Three Brief Proofs of

ARROW'S IMPOSSIBILITY THEOREM

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ARROW'S IMPOSSIBILITY THEOREM

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Abstract

Arrow's original proof of his impossibility theorem proceeded in two steps: showing the existence of a decisive voter, and then showing that a decisive voter is a dictator. Barbera replaced the decisive voter with the weaker notion of a pivotal voter, thereby shortening the first step, but complicating the second step. I give three brief proofs, all of which turn on replacing the decisive/pivotal voter with an extremely pivotal voter (a voter who by unilaterally changing his vote can move some alternative from the bottom of the social ranking to the top), thereby simplifying both steps in Arrow's proof.

My first proof is the most straightforward, and the second uses Condorcet preferences (which are transformed into each other by moving the bottom alternative to the top). The third (and shortest) proof proceeds by reinterpreting Step 1 of the first proof as saying that all social decisions are made the same way (neutrality).

Keywords: Arrow Impossibility Theorem, pivotal, neutrality

JEL Classification: D7, D70, D71

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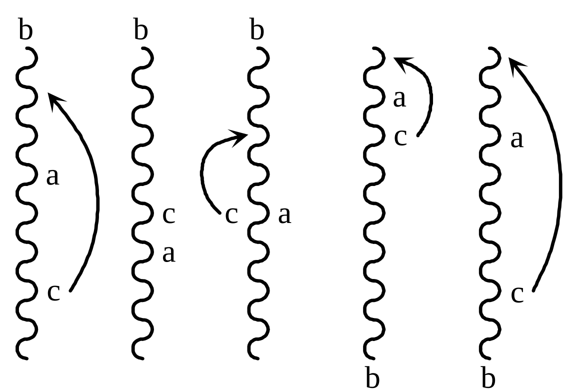
1 Statement and First Proof of Arrow's Theorem

Let $\mathbf{A} = \{A, B, ..., C\}$ be a finite set of at least three alternatives. A transitive preference over $\mathbf{A}$ is a ranking of the alternatives in $\mathbf{A}$ from top to bottom, with ties allowed. We consider a society with N individuals, each of whom has a (potentially different) transitive preference. A constitution is a function which associates with every N-tuple (or profile) of transitive preferences a transitive preference called the social preference.

A constitution respects **unanimity** if society puts alternative α strictly above β whenever every individual puts α strictly above β. The constitution respects **independence of irrelevant alternatives** if the social relative ranking (higher, lower, or indifferent) of two alternatives α and β depends only on their relative ranking by every individual. The constitution is a **dictatorship** by individual n if for every pair α and β, society strictly prefers α to β whenever n strictly prefers α to β.

ARROW'S THEOREM: *Any constitution that respects transitivity, independence of irrelevant alternatives, and unanimity is a dictatorship.*

FIRST PROOF: Let alternative b be chosen arbitrarily. We argue first that at any profile in which every voter puts alternative b at the very top or very bottom of his ranking of alternatives, society must as well (even if half the voters put b at the top and half put b at the bottom). Suppose to the contrary that for such a profile and for distinct a, b, c, the social preference put $a \geq b$ and $b \geq c$. By independence of irrelevant alternatives, this would continue to hold even if every individual moved c above a, because that could be arranged without disturbing any ab or cb votes (since b occupies an extreme position in each individual's ranking, as can be seen from the diagram). By transitivity the social ranking would then put $a \geq c$, but by unanimity it would also put $c > a$, a contradiction.



We argue that there is a voter $n^* = n(b)$ who is extremely pivotal in the sense that by changing his vote at some profile he can move b from the bottom of the social ranking to the top. To see this, let each voter put b at the bottom of his (otherwise arbitrary) ranking of alternatives. By unanimity, society must as well. Now let the individuals from voter 1 to N successively move b from the bottom of their rankings to the top, leaving the other relative rankings in place. Let n^* be the first voter whose change causes the social ranking of b to change. (By unanimity, a change must occur at the latest when $n^* = N$.) Denote by profile I the list of all voter rankings just before n^* moves b, and denote by profile II the list of all voter rankings just after n^* moves b to the top. Since in profile II b has moved off the bottom of the social ranking, we deduce from our first argument that the social preference corresponding to profile II must put b at the top.

We argue third that $n^* = n(b)$ is a dictator over any pair ac not involving b. To see this, choose one element, say a, from the pair ac. Construct profile III from profile II by letting n^* move a above b, so that $a >_{n^*} > b >_{n^*} c$, and by letting all the agents $n \neq n^*$ arbitrarily rearrange their relative rankings of a and c while leaving b in its extreme position. By independence of irrelevant alternatives, the social preferences corresponding to profile III would necessarily put $a > b$ (since all individual ab votes are as in profile I where n^* put b at the bottom), and $b > c$ (since all individual bc votes are as in profile II where n^* put b at the top). By transitivity, society must put $a > c$. By independence of irrelevant alternatives, the social preference over ac must agree with n^* whenever $a >_{n^*} c$.

We conclude by arguing that n^* is also a dictator over every pair ab. Take a third distinct alternative c to put at the bottom in the construction of paragraph 2. From the third argument, there must be a voter $n(c)$ who is an $\alpha\beta$ dictator for any pair $\alpha\beta$ not involving c, such as ab. But agent n^* can affect society's ab ranking, namely at profiles I and II, hence this ab dictator $n(c)$ must actually be n^*. ■

SECOND PROOF: In a Condorcet preference assignment each voter $n \in N$ is assigned one of the Condorcet preferences described below:

$$\begin{array}{cccc} \underline{\mathbb{C}_A} & \underline{\mathbb{C}_B} & \cdots & \underline{\mathbb{C}_C} \\ A & B & & C \\ B & C & & A \\ & & & B \\ \vdots & \vdots & & \\ C & A & & \vdots \end{array}$$

If all $n \in N$ are assigned to the first preference $\mathbb{C}_A$, then by unanimity, $\mathbb{C}_A$ must be the social preference. Among Condorcet assignments π such that the social preference is $\mathbb{C}_A$, find π_A that minimizes the number of voters with preferences $\mathbb{C}_A$. There must be at least one voter n^* in π_A with preferences $\mathbb{C}_A$, for otherwise C would be unanimously preferred to A.

Suppose alternative β immediately follows α alphabetically. Suppose at π_A n^* unilaterally switches to $\mathbb{C}_\beta$, giving the Condorcet assignment π_β. By IIA, we still have $A >_{\pi_\beta} \cdots >_{\pi_\beta} \alpha$ and $\beta >_{\pi_\beta} \cdots >_{\pi_\beta} C$. Hence, for the social order to change, we must get $\alpha \leq_{\pi_\beta} \beta$. (Furthermore, if $\alpha =_{\pi_\beta} \beta$, then by transitivity and the fact $N \geq 3$, $A >_{\pi_\beta} C$.)

Suppose now that n^* switches to $-\mathbb{C}_A$, where $A < B < \cdots < C$, giving the non-Condorcet assignment $\pi_{\bar{A}}$. Take two alphabetically consecutive alternatives α, β. Then $\mathbb{C}_\beta$ and $-\mathbb{C}_A$ agree on $\alpha\beta$ ($\beta > \alpha$) and on AC ($C > A$). Hence by IIA, $\alpha \leq_{\pi_{\bar{A}}} \beta$ since $\alpha \leq_{\pi_\beta} \beta$. Since α, β are arbitrary, this gives $A \leq_{\pi_{\bar{A}}} \cdots \leq_{\pi_{\bar{A}}} C$. Furthermore, if $\alpha =_{\pi_{\bar{A}}} \beta$, then by IIA, $\alpha =_{\pi_\beta} \beta$, and from the last paragraph, this would imply that $A >_{\pi_\beta} C$, and thus by IIA, $A >_{\pi_{\bar{A}}} C$, contradicting $A \leq_{\pi_{\bar{A}}} B \leq_{\pi_{\bar{A}}} \cdots \leq_{\pi_{\bar{A}}} C$. We conclude that $A <_{\pi_{\bar{A}}} B <_{\pi_{\bar{A}}} \cdots <_{\pi_{\bar{A}}} C$.

We finish the proof by showing that n^* is a dictator. Suppose that at an arbitrary preference assignment π, agent n^* can unilaterally arrange any strict preference by adopting that preference himself. (Note that by IIA, we have proved this is the case at $\pi = \pi_A$.) Change π to π' by letting a single voter $n \neq n^*$ raise some alternative half a step higher in his ranking in such a way that either he breaks a single tie $\alpha\beta$ or creates a single tie $\alpha\beta$ (but not both). By IIA, this change by n cannot change the social ranking of any pair except possibly $\alpha\beta$. Let n^* rank $\alpha > \gamma > \beta$ at π, for some third alternative γ. By hypothesis the social ranking at π has $\alpha >_\pi \gamma >_\pi \beta$. Hence $\alpha >_{\pi'} \gamma$ and $\gamma >_{\pi'} \beta$, so by transitivity $\alpha >_{\pi'} \beta$. Thus by IIA, the half-step move by n cannot change the power of n^* to enforce his strict preference over every pair at π'. Since π_A has no ties, for any pair $\alpha\beta$, a sequence of such half-moves can always be found to achieve arbitrary preferences for every voter $n \neq n^*$ over the given pair $\alpha\beta$. By IIA, n^* is a dictator. ■

THIRD PROOF

STRICT NEUTRALITY LEMMA: *All binary social rankings are made the same way. Consider two pairs of alternatives ab and $\alpha\beta$. Suppose each voter strictly prefers a to b, or b to a, and suppose each voter has the same relative ranking of $\alpha\beta$ as he does of ab. Then the social preference between ab is identical to the social preference between $\alpha\beta$, and both social preferences are strict.*

PROOF: Assume the pair $\alpha\beta$ is not identical to the pair ab. (If $N \geq 3$, such a distinct pair exists.) Suppose WLOG that socially $a \geq b$. Move α just above a for each voter n (if $\alpha \neq a$), and move β just below b for each voter n (if $\beta \neq b$). Since all ab preferences are strict, this can be arranged while maintaining the same $\alpha\beta$ preferences, as the diagram make clear.

α	α	α	b	b
a	a	a	β	β
b	b	b	α	α
β	β	β	a	a

By unanimity, $\alpha > a$ (if $\alpha \neq a$) and $b > \beta$ (if $\beta \neq b$). By transitivity, $\alpha > \beta$. Reversing the roles of ab and $\alpha\beta$, we conclude by IIA that socially $a > b$ in the original profile, proving the lemma.

Next, take two distinct alternatives a and b and start with $b >_n a$ for all n. Beginning with $n = 1$, let each voter successively move a above b. By unanimity and the strict neutrality lemma, there will be a voter n^* who moves the social preference from $b > a$ to $a > b$ when he moves a up. The situation is described below.

$$
\begin{array}{ccccccc}
\underline{1} & & \underline{n^*} & & \underline{N} & & \\
a & a & a & b & b & \multirow{2}{*}{$\rightarrow$} & a \\
b & b & b & a & a & & b
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
\underline{1} & & \underline{n^*} & & \underline{N} & & \\
a & a & b & b & b & \multirow{2}{*}{$\rightarrow$} & b \\
b & b & a & a & a & & a
\end{array}
$$

We now show that n^* is a dictator. Take an arbitrary pair of alternatives α, β and let n^* rank $\alpha >_{n^*} \beta$. Let the $\alpha\beta$ rankings for $n \neq n^*$ be arbitrary. Take $c \notin \{\alpha, \beta\}$ and put c above everything for $1 \leq n < n^*$, c below everything for $n^* < n \leq N$, and $\alpha >_{n^*} c >_{n^*} \beta$. By IIA, neutrality, and the profile discovered in paragraph 2, socially $\alpha > c$ and $c > \beta$, and so by transitivity, $\alpha > \beta$.

$$
\begin{array}{ccccc}
\underline{1} & & \underline{n^*} & & \underline{N} \\
c & c & \alpha & \beta & \alpha \\
\beta & \multirow{2}{*}{$\alpha\beta$} & c & \alpha & \beta \\
\alpha & & \beta & c & c
\end{array}
$$

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