

Maximal Elements of Acyclic Relations on Compact Sets

A binary relation P on a set X said to be acyclic if there is no finite set $\{x_1, \dots, x_n\} \subset X$ such that $x_i P x_{i+1}$ for $i = 1, \dots, n - 1$ and such that $x_n P x_1$. Sen [5] demonstrates that, if X is a finite set and P is acyclic, then there exists an element of X which is maximal relative to P . This result can be extended in a natural way to compact sets in a topological space when P satisfies a certain continuity property.^{1,2}

Where R is any binary relation and $x \in X$, define $R_x(x) \equiv \{y \in X \mid yRx\}$ and $R_x^{-1}(x) \equiv \{y \in X \mid xRy\}$.

THEOREM. *Let X be a compact set in a topological space and let P be an acyclic relation on X . If, for all $x \in X$, $P_x^{-1}(x)$ is open in the relative topology of X , then there exists $\bar{x} \in X$ such that $P_{\bar{x}}(\bar{x}) = \emptyset$.*

To prove the theorem we use the following lemma due to Rader [4, p. 134].

LEMMA. *Let X be a compact set and let Q be an irreflexive, transitive relation on X such that for all $x \in X$, $Q_x^{-1}(x)$ is open (in the relative topology on X). Then there exists $\bar{x} \in X$ such that $Q_{\bar{x}}(\bar{x}) = \emptyset$.*

Proof of Theorem. Let X and P satisfy the hypothesis of the theorem. Let Q be the transitive closure on X of P ; that is, xQy if and only if xPy or there exists a finite set $\{x_1, \dots, x_n\} \subset X$ such that xPx_1 , x_iPx_{i+1} for $i = 1, \dots, n - 1$, and x_nPy . Clearly Q is transitive on X . Since P is acyclic, Q must be asymmetric and hence irreflexive. Furthermore, for all $x \in X$, $Q_x^{-1}(x)$ is open in the relative topology. The latter is true since, if $y \in Q_x^{-1}(x)$,

¹ I am informed that Kotaro Suzumura has independently discovered this result. The theorem also appears in [1], where it is applied to the problem of existence of competitive equilibrium.

² Fishburn [2] treats acyclic relations (which he calls suborders). He shows that, given certain monotonicity and Archimedean assumptions, if P is acyclic there exists an upper semicontinuous function U such that xPy implies $U(x) > U(y)$. Since U is upper semicontinuous, it takes a maximum on compact subsets of its domain. Where \bar{x} maximizes U on a compact set X , $P(\bar{x}) \cap X = \emptyset$. Thus Fishburn's assumptions are sufficient to ensure the existence of maximal elements on compact sets. The treatment here allows one to avoid monotonicity and to substitute the topological assumption that $P_x^{-1}(x)$ is open for the Archimedean continuity conditions of Fishburn.

then there exists $x_n \in X$ such that xQx_n and $y \in P_X^{-1}(x_n) \subset Q_X^{-1}(x)$. But $P_X^{-1}(x_n)$ is, by assumption, open in the relative topology. Therefore $Q_X^{-1}(x)$ must be open since it contains a neighborhood of each of its points. According to the Lemma, then, there exists $\bar{x} \in X$ such that $Q_X(\bar{x}) = \emptyset$. But $P_X(\bar{x}) \subset Q_X(\bar{x})$. Therefore, $P_X(\bar{x}) = \emptyset$. Q.E.D.

Since the lemma borrowed from Rader is important and its proof is brief and elegant, we present a proof here. This proof differs slightly in format from that of Rader but is similar in spirit.

Key to the proof are the following standard results of topology and set theory, respectively. Both can be found in Kelley [3].

FINITE INTERSECTION PRINCIPLE. Let S be a collection of compact sets. If the intersection of every finite subcollection of S is nonempty, then the intersection of all sets in S is nonempty.

KURATOWSKI'S LEMMA. Every totally ordered subset of a partially ordered set is contained in a maximal totally ordered subset.

Proof of Lemma. Since Q is transitive, Q is a partial order. According to Kuratowski's lemma, there exists a maximal totally ordered subset of X . Call this set A . Let B be an arbitrary finite subset of A . For all $x \in X$, define $S(x) \equiv X \cap [Q_X^{-1}(x)]^c$. Since B is finite and totally ordered, there exists $\hat{x} \in B$ such that $\hat{x}Px$ if $x \in B$ and $x \neq \hat{x}$. Since Q is transitive and irreflexive, $\hat{x} \in \bigcap_{x \in B} S(x)$. Since $Q^{-1}(x)$ is open, $S(x)$ is closed and hence compact for all $x \in X$. It follows from the finite intersection principle that there exists $\bar{x} \in \bigcap_{x \in A} S(x)$. Clearly, $Q_X(\bar{x}) \cap A = \emptyset$. Suppose that $x \in X \cap A^c$. If $xQ\bar{x}$, then $A \cup \{x\}$ is a totally ordered subset of X . But this is impossible since A is a maximal totally ordered subset. It follows that $Q_X(\bar{x}) = \emptyset$. Q.E.D.

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