where the population is large and the road is also large, this difference will typically very small. (Although congestion costs imposed by any one individual on a single other individual are small, total congestion costs imposed by one individual may be still be large because this small cost is imposed on each of a large number of individuals.)

Since for $j \neq i$, i's driving affects j's utility only through its effect on congestion, we have for all $j \neq i$,

$$MRS_{ij}^{D}(M_i, D_i, C) = \left(\frac{U_C^j(M_j, D_j, C)}{U_M^j(M_j, D_j, C)} \right) C_D(D, H) \tag{3.6}$$

The Samuelson condition for an efficient amount of driving by resident i is that the sum of all residents' marginal rates of substitution between i's driving and their own consumptions of Big Macs equals the resource cost p_F of driving by i. That is:

$$\sum_{j=1}^{n} MRS_{ij}^{D}(M_i, D_i, C) = p_F. \tag{3.7}$$

From Equations 3.5 and 3.6 it follows that Equation 3.7 is equivalent to:

$$\frac{U_D^i(M_i, D_i, C)}{U_M^i(M_i, D_i, C)} = p_F - C_D(D, H) \sum_{j=1}^{n} \left(\frac{U_C^j(M_j, D_j, C)}{U_M^j(M_j, D_j)} \right) \tag{3.8}$$

In words, this condition requires that i's marginal rate of substitution between driving and Big Macs *holding constant the level of congestion* must equal the fuel cost of driving plus the sum of individual valuations of the damage caused by the marginal congestion from extra driving.

The Samuelson condition for an efficient amount of highway expenditures is that the sum of residents' marginal rates of substitution between highway expenditures and Big Macs equals the cost of highway expenditures. This Samuelson condition can be written as:

$$C_H(D, H) \sum_{j=1}^{n} \left(\frac{U_C^j(M_j, D_j, C)}{U_2^j(M_j, D_j, C)} \right) = p_H \tag{3.9}$$

From Equations 3.8 and 3.9, it follows that for all i,

$$\frac{U^i_D(M_i, D_i, C)}{U^i_M(M_i, D_i, C)} = p_F - p_H\left(\frac{C_D(D,H)}{C_H(D,H)}\right). \tag{3.10}$$

Equation 3.10 has a nice interpretation. Suppose the congestion level is set at some fixed level. If D is increased by a small amount, then in order to hold congestion constant, we would have to increase H. In fact if we totally differentiate the expression $C(D,H) = k$, it must be that

$$\frac{dH}{dD} = -\frac{C_D(D,H)}{C_H(D,H)} \tag{3.11}$$

With the aid of Equation 3.11 we see that Equation 3.10 tells us that i's marginal rate of substitution between driving and Big Macs (holding congestion constant) should be equal to the fuel cost of driving plus the cost of adding enough extra highway to eliminate the extra congestion caused by the marginal bit of driving.

Optimal Tolls

Let us consider a system of uniform tolls that results in approximately Pareto efficient amounts of driving by each resident. From equation 3.8 it appears that a likely candidate for this toll rate is

$$T = -p_H\left(\frac{C_D(D,H)}{C_H(D,H)}\right). \tag{3.12}$$

If the toll is set at $p_F + T$, resident i will drive enough to equate her marginal rate of substitution between driving and Big Macs to to the cost of driving. In this case

$$MRS^D_i(D_i, M_i, C) = p_F + T = p_F - p_H\left(\frac{C_D(D,H)}{C_H(D,H)}\right) \tag{3.13}$$

The right side Equation 3.13 is the same as the right side of the Samuelson condition stated in Equation 3.8. But the left side of Equation 3.8 is i's marginal rate of substitution between driving and Big Mac's *holding congestion constant*, while the left side of Equation 3.13 is i's marginal rate of substitution *holding the amount of driving by others* constant. Where the population is large, the amount of congestion that i imposes on himself is small and so the difference between these two concepts of marginal rate of substitution is small. Thus the amount of driving by each individual is close to the efficient amount.

Will Optimal Tolls Finance Optimal Highways?

The government in this model has two tasks. Collecting (and returning) tolls and providing (and taxing for) highways. An interesting question is whether the revenue collected from efficient tolls would be sufficient to pay for highway construction.

Let us suppose that the congestion function is homogeneous of some degree, k. If $k = 0$, there are "constant returns to scale" in the sense that doubling both the amount of driving and the amount of highways leaves the level of congestion unchanged. If $k > 0$, there are "decreasing returns". Doubling driving and highways makes congestion worse. If $k < 0$, there are increasing returns. Doubling driving and highways leads to less congestion. According to Euler's theorem on homogeneous functions, it must be that

$$DC_D + HC_H = kC. \tag{3.14}$$

Where the government charges the approximately optimal uniform toll given in Equation 3.12, the government's revenue is

$$TD = -p_H \left(\frac{C_D}{C_H} \right) D \tag{3.15}$$

From Equations 3.14 and 3.15 it follows that when $C(D, H)$ is homogeneous of degree k,

$$TD = p_H H - k \frac{C}{C_H} p_H \tag{3.16}$$

Therefore in the case of constant returns to scale where $k = 0$, the revenue from tolls exactly covers construction costs. In the case of decreasing returns to scale, where $k > 0$, revenue from optimal tolls more than covers the cost of the optimal highways. Finally, in the case of increasing returns to scale, where $k < 0$, efficient toll revenue will not cover total highway construction costs.

Appendix: Euler's Theorem on Homogeneous Functions

This theorem is a useful tool for economic analysis with a simple and beautiful proof. A function $f(x_1, \ldots, x_n)$ whose domain is a subset of $\Re^n$ is said to be homogeneous of degree k if $f(\lambda x) = \lambda^k f(x)$ for all $\lambda > 0$ and all x in the domain of f.

Theorem 1 *If a function f is homogeneous of degree k, then*

$$\sum_i x_i f_i(x) = k f(x)$$

for all x in the domain of f.

Proof: Simply differentiate both sides of the identity $f(\lambda x_1, \ldots, \lambda x_n) = \lambda^k f(x_1, \ldots, x_n)$ with respect to λ. This yields $\sum x_i f_i(\lambda x_1, \ldots, \lambda x_n) = k\lambda^{k-1} f(x_1, \ldots, x_n)$. Evaluate this last expression at $\lambda = 1$ to find that $\sum_i x_i f_i(x) = k f(x)$.
■

Exercises

3.1 Residents of Carburetor, Ohio (pop. n), have utility functions

$$U_i(D_i, M_i, C) = A_i D_i - \frac{1}{2} D_i^2 - D_i \frac{D}{H} + M_i \tag{3.17}$$

where for each i, D_i is driving by i, $D = \sum_{j=1}^{n} D_j$, M_i is money expenditure by i on goods , H is total highway expenditures in Carburetor and where $A_i > 1$ is a parameter for each i. Gasoline is available for free in Carburetor, and it costs nothing to maintain cars. The only goods that money can buy in Carburetor are Big Macs and highway improvements. The initial endowment of income is W_i for each i. The price of Big Macs is 1.

1. Since preferences are quasilinear, the Pareto optimal amount of driving for each i and the Pareto optimal total highway expenditures must be independent of income distribution (except for the case of Pareto optimal allocations where some consumers consume no Big Macs). Find these Pareto optimal quantities.

2. If no tolls are charged, find the Nash equilibrium amount of driving by each resident of Carburetor.

3. Suppose that each resident of Carburetor is charged a uniform toll according to the rule suggested in the text of this lecture. What will this toll be? How much driving will each i do?