Econometrica, Vol. 51, No. 6 (November, 1983)

INDEPENDENCE OF ALLOCATIVE EFFICIENCY FROM DISTRIBUTION IN THE THEORY OF PUBLIC GOODS

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When is the Pareto optimal amount of public goods independent of income distribution? Subject to certain simple regularity conditions, the answer is "when preferences of each individual i can be represented by a utility function of the form $U_i(X_i, Y) = A(Y)X_i + B_i(Y)$ where X_i is the amount of the (one) private good consumed by i and Y is the vector of public goods." Besides proving necessity and sufficiency conditions for utility to be of this special form, we show implications of this form for Lindahl equilibrium, majority voting, and the Groves–Clarke mechanism for preference revelation.

If all the seas were one sea,
What a *great* sea that would be!
If all the trees were one tree,
What a *great* tree that would be!
And if all the axes were one axe,
What a *great* axe that would be!
And if all the men were one man,
What a *great* man that would be!
And if the *great* man took the *great* axe,
And cut down the *great* tree,
And let it fall into the *great* sea,
What a splish-splash that would be!

NORMALLY AGGREGATE DEMAND for private goods cannot be treated as if it were the demand of a single gigantic rational consumer. This is possible only if "income distribution doesn't affect aggregate demand." Gorman [**11**] discovered restrictions on the form of indirect utility functions that are necessary and sufficient to allow such aggregation. In early partial equilibrium treatments of public goods theory by Lindahl [**15**] and Bowen [**6**] the efficient amount of public goods appears to be determined independently of income distribution. Samuelson [**18, 19**] observes that generally an efficient amount of public goods cannot be determined independently of the distribution of private goods. He points out that such separation is possible in the special case where preferences of all consumers are *quasi-linear*, that is, representable by utility functions that are linear in private goods. Musgrave [**17**] responds that although independence of allocation from distribution is not legitimate in a strict logical sense, separation of allocational decisions from distributional decisions is a useful simplification of reality that may in practical situations lead to better decision making than attempts to simultaneously determine allocation and distribution.

If quasi-linear preferences were necessary for separation of allocation from distribution, then Musgrave's case for separation, even as an approximation, would not be good. Quasi-linearity has strong and rather easily refutable implications. For example, it implies a zero income elasticity of demand for public goods. Several recent studies of the demand for public goods strongly reject the hypothesis that the income elasticity of demand for local public goods is close to zero.[1] As it turns out, however, separation of allocation from distribution is possible for a broader class of preferences. This class is essentially dual to the class of preferences found by Gorman to admit construction of a "representative consumer" in the theory of demand for private goods[2] and consists of preferences that can be represented by a utility function of the form $A(Y)X_i + B_i(Y)$ for each individual i.

In the first section we develop a rigorous theory of when allocation can be separated from income distribution, or equivalently of when there is a representative consumer of public goods. In the second section of this paper we show that the assumption that preferences belong to the class that allows a representative consumer has interesting implications for Lindahl's allocation theory and for Bowen's majority voting theory. We also demonstrate that demand revealing mechanisms of the kind introduced by Clarke **[8]** and Groves and Loeb **[13]** for the case of quasi-linear utility can be extended in a simple way to this broader class of preferences.

1. GENERAL THEOREMS

Let there be n public goods, one private good, and m consumers. Consumer i's preferences are represented by a utility function of the form $U_i(X_i, Y)$ where X_i is his consumption of the private good and Y is the vector of public goods supplied. An *allocation* is a vector $(X_1, \ldots, X_m, Y) \in R_+^{m+n}$ specifying each citizen's consumption of private goods and the vector of public goods. The set of *feasible aggregate outputs* is a set $\mathscr{F} \subset R_+^{1+n}$ and the set of *feasible allocations* is the set $\{(X_1, \ldots, X_n, Y) \geqq 0 \mid (\sum_{i=1}^m X_i, Y) \in \mathscr{F}\}$ of allocations that can be achieved by distributing the private goods from a feasible aggregate output.

A feasible aggregate output (X, Y) is defined to be *always Pareto efficient* if every allocation $(X_1, \ldots, X_n, Y)$ such that $\sum_{i=1}^n X_i = X$ is Pareto optimal. Let $\mathscr{E} \subset \mathscr{F}$ be the set of always Pareto efficient aggregate outputs. In general, even where Pareto optimal allocations exist, $\mathscr{E}$ can be empty because the efficient amount of public goods is not independent of the distribution of private goods. A minimal requirement for determining efficient outputs of public goods independently of income distribution is that the set $\mathscr{E}$ is nonempty. We define allocation and distribution to be *weakly independent* if $\mathscr{E}$ is a nonempty set. As it turns out, subject to certain technical qualifications, a necessary and sufficient condition for allocation and distribution to be weakly independent is that for each i, preferences of consumer i are representable by a utility function of the form $A(Y)X_i + B_i(Y)$. Sufficiency is very easily proved. In particular:

[1] Examples of such studies are Borcherding and Deacon **[5]** and Bergstrom and Goodman **[2]**. Other similar studies are reviewed by Inman **[14]**.

[2] This duality is discussed explicitly in Bergstrom and Cornes **[4]**.

THEOREM 1: *If preferences of every consumer i are representable by a utility function of the form $A(Y)X_i + B_i(Y)$, then any aggregate allocation $(\overline{X}, \overline{Y})$ that maximizes $A(Y)X + \sum_i B_i(Y)$ on $\mathscr{F}$ is always Pareto efficient. If, furthermore, the set $\mathscr{F}$ is compact and utility functions are continuous, then allocation and distribution are weakly independent.*

PROOF: Suppose that $(\overline{X}, \overline{Y})$ maximizes $A(Y)X + \sum_i B_i(Y)$ on $\mathscr{F}$. Let $(\overline{X}_1, \ldots, \overline{X}_n, \overline{Y})$ be any allocation such that $\sum_i \overline{X}_i = \overline{X}$ and suppose that $(X_1, \ldots, X_n, Y)$ is Pareto superior to $(\overline{X}_1, \ldots, \overline{X}_n, \overline{Y})$. Then $A(Y)X_i + B_i(Y) \geqq A(\overline{Y})\overline{X}_i + B_i(\overline{Y})$ for all i with strict inequality for some i. Therefore $A(Y) \sum_i X_i + \sum_i B_i(Y) > A(\overline{Y})\overline{X} + \sum_i B_i(\overline{Y})$. Since $(\overline{X}, \overline{Y})$ maximizes $A(Y)X + \sum_i B_i(Y)$ on $\mathscr{F}$ it must be that $(\sum X_i, Y) \notin \mathscr{F}$. Therefore $(\overline{X}_1, \ldots, \overline{X}_n, \overline{Y})$ must be Pareto optimal. If the functions $A(Y)X_i + B_i(Y)$ are all continuous and $\mathscr{F}$ is compact, there exists $(\overline{X}, \overline{Y})$ that maximizes $A(\overline{Y})\overline{X} + \sum_i B_i(\overline{Y})$ on $\mathscr{F}$. By the previous argument $(\overline{X}, \overline{Y})$ is always Pareto efficient so that $\mathscr{E}$ is nonempty. *Q.E.D.*

The converse result is more difficult and requires some qualification. Evidently if the feasible set is very special, allocation and distribution can be independent for a very wide class of preferences. We are interested in the implications of the stronger assumption that allocation and distribution are independent for every finite feasible set $\mathscr{F} \subset R_+^{1+n}$. It will be helpful to consider a binary relation $\circledS$ with the property that the set $\mathscr{E}$ of always Pareto efficient aggregate outputs is equal to the set of maximal elements of $\circledS$ on $\mathscr{F}$. Let us define $\circledS$ so that $(X, Y) \circledS (X', Y')$ if and only if there exist allocations $(X_1, \ldots, X_n, Y)$ and $(X_1', \ldots, X_n', Y')$ such that $\sum_i X_i = X$, $\sum_i X_i' = X'$, and $(X_1, \ldots, X_n, Y)$ is Pareto superior to $(X_1', \ldots, X_n', Y')$. From the definitions of $\mathscr{E}$ and $\circledS$ the following is immediate.

REMARK 1: The set $\mathscr{E}$ of always Pareto optimal aggregate outputs is equal to the set of maximal elements of $\circledS$ on $\mathscr{F}$.

Therefore allocation and distribution are weakly independent precisely when $\circledS$ has at least one maximal element on $\mathscr{F}$. If a binary relation has a cycle, then the finite set whose elements constitute the cycle does not have a maximal element.[3] These facts together with Remark 1 enable us to assert the following.

REMARK 2: If allocation and distribution are weakly independent for every finite feasible set $\mathscr{F} \subset R_+^{1+n}$, then the binary relation $\circledS$ has no cycles in R_+^{1+n}.[4]

[3] A binary relation $\circledS$ has a cycle if there is a finite set $\{x_1, \ldots, x_k\}$ such that $x_{j+1} \circledS x_j$ for $j = 1, \ldots, k-1$ and such that $x_1 \circledS x_k$.

[4] The converse is also true. That is, if $\circledS$ has no cycles then $\circledS$ has at least one maximal element on any finite set (Sen [**20**]). In fact if $\circledS$ has no cycles and is continuous then it has at least one maximal element on any compact set (Bergstrom [**3**]).

The relation $\circledS$ should not be confused with the Kaldor–Hicks–Samuelson partial order which was central to the "new welfare economics" (see Chipman and Moore [7]). The K.H.S. relation has (X, Y)K.H.S.(X', Y') if and only if for *every* $(X_1', \ldots, X_m', Y')$ such that $\sum_i X_i' = X'$ there exists *some* $(X_1, \ldots, X_m, Y)$ such that $\sum_i X_i = X$ and such that $(X_1, \ldots, X_m, Y)$ is Pareto superior to $(X_1', \ldots, X_m', Y)$. The set of K.H.S. maximal elements on $\mathscr{F}$ is the set of aggregate outputs which are potentially efficient in the sense that if (X, Y) is K.H.S. maximal then *some* distribution of X accompanied by Y would be Pareto optimal. The set of K.H.S. maximal elements generally includes $\mathscr{E}$ as a proper subset.

Two more definitions will help to state our necessity theorem succinctly. A vector of public goods Y^0 is a *worst supply of public goods* if for all i and all $X_i \geqq 0$, $Y \geqq 0$, $U_i(X_i, Y) \geqq U_i(X_i, Y^0)$. For example, if preferences are monotone increasing in Y, then $Y^0 = 0$ is a worst supply of public goods. *Differences in public goods are always compensatable* by private goods if for all i and every $Y \geqq 0$, $Y' \geqq 0$, and $X_i \geqq 0$ there exists $X_i' \geqq 0$ such that $U_i(X_i', Y') \geqq U_i(X_i, Y)$.[5]

THEOREM 2: *Let preferences of every consumer be representable by a continuous utility function that is monotone increasing in X_i. Assume that there exists a worst supply Y^0 of public goods and that differences in public goods are always compensatable by private goods. If allocation and distribution are weakly independent for every finite feasible set, then there exist continuous functions $A(Y), B_1(Y), \ldots, B_m(Y)$ such that preferences of each consumer i are representable by a utility function of the form $A(Y)X_i + B_i(Y)$.*

PROOF: From our assumptions it follows that for every (X_i, Y), there exists a unique X_i' such that $U_i(X_i', Y^0) = U_i(X_i, Y)$. Therefore we can define $U_i^*(X_i, Y)$ so that $U_i(U_i^*(X_i, Y), Y^0) = U_i(X_i, Y)$. Since preferences are monotone increasing, $U_i^*(X_i, Y)$ represents i's preferences. Furthermore, we see from the definition that $U_i^*(X_i, Y^0) = X_i$ for all X_i.

We can show that if $\sum X_i = \sum X_i' = X$, then $\sum_i U_i^*(X_i, Y) = \sum_i U_i^*(X_i', Y)$. For suppose not. Then without loss of generality, let $\sum_i U_i^*(X_i, Y) - \sum_i U_i^*(X_i', Y) = \delta > 0$. Let $Z_i = U_i^*(X_i, Y) - (\delta/2n)$ and let $Z_i' = U_i^*(X_i', Y) + (\delta/2n)$. Then $\sum_i Z_i = \sum Z_i' \equiv Z$. Now $U_i^*(Z_i, Y^0) = Z_i < U_i^*(X_i, Y)$ for all i and $U_i^*(Z_i', Y^0) = Z_i' > U_i^*(X_i', Y)$ for all i. Therefore $(X, Y)\circledS(Z, Y^0)$ and $(Z, Y^0)\circledS(X, Y)$. But according to Remark 2, our assumption that allocation and distribution are weakly independent implies that $\circledS$ has no cycles. This is a contradiction.

From the result of the previous paragraph it follows that $\sum_i U_i^*(X_i, Y) = U(\sum_i X_i, Y)$ for some function U. An equation of this functional form is

[5] An example of a utility function where differences in public goods are not always compensatable would be $U(X, Y) = Y - (1/(1 + X))$. Let $Y = 2$, $X = 1$, and $Y' = 1$. Then $U(X, Y) = 3/2$ and $U(X', Y') < 1$ for all $X' \geqq 0$.

known as Pexider's functional equation. It is well known (see Aczel [1]) that Pexider's functional equation implies that $U_i^*(X_i, Y) = A(Y)X_i + B_i(Y)$ for all i. Since $U_i^*(X_i, Y)$ represents i's preferences, Theorem 2 is established. *Q.E.D.*

According to Theorem 1, if utility functions are of the form $A(Y)X_i + B_i(Y)$, then every aggregate output $(\overline{X}, \overline{Y})$ that maximizes the "aggregate utility function" $A(Y)X + \sum_i B_i(Y)$ is always Pareto efficient. It would be convenient if every Pareto optimal allocation $(X_1^*, \ldots, X_n^*, Y^*)$ had the property that $(\sum_i X_i^*, Y^*)$ maximizes $A(Y)\sum_i X_i + \sum_i B_i(Y)$ on $\mathscr{F}$. Then we could claim, not only that there exist always Pareto efficient aggregate outputs, but also that a redistribution from *any* Pareto efficient allocation leads to another Pareto efficient allocation. This is not the case, however, as we see from Example 1.

EXAMPLE 1: There are two consumers with identical (quasi-linear) utilities of the form $A(Y)X_i + B_i(Y)$ where $A(Y) \equiv 1$ and $B_i(Y) = \sqrt{Y}$ for $i = 1$ and 2. The set of feasible aggregate outputs is $\mathscr{F} = \{(X, Y) \geqq 0 \mid X + Y = 3\}$.

In Example 1, $A(Y)X + \sum_i B_i(Y)$ is maximized on $\mathscr{F}$ when $X = 2$ and $Y = 1$. Every allocation of the form $(X_1, X_2, 1)$ where $X_1 \geqq 0$, $X_2 \geqq 0$ and $X_1 + X_2 = 2$ is Pareto optimal. But there are some other Pareto optima for which aggregate outputs do not maximize $A(Y)X + \sum_i B_i(Y)$ on $\mathscr{F}$. These are the boundary optima in which one consumer or the other receives no private goods. In fact it is not hard to show that every outcome in which one consumer or the other receives no private goods and where $1/4 \leqq Y < 1$ and $X = 3 - Y$ is Pareto optimal. However any redistribution of private goods from one of these boundary optima yields an allocation which is not Pareto optimal.

In Example 1, the only troublesome Pareto optimal allocations are boundary optima in which at least one consumer receives no private goods. It would be useful to know whether it is generally true that the only Pareto optima which do not remain Pareto optimal under redistribution of private goods are boundary optima of this type. We define an *interior* allocation to be an allocation $(X_1, \ldots, X_m, Y)$ such that $X_i > 0$ for all i. We define an *interior Pareto optimum* to be a Pareto optimal interior allocation. We will say that allocation and distribution are *essentially independent* if for every interior Pareto optimum $(\overline{X}_1, \ldots, \overline{X}_m, \overline{Y})$, the aggregate output $(\sum_i \overline{X}_i, \overline{Y})$ belongs to the set $\mathscr{E}$ of always Pareto efficient aggregate outputs. We are able to show the following.

THEOREM 3: *Let all consumers have preferences representable by quasi-concave utility functions of the form $A(Y)X_i + B_i(Y)$ and let the set $\mathscr{F}$ be convex. Then allocation and distribution are essentially independent. The set of interior Pareto optima consists of those Pareto optimal allocations $(\overline{X}_1, \ldots, \overline{X}_m, \overline{Y})$ such that $(\sum_i \overline{X}_i, \overline{Y})$ maximizes $A(Y)X + \sum_i B_i(Y)$ on $\mathscr{F}$.*

PROOF OF THEOREM 3: Suppose that $(\bar{X}_1, \ldots, \bar{X}_m, \bar{Y})$ is an interior Pareto optimum and $(\sum_i \bar{X}_i, \bar{Y})$ does not maximize $A(Y)X + \sum_i B_i(Y)$ on $\mathscr{F}$. Then there exists $(\bar{\bar{X}}, \bar{\bar{Y}}) \in \mathscr{F}$ such that

$$\left[A(\bar{\bar{Y}})\bar{\bar{X}} + \sum_i B_i(\bar{\bar{Y}})\right] - \left[A(\bar{Y})\sum_i \bar{X}_i + \sum_i B_i(\bar{Y})\right]$$

$$= \delta > 0 \qquad \text{for some } \left(\bar{\bar{X}}, \bar{\bar{Y}}\right) \in \mathscr{F}.$$

Define

$$\bar{\bar{X}}_i = \frac{A(\bar{Y})\bar{X}_i + B_i(\bar{Y}) + \frac{\delta}{n} - B_i(\bar{\bar{Y}})}{A(\bar{\bar{Y}})} \qquad \text{for each } i.$$

Then

$$A(\bar{\bar{Y}})\bar{\bar{X}}_i + B_i(\bar{\bar{Y}}) > A(\bar{Y})\bar{X}_i + B_i(\bar{Y}) \qquad \text{for each } i$$

and $\sum_i \bar{\bar{X}}_i = \bar{\bar{X}}$. If $\bar{\bar{X}}_i \geqq 0$, for each i, then the allocation $(\bar{\bar{X}}_1, \ldots, \bar{\bar{X}}_m, \bar{\bar{Y}})$ would be feasible and Pareto superior to $(\bar{X}_1, \ldots, \bar{X}_m, \bar{Y})$ which would lead immediately to a proof by contradiction. In general, however, it may be that $\bar{\bar{X}}_i < 0$ for some i. Therefore we proceed as follows. Define $X_i(\lambda) = \bar{X}_i + \lambda(\bar{\bar{X}}_i - \bar{X}_i)$. Since $(\bar{X}_1, \ldots, \bar{X}_n, \bar{Y})$ is an interior Pareto optimum, there exists a sufficiently small $\hat{\lambda}$ such that $X_i(\hat{\lambda}) > 0$ for all i. Let $Y(\hat{\lambda}) = \bar{Y} + \hat{\lambda}(\bar{\bar{Y}} - \bar{Y})$. Since $\mathscr{F}$ is a convex set, $(X_2(\hat{\lambda}), \ldots, X_n(\hat{\lambda}), Y(\hat{\lambda})) \in \mathscr{F}$ and since utility functions are quasi-concave,

$$A(Y(\hat{\lambda}))X_i(\hat{\lambda}) + B_i(Y(\hat{\lambda})) > A(\bar{Y})\bar{X}_i + B_i(\bar{Y}) \qquad \text{for all } i.$$

Therefore $(X_1(\hat{\lambda}), \ldots, X_n(\hat{\lambda}), Y(\hat{\lambda}))$ is feasible and Pareto superior to $(\bar{X}_1, \ldots, \bar{X}_m, \bar{Y})$. This contradicts our assertion that $(\bar{X}_1, \ldots, \bar{X}_n, \bar{Y})$ is Pareto optimal. It follows that $(\sum_i \bar{X}_i, \bar{Y})$ maximizes $A(Y)X + \sum_i B_i(Y)$ on $\mathscr{F}$. According to Theorem 1, it then must be that $(\sum_i \bar{X}_i, \bar{Y})$ is an always Pareto efficient aggregate output. Therefore allocation and distribution are essentially independent. *Q.E.D.*

To see that the convexity assumption is needed for Theorem 3, consider the following example.

EXAMPLE 2: There are two consumers. Consumer i has utility of the form $A(Y)X_i + B_i(Y)$ where $A(Y) \equiv 1$ and $B_1(0) = B_2(0) = 0$, $B_1(1) = 1$, $B_2(1) = \frac{1}{2}$. Let $\mathscr{F} = \{(1, 1), (2, 0)\}$.

If aggregate output is (1, 1) then the set of possible distributions of utility is $\{(u_1, u_2) \mid 1 \leqq u_1 \leqq 2, \frac{1}{2} \leqq u_2 \leqq 1\frac{1}{2}$, and $u_1 + u_2 = 2\frac{1}{2}\}$. If aggregate output is (2, 0) then the set of possible distributions of utility is $\{(u_1, u_2) \mid 0 \leqq u_1 \leqq 2, 0 \leqq u_2 \leqq 2$, and $u_1 + u_2 = 2\}$. It is easily verified that every allocation obtained by distributing private goods from the aggregate output (1, 1) is Pareto optimal. Thus (1, 1) is an always Pareto efficient aggregate output and allocation and distribution are weakly independent. Some of the interior allocations obtained by distribution from the aggregate output (2, 0) are also Pareto efficient. In fact all allocations $(x_1, x_2, 0)$ such that $x_2 > \frac{3}{2}$ and $x_1 = 1 - x_2$ are Pareto efficient. But (2, 0) is not an always Pareto efficient aggregate output, since all of the allocations $(x_1, x_2, 0)$ such that $x_2 \leqq \frac{3}{2}$ and $x_1 = 1 - x_2$ are Pareto dominated by allocations obtainable from the aggregate output vector (1, 1). Therefore allocation and distribution are not essentially independent. Furthermore, we see that interior Pareto optima of the form $(x_1, x_2, 0)$ do not maximize $A(Y)\sum_i X_i + \sum_i B_i(Y)$ on $\mathscr{F}$.

Since Theorem 3 requires that $A(Y)X_i + B_i(Y)$ be quasi-concave, it would be useful to have a simple recipe for constructing quasi-concave functions of this form and a simple diagnostic test to tell us whether a given function of this form is quasi-concave. Remark 3 supplies both.

REMARK 3: Let $U(X, Y) = A(Y)X + B(Y)$ where $A(Y) > 0$ for all $Y \geqq 0$. Let $\alpha(Y) = 1/A(Y)$ and $\beta(Y) = B(Y)/A(Y)$. Then $U(X, Y)$ is quasi-concave on R_+^{1+n} if and only if $\alpha(Y)$ is a convex function and $\beta(Y)$ is a concave function.[6]

PROOF: Quasi-concavity of $h(Y)$ on R_+^{1+n} is seen to be equivalent to convexity of the function $h(Y) \equiv u \cdot \alpha(Y) - \beta(Y)$ for all $u \geqq 0$. But $h(Y)$ is convex for all $u \geqq 0$ if and only if $\alpha(Y)$ is convex and $\beta(Y)$ is concave. *Q.E.D.*

2. APPLICATIONS

If preferences are representable by utility functions of our special form, then several of the standard problems in the theory of public finance and welfare economics have interesting special solutions.

A. Lindahl Equilibrium

A Lindahl equilibrium occurs when individual "tax prices" are adjusted in such a way that, given their tax prices, consumers agree unanimously on the amount of public goods to be provided. Lindahl equilibrium is known to be Pareto optimal and to belong to the "core" when public goods are desirable (Foley [**10**]). In case utility functions are of the special form

(1) $\quad A(Y)X_i + B_i(Y)$

[6] We are indebted to Professor J. A. Mirrlees for suggesting this condition.

Lindahl equilibrium has a very special structure. As it turns out, in this case Lindahl tax schedules will be affine in wealth. This means that such taxes could be collected by means of a proportional wealth tax (at the same rate for everyone) augmented by a "head tax" that may be positive or negative for an individual depending on the private functions, $B_i(Y)$.

We conduct this discussion with a simplified formal model which could be extended in a straightforward way to more general environments. Let there be one private good and one public good. Each i has an initial endowment W_i of private good. Public goods can be made from private goods at constant unit cost c. The set of feasible allocations, then, is

$$\left\{(X_1, \ldots, X_m, Y) \mid \sum_i X_i + cY = \sum_i W_i\right\}.$$

A Lindahl equilibrium consists of tax shares t_i for each i where $\sum_i t_i = 1$ and a feasible allocation $(\overline{X}_1, \ldots, \overline{X}_m, \overline{Y})$ such that for all i, $(\overline{X}_i, \overline{Y})$ maximizes $U_i(X_i, Y)$ subject to the budget constraint $X_i + t_i cY = W_i$. In Lindahl equilibrium, therefore, each consumer's marginal rate of substitution between public and private goods equals his tax price $t_i c$.

If utility functions are of the form (1), then marginal rates of substitution take the form $\alpha(Y)X_i + \gamma_i(Y)$ where $\alpha(Y) = A'(Y)/A(Y)$ and $\gamma_i(Y) = B_i'(Y)/A(Y)$. Therefore, in Lindahl equilibrium,

$$t_i c = \alpha(\overline{Y})\overline{X}_i + \gamma_i(\overline{Y}) = \alpha(\overline{Y})\left[W_i - t_i c\overline{Y}\right] + \gamma_i(\overline{Y}). \tag{2}$$

Rearranging equation (2), we have

$$t_i c = \frac{\alpha(\overline{Y})}{1 + \alpha(\overline{Y})\overline{Y}} W_i + \frac{\gamma_i(\overline{Y})}{1 + \alpha(\overline{Y})\overline{Y}}. \tag{3}$$

From (2) we see that, as promised, each consumer's tax share is an affine function of his wealth. Summing equation (2) over the i's, recalling that $\sum_i t_i = 1$, and rearranging terms, we find that:

$$c = \alpha(\overline{Y})\left[\sum_i W_i - c\overline{Y}\right] + \sum_i \gamma_i(\overline{Y}) \tag{4}$$

which is just the Samuelson first order condition for efficiency applied to this case. Thus if the government knows the utility functions, it could compute Lindahl equilibrium simply by solving equation (4) for $\overline{Y}$ and then assessing taxes $t_i c\overline{Y}$ where $t_i c$ is found from equation (3). These taxes will just pay for $\overline{Y}$ and all consumers, given their tax rates, will agree that $\overline{Y}$ is the "right" amount of public goods.

B. Majority Voting Equilibrium

A serious disadvantage of the Lindahl allocation method is that it requires the government to know details of individual preferences which are private information and which individuals may have an incentive to conceal. A less stringent requirement would be that the government knows $A(Y)$ and has a good estimate of $\bar{B}(Y) = (1/n)\sum_i B_i(Y)$. Then the government would know an "average utility function"

$$A(Y)X_i + \bar{B}(Y) \tag{5}$$

although it would not know detailed individual preferences. The government would know enough to find an efficient amount of public goods since it needs only to choose Y to maximize

$$A(Y)\sum_i X_i + \sum_i B_i(Y) = A(Y)\sum_i X_i + n\bar{B}(Y) \tag{6}$$

subject to the feasibility constraint.

If taxes are assessed according to an "average" Lindahl schedule, we have:

$$t_i c = \frac{\alpha(\bar{Y})}{1 + \alpha(\bar{Y})\bar{Y}} W_i + \frac{\bar{\gamma}(\bar{Y})}{1 + \alpha(\bar{Y})\bar{Y}} \tag{7}$$

where

$$\alpha(\bar{Y}) = \frac{A'(Y)}{A(\bar{Y})}$$

and

$$\bar{\gamma}(\bar{Y}) = \frac{\bar{B}'(\bar{Y})}{A(\bar{Y})} = \frac{1}{n}\sum_i \gamma_i(\bar{Y}).$$

Suppose tax shares are set by the schedule (7) and consumers are allowed to vote on the amount of public goods. If the amount of public goods were Y, then the utility of consumer i after paying his taxes would be

$$\tilde{U}_i(Y) = A(Y)[W_i - t_i c Y] + B_i(Y). \tag{8}$$

If consumer i has convex preferences, then the function $\tilde{U}_i(Y)$ will be quasi-concave in Y and hence single-peaked. Consumer i's "peak" is Y_i^* where Y_i^* maximizes $\tilde{U}_i(Y)$. Let $\hat{Y}^*$ be the median of the Y_i^*'s. Since preferences are single peaked, $\hat{Y}^*$ would be the only stable outcome of a pairwise majority voting process. By straightforward calculation we see that

$$U_i'(\bar{Y}) \gtreqless 0 \quad \text{as} \quad \alpha(\bar{Y})\left[W_i - t_i c\bar{Y}\right] + \gamma_i(\bar{Y}) \gtreqless t_i c. \tag{9}$$

Substituting from (7) into (9) and rearranging terms we have

$$U_i'(\bar{Y}) \gtreqless 0 \quad \text{as} \quad \gamma_i(\bar{Y}) \gtreqless \gamma(\bar{Y}). \tag{10}$$

Therefore $Y_i^* \gtreqless \bar{Y}$ as $\gamma_i(\bar{Y}) \gtreqless \bar{\gamma}(\bar{Y})$.

Suppose, now, that the functions $\gamma_i(\cdot)$ are symmetrically distributed over the population. Then the mean, $\bar{\gamma}(\bar{Y})$, of the terms, $\gamma_i(\bar{Y})$, will equal their median. This fact, together with (10), implies that just as many people will want more as will want less public good. Therefore $\bar{Y} = \hat{Y}^*$, the median of the favorite amounts. It follows that if taxes are assessed according to (7) and if preferences are symmetrically distributed in this sense, then majority vote will select the Pareto efficient quantity, $\bar{Y}$. This generalizes a result of Bowen [**6**] who showed that if preferences are symmetrically distributed and quasi-linear and if taxes are the same for everyone, then the majority rule outcome is Pareto optimal.

C. Demand-Revealing Mechanisms[7]

Clarke [**8**] and Groves and Loeb [**13**] have demonstrated that if utility is quasi-linear, then there exists an "incentive compatible" mechanism that determines the supply of public goods and individual tax rates. This mechanism uses information supplied by consumers about their own preferences and has the property that honest revelation of preferences is a dominant strategy for each consumer. The amount of public goods selected will satisfy the Samuelson marginal rate of substitution conditions. Groves and Ledyard [**12**] suggest that in more realistic cases where the demand for public goods is income responsive, it may be necessary to settle for a preference revelation mechanism in which honest revelation is a Nash equilibrium but not a dominant strategy. They concede, however, that Nash equilibrium in this context is a less persuasive game theoretic "solution" than dominant strategy. We show here that the Clarke, Groves–Loeb results generalize to the case where utility functions are of the form:

$$(11) \qquad U_i(X_i, Y) = A(Y)X_i + B_i(Y)$$

where U_i is strictly quasi-concave.[8]

The procedure in its simplest form assumes that the function $A(Y)$ is public information.[9] The mechanism induces consumers to honestly reveal the "private information" $B_i(Y)$, in their utility functions. Let the technology be as follows. Each consumer i has an initial endowment of private goods, W_i. Public good is produced from private goods at a total cost $C(Y)$ where $C'(Y) > 0$ and $C''(Y) \geqq 0$. The set of feasible allocations is then the convex set: $\{(X_1, \ldots, X_m, Y) \mid \sum_i X_i + C(Y) = W\}$. All consumers are asked to reveal their functions $B_i(\cdot)$. Each i then reports a function $M_i(\cdot)$ (possibly different from $B_i(\cdot)$). Let $M = (M_i(\cdot), \ldots, M_n(\cdot))$ be the vector of functions reported.

[7] After this paper was written, we discovered a recent paper by Joseph Sicilian [**21**] which reports results similar to the results of this section.

[8] Conn [**9**] has shown a different way in which the Clarke–Groves–Loeb results can be extended beyond the quasi-linear case.

[9] This does not seem unreasonable since if $A(Y)$ is common to everyone's utility function, anyone could discover $A(Y)$ by introspection. If one wished, however, it would not be difficult to devise a mechanism in which honest revelation of $A(Y)$ is a Nash equilibrium and honest revelation of $B_i(Y)$ is dominant strategy.

The government chooses an amount of public goods $Y(M)$ so as to maximize

$$(12)\qquad A(Y)\Big(\sum_j W_j - C(Y)\Big) + \sum_j M_j(Y).$$

Consumer i is assessed a tax bill equal to

$$(13)\qquad T_i(M) = W_i - \sum_j W_j + C(Y(M)) - \left(\frac{\sum_{j\neq i} M_j(Y) + R_i(M)}{A(Y(M))}\right)$$

where $R_i(M)$ is a function that may depend on the information sent by all consumers other than i but must be entirely independent of i's own message. Since for each i,

$$(14)\qquad X_i(M) = W_i - T_i(M),$$

it follows from (11), (13), and (14) that if the vector of functions reported is M, then i's utility is

$$(15)\qquad A(Y(M))\Big[\sum_j W_j - C(Y(M))\Big] + \sum_{j\neq i} M_j(Y(M)) + B_i(Y(M)) + R_i(M).$$

Since $R_i(M)$ is independent of $M_i(\cdot)$, we notice from (15) that the only way in which i's stated function $M_i(\cdot)$ affects his final utility is through the dependence of (15) on $Y(M)$. Therefore, given any choice of messages by the other players, the best choice of M_i for i is the one that leads the government to choose $Y(M)$ to maximize

$$(16)\qquad A(Y)\Big[\sum_j W_j - C(Y)\Big] + \sum_{j\neq i} M_j(Y) + B_i(Y).$$

But recall that the government seeks to maximize (13). Therefore if i reports his true function, so that $M_i(Y) \equiv B_i(Y)$, then the government in maximizing (13) will also maximize (16). It follows that regardless of the message sent by others, consumer i can do no better than to report the truth. Honest revelation is therefore a dominant strategy.

Let $(\bar{T}_1, \ldots, \bar{T}_n, \bar{Y})$ be an equilibrium for this process. That is, $\bar{Y}$ maximizes (12) where $M_i(\cdot) = B_i(\cdot)$ for all i and $\bar{T}_i = T_i(B_1(\cdot), \ldots, B_n(\cdot))$. If it happened that $\sum_i \bar{T}_i = C(\bar{Y})$, then the allocation $(\bar{X}_1, \ldots, \bar{X}_n, \bar{Y})$ where $\bar{X}_i = W_i - \bar{T}_i$ would be Pareto optimal. This is a consequence of Theorem 3 and the fact that $\bar{Y}$ maximizes $A(Y)[\sum_j W_j - C(Y)] + \sum_j B_j(Y)$.

Here, as in the case of quasi-linear utility, it is in general impossible to find functions, $R_i(\cdot)$, that guarantee that $\sum_i \bar{T}_i = C(\bar{Y})$. For the quasi-linear case, Clarke and Groves–Loeb were able to find functions $R_i(M)$ that guarantee feasibility in the sense that tax revenues at least cover costs.[10] We can extend

[10] For expository simplicity, we extend the Clarke tax. The Groves–Loeb tax includes the Clarke tax as a special case (Loeb [**16**]). The generalization of the Groves–Loeb tax is a straightforward extension of the argument used here.

their idea to our broader class of preferences. Suppose that for each i, the government sets a "target share", $\Theta_i \geqq 0$ where $\sum_i \Theta_i = 1$. The government tries to fix $R_i(M)$ so that $T_i(M) \geqq \Theta_i C(Y(M))$ for each i. From (13) we see that

$$(17) \qquad A(Y(M))[T_i(M) - \Theta_i C(Y(M))] = A(Y(M))\Big[(1-\Theta_i)C(Y(M)) - \sum_{j\neq i} W_j\Big] - \sum_{j\neq i} M_j(Y(M)) - R_i(M).$$

From equation (17) and the assumption that $A(Y(M)) > 0$, we see that the government could guarantee that $T_i(M) \geqq \Theta_i C(Y(M))$ if it could set

$$(18) \qquad R_i(M) \leqq A(Y(M))\Big[(1-\Theta_i)C(Y(M)) - \sum_{j\neq i} W_j\Big] - \sum_{j\neq i} M_j(Y(M)).$$

To this end, the government may choose

$$(19) \qquad R_i(M) = \min_Y \Big\{A(Y)\Big[(1-\Theta_i)C(Y) - \sum_{j\neq i} W_j\Big] - \sum_{j\neq i} M_j(Y).$$

It can be seen that $R_i(M)$ as defined in (19) does not depend in any way on i's stated function $M_i(\cdot)$. Furthermore, it is clear that $R_i(M)$ defined in this way satisfies the inequality (18). From (17) it follows that $T_i(M) \geqq \Theta_i C(Y(M))$. Since $\sum_i \Theta_i = 1$, it must be that $\sum_i T_i(M) \geqq C(Y(M))$.

The fact that a simple extension of the Clarke tax performs equally satisfactorily on a larger class of preferences than the quasi-linear reduces the sting of one of the list of criticisms of this mechanism found in Groves and Ledyard [**12**]. Whether in this environment the Clarke tax is likely to perform as well as alternative mechanisms in which honest preference revelation is a Nash equilibrium rather than a dominant strategy equilibrium remains an open question.

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Manuscript received November, 1981; final revision received September, 1982.

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