 NAME_______________

VCG Mechanism

Calculus **1.1 (2)** Three friends, Archie, Betty, and Veronica, are planning a party. They disagree about how many people to invite. Each person i has a quasilinear utility function of the form $m_i + u_i(x)$ where m_i is the number of dollars that i has to spend and x is the number of guests invited to the party. Suppose that for each i,

$$u_i = a_i x - \frac{1}{2}x^2.$$

Everyone knows the functional form of the others' utility functions and knows his own value of a_i but does not know anyone else's value. Let us suppose that the actual values of a_i are 20 for Archie, 40 for Betty, and 60 for Veronica.

(a) How many guests should be invited to maximize the sum of the three persons' utilities? 40 guests

(b) Suppose that the three friends decide to use the VCG mechanism to determine the number of guests. If each plays his or her best strategy, how many guests will be invited? 40 guests

(c) In the VCG mechanism, if the amount of public good supplied is x, Archie would receives a sidepayment equal to the sum of Betty's and Veronica's utility for x. If Betty and Veronica play their best strategies (without colluding) and if the amount of public good is x, this sidepayment will be $100x - x^2$. If everybody plays their best strategy, the amount of this sidepayment in dollars is $2400 .

(d) In addition to receiving sidepayments, the VCG mechanism requires that each person must pay an amount equal to the maximum possible sum of the other two persons' utilities. If Betty and Veronica play their best strategies this amount is $2500 . On net, Archie has to pay the difference between this amount and the sidepayment that he receives. If everybody plays their best strategy, what is the net amount that Archie must pay? $100

(e) If everybody plays their best strategy, what is the net amount that Betty has to pay? $0 What is the net amount that Veronica has to pay? $100

(f) Suppose that the party is organized not by just three people, but by a dormitory with 21 residents. All of these residents have utility functions of the same form as Archie, Betty, and Veronica. Seven of them have $a_i = 20$, seven have $a_i = 40$, and seven have $a_i = 60$. In order to maximize the sum of the residents' utilities, how many guests should be invited? 40 guests If there were only six persons with $a_i = 20$, seven with $a_i = 40$ and seven with $a_i = 60$, how many guests would have to be invited in order to maximize the sum of utilities? 41 guests

(g) If everybody plays their best strategy in the VCG game, then after all sidepayments and taxes are collected, how much net tax will each of the people with $a_i = 20$ have to pay? $10 How much net tax will each of the people with $a_i = 40$ have to pay? $0 How much net tax will each of the people with $a_i = 60$ have to pay? $10

1.2 (2) Homeowners 1, 2, and 3 live at the end of a badly deteriorated road. Fixing the road would cost \$$C$. The value to Homeowner 1 of fixing the road is \$3,000, the value to Homeowner 2 is \$5,000, and the value to Homeowner 3 is \$8,000. Each homeowner claims that fixing the road is not worth much to him, because each wants the others to pay the cost. The local government suspects that the total value to these homeowners of fixing the road is greater than \$$C$ and has decided to require the three homeowners to use the VCG mechanism to determine whether to fix the road. Since the government had no idea of the individual values for fixing the road, it decided to allocate the costs equally among the three homeowners. Each homeowner is asked to report his value for fixing the road. If the sum of the reported values is greater than C, the road will be fixed and each homeowner will have to pay \$$C/3$ and also will have to pay an additional tax as calculated by the VCG mechanism.

(a) Suppose that $C = \$13,500$, so that each homeowner has to pay \$4,500 as his share of the cost. If homeowners report their values accurately, the sum of the reported values will be $16,000 . Since the sum of reported values is greater than $C, the government will choose to build the road. To calculate Homeowner 3's VCG tax, we reason as follows. If the road is repaired, Homeowners 1 and 2 will have to pay a total of $9,000 while the sum of their values for the repairs is $8,000 , so their net change in utility is -$1000 . Since Homeowner 1's

response changes the decision about whether the road is built, she would be assessed a VCG tax of $1,000 in addition to her \$4,000 share of the costs. What would be Homeowner 1's VCG tax? 0 Homeowner 2's VCG tax? 0 Would Homeowner 1 be better off or worse off with the outcome of the VCG mechanism than if the road is left unrepaired? Worse off Would Homeowner 2 be better off or worse off than if the road is left unrepaired? Better off Would Homeowner 3 be better off or worse off with the VCG outcome than if the road is left unrepaired? Better off When the VCG tax is used, would the sum of the utilities of the three homeowners be higher or lower than if the road were left unrepaired? Higher

(b) Suppose that $C = \$18,000$ and so each homeowner will have to pay a \$6,000 share of the cost. If homeowners report their values accurately, the sum of the reported values will be \$16,000 . Since the sum of reported values is less than \$C, the government will choose not to repair the road. Let us calculate Homeowner 1's VCG tax. If the road is repaired, the total amount that Homeowners 2 and 3 have to pay is \$12,000 and the sum of the values to Homeowners 2 and 3 of fixing the road will be \$13,000 , so their net gain from the project is \$1000 . Since Homeowner 3's response changes the outcome of whether the road is repaired, she would be assessed a VCG tax of \$1,000 .

(c) Suppose that instead of 3 homeowners at the end of the road, there were 30 homeowners, 10 of type 1 who valued repairing the road at \$3,000, 10 of type 2 who valued repairing the road at \$5,000 and ten of type 3 who valued repairing the road at \$8,000. Suppose that the cost of repairing the road is $C = \$135,500$. If the road is repaired, each homeowner would have to pay a tax of $\$135,000/30 = \$4,500$. If the VCG mechanism is used and each individual reports his true valuation, the road will be repaired, since the sum of reported valuations will be \$160,000. To calculate the VCG tax of a type 3 consumer, we observe that the sum of the values of the project to all other consumers is \$152,000 . The total amount that other consumers would have to pay if the project is undertaken is \$131,000 . Would a type 3 homeowner have to pay a positive amount of VCG tax? No How about type 1 and 2 homeowners? They

would have 0 VCG tax as well.

Calculus **1.3 (2)** (For this problem, you will want to use a calculator.) A developer would like to build an amusement park in the midst of the once-thriving city of Broken Axle, Michigan. In order to build this park, he must purchase all of the land in a specific 10-acre area. If he does not get all of this land, he cannot build the park. Nobody except the developer knows exactly how much he would be willing to pay for the land. All persons other than the developer believe that the probability that he is willing to pay at least $\$p$ for the land is $G(p) = e^{-p/k}$ where $k = 200,000$.

(a) Suppose that all of the land belongs to a single owner. This owner values the land at $50,000 in its current use. He must make a take-it-or-leave-it offer to the developer. The owner wants to make an offer p that maximizes his expected profit from the sale, which is $(p - \$50,000)G(p)$.

At what price should the owner offer the land? $250,000 At this price, what is the probability that the developer will buy the land? $e^{-1.25} = .287$ How much is the owner's expected profit? $\$200,000 \times .287 = \$57,301$

(b) Suppose that instead the land consists of 10 separate parcels with 10 different owners. Let v_i be owner i's value for his own parcel if he keeps it. These owners have different values v_i and only the owner knows what it is worth to himself. Let us assume that $v = \sum_{i=1}^{10} v_i = \$50,000$. Suppose that each owner i sets a price p_i for his land. Let us define p to be the sum of these individual offer prices. That is, $p = \sum_{i=1}^{10} p_i$. If the developer is willing to pay at least p for the entire 10-acre parcel, then he buys it and pays each owner i, the amount p_i. In equilibrium, each owner makes the offer that maximizes his expected profit given the offers made by the other owners. That is, he chooses p_i to maximize

$$(p_i - v_i)G(p) = (p_i - v_i)e^{-\sum p_j/k}$$

where $k = \$200,000$. What price p_i will owner i set? $v_i + k = v_i + \$200,000$ What is the sum of the prices asked by the 10 owners? $2,050,000 What is the probability that the developer will buy the land at this price? $e^{-10.25} = .000035$ What is the sum over all 10 owners of expected profits? $72.5

(c) The 10 separate owners recognize that if they could somehow coordinate their offers, they would make much larger expected total profits. They decide to use the VCG mechanism to do so. In this instance the VCG mechanism works as follows. Each individual states an amount r_i that the land is worth to him. All of the land will be offered to the developer at a price p that would maximize total profits of owners if each owner reports his actual value, $r_i = v_i$. If the developer chooses to purchase the land, then each owner will receive an equal share of the sales price p. In addition to receiving this share if the sale is made, each owner will pay a "tax" that depends on his response r_i and on the responses of the other owners in the following way. Person i will receive an amount the sum of the expected profits of the other 9 owners, but must pay an amount equal to the maximum expected profits that would be received by the other owners if the price were chosen to maximize the expected total profits of these other owners. With this mechanism, the best strategy for each owner is to state his true value $r_i = v_i$. If each owner uses this best strategy, at what price will the land be offered to the developer?

$250,000 What is the probability that the developer will buy the land? $e^{-1.25} = .287$

(d) Suppose that person i has a reservation price $v_i = \$5,900$. Let us calculate the tax that the VCG mechanism would assign to i if each individual responds with $r_i = v_i$. If the land is offered at price p to the developer, then expected profits of persons other than i will be

$$\left(\frac{9}{10}p - v_{\sim i}\right) e^{-p/200,000}$$

where $v_{\sim i} = \sum_{j \neq i} v_j =$ $44,100 is the sum of the reservation prices of persons other than i. We found that if everyone responds optimally in the VCG mechanism, the land will be offered at a price of

$250,000 . Therefore the expected profits of persons other than i will be $(\$225,000 - \$44,100)e^{-1.25} = \$51,829$. The price p that maximizes the sum of expected profits of owners other than i is $245,556. At this price, total expected profits of these owners would be $59,014 . The net tax that i pays is the difference between this amount and the actual expected profits of other owners. This difference is $14 .

(e) Let us calculate total expected profit of the individual with $v_i = \$5900$ under the VCG mechanism. Whether or not there is a sale, person i pays the amount of tax that we calculated in the previous section. With probability .287 , the land is sold at price $250,000 and each owner

will receive 1/10 of the sales price, which is `$25,000` , giving him a profit of `$25,000-$5900=$19,100` over his reservation value. Therefore the expected profit for person i is `$.287*19,100-14=$5467`

(f) If individual j has $v_j = \$5,000$, what is the net amount of tax that he pays under the VCG mechanism? `$0` (Hint: there is an easy answer.)