Journal of Mathematical Economics 3 (1976) 265–268. © North-Holland Publishing Company

PREFERENCES WHICH HAVE OPEN GRAPHS

Theodore C. BERGSTROM, Robert P. PARKS, and Trout RADER

Washington University, St. Louis, MO 63130, U.S.A.

Received June 1975, revised version received January 1976

Recent papers by Mas-Colell (1974), Gale and Mas-Colell (1975), Shafer (1974), and Shafer and Sonnenschein (1975), have shown useful implications of the assumption that P has an open graph. Unless additional conditions are imposed, P may have open sections, but not open graph. We examine some conditions which are sufficient for equivalence of these properties.

Let X be a topological space and $P \subset X \times X$ a relation on X. If $(x, y) \in P$, we write xPy. The *upper* and *lower sections* of P are the sets $P(x) = \{y \in X : yPx\}$ and $P^{-1}(x) = \{y \in X : xPy\}$. If the set P is open in the product topology on $X \times X$, P is said to have an *open graph*. If for all $x \in X$, $P(x)$ and $P^{-1}(x)$ are open in X, P is said to have *open sections*.

Remark 1. If P has an open graph, then P has open sections.

Proof. Let $(x, y) \in P$. Then for some pair of neighborhoods, $N(x)$[1] and $N(y)$ in X, $(x, y) \in N(x) \times N(y) \subset P$. Thus $N(x) \subset P(y)$ and $N(y) \subset P^{-1}(x)$. Therefore $P(y)$ is open for all $y \in X$ and $P^{-1}(x)$ is open for all $x \in X$. It follows that P has open sections. Q.E.D.

Remark 2. If P is transitive and has open sections, and if xPz and zPy for some $z \in X$, then $(x, y) \in \operatorname{Int} P$.

Proof. Let xPz and zPy. If P has open sections, then there exist neighborhoods $N(x)$ and $N(y)$ such that for all $(x', y') \in N(x) \times N(y)$, $x'Pz$ and zPy'. Then transitivity implies $x'Py'$ and $N(x) \times N(y) \subset P$. Therefore $(x, y) \in \operatorname{Int} P$. Q.E.D.

Definition. P is *order dense* if for all $(x, y) \in P$, there exists $z \in X$ such that xPz and zPy.

[1] In this paper '$N(x)$' always denotes a neighborhood of x in X.

Theorem 1. *If P is transitive and order dense, then P has an open graph if and only if P has open sections.*[2]

Theorem 1 is immediate from Remarks 1 and 2.

Definition. Where $P \subset X \times X$, $P^c = \{(x, y) \in X \times X: (x, y) \notin P\}$.

Theorem 2. *If P and P^c are transitive, then P has an open graph if and only if P has open sections.*

Proof. Let P and P^c be transitive. If P has open sections and $(x, y) \in P$, there exist $N(x)$ and $N(y)$ such that $N(x) \subset P(y)$ and $N(y) \subset P^{-1}(x)$. If $N(x) \times N(y) \subset P$, then $(x, y) \in \operatorname{Int} P$. If not, then $x'P^cy'$ for some $(x', y') \in N(x) \times N(y)$. If xP^cx', then by transitivity of P^c, xP^cy'. But this contradicts $y' \in N(y) \subset P^{-1}(x)$. Therefore xPx' and, since $x' \in N(x)$, $x'Py$. From Remark 2, it follows that $(x, y) \in \operatorname{Int} P$. We have shown that if P has open sections, $P = \operatorname{Int} P$ and hence P has an open graph. Remark 1 shows the converse. Q.E.D.

Definition. P is a *weak order* if P is asymmetric and P^c is transitive.

Remark 3. If P is a weak order, then P is transitive. Where $I = \{(x, y): xP^cy$ and $yP^cx\}$ and $R = \{(x, y): yP^cx\}$, P is a weak order if and only if I is an equivalence relation and R is reflexive, transitive and complete.

The proof of Remark 3 is straightforward and can be found in Fishburn (1970)

The following is an immediate consequence of Theorem 2 and Remark 3.

Corollary. *If P is a weak order, then P has an open graph if and only if P has open sections.*

The following remark shows that transitivity alone is not sufficient for equivalence of the open graph and open sections properties.

Remark 4. There are transitive relations with open sections which do not have an open graph.

Proof. We construct an example. Let X be the real line and $P = \{(x, y) \in X \times X: x - y > 1$. Clearly P is transitive and has open sections. (In fact P has an open graph although P is not order dense and P^c is not transitive.) Construct the relation P^* by deleting from P the set $\{(2+1/n, 1/n): n$ is a positive integer$\}$. Since $((2+1/n, 1/n) \to (2, 0)$ and $(2, 0) \in P^*$, P^* does not have an open graph. The sections of P^* are obtained by deleting at most one point from the sections

[2]Schmeidler (1969) proves a similar but less general result.

of P. Therefore P^* has open sections. Suppose that xP^*y and yP^*z. Then xPy and yPz and hence $x-z > 2$. But if $x-z > 2$, then xP^*z. Therefore P^* is transitive. Thus P^* is a transitive relation with open sections which does not have an open graph. Q.E.D.

The following result involves only trivial modifications of a theorem due to Shafer (1974).

Theorem 3. If X is the non-negative orthant in $\mathbf{R}^n$ (with the usual relative topology) and $P(x)$ is a convex set for all $x \in X$ [or $P^{-1}(x)$ is a convex set for all $x \in X$] then P has an open graph if and only if P has open sections.[3]

Another characterization of the open graph property, due to Shafer (1974) is the following:

Theorem 4. If X is a metric space and P is asymmetric, then P has an open graph if and only if there exists a real, continuous function k: $X \times X \rightarrow R$ such that $k(x, y) > 0$ if and only if xPy.

Proof. Shafer (1974, Theorem 1) demonstrates that the open graph property implies the existence of such a function. Where k is continuous and $k(x, y) > 0$ if and only if xPy, $P = k^{-1}\{z \in R: z > 0\}$ is the inverse image of an open set and hence is open in $X \times X$. Q.E.D.

Theorem 4 suggests a simple way of constructing preferences which have both open graph and open sections but satisfy none of the conditions shown above to be sufficient for equivalence of these properties. For example, let $k(x, y) = \text{Min}\,\{\{\frac{3}{2} - \sin(x) - \sin(y), \sin(x) - \sin(y) - \frac{1}{2}\}$, and $P = \{(x, y) \in R \times R: k(x, y) > 0\} = \{(x, y) \in R \times R: \frac{3}{2} > \sin(x) - \sin(y) > \frac{1}{2}\}$. Since k is continuous, P has an open graph. It is easy to show that P is neither order dense nor transitive, that P^c is not transitive and that for all $x \in X$, neither $P(x)$ nor $P^{-1}(x)$ is convex.

Conclusion

From the above remarks we may conclude the following. In the case of the classical 'rational consumer' for whom P is a weak order, P has an open graph if and only if P has open sections. These properties are also equivalent if P is order dense and transitive. If, however, P is merely transitive, P may have open sections but not an open graph. In finite dimensional Euclidean space, if $P(x)$ is a convex set for all $x \in X$, the open graph property and the open sections property are equivalent even if P is not transitive.

[3]Whether this result can be generalized to cases where X is infinite dimensional or to the case where X is an arbitrary convex subset of $\mathbf{R}^n$ is not known to us.

The assumption, '$P(x)$ is convex' demands more convexity than is needed for many purposes in general equilibrium analysis. For example, Bergstrom (1975) shows the existence of equilibrium in an economy where P has open sections and for all $x \in X$, x does not belong to the convex hull of $P(x)$. The preference relation in Remark 4 has both of these properties but $P(x)$ is not always convex and P does not have an open graph.

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