Characterization and Implementation of Single-valued Nash Bargaining Solutions with Non-convex Problems

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Nash bargaining solution: Quick review

- Nash’s (1950) axiomatic analysis of 2-person bargaining
- Bargaining problem: \((S, d)\) where \(S \subset \mathbb{R}^2\) is set of feasible payoff pairs, \(d \in \text{int}(S)\) is disagreement point, \(S\) is compact and convex
- Bargaining solution: select a feasible payoff pair for every bargaining problem
- Axioms: Invariance to positive affine transformation (INV), Pareto optimality (PO), Independence of irrelevant alternatives (IIA), Symmetry (SYM)
- Nash bargaining solution (NBS):
  \[
  f(S, d) = \arg\max_{u \in S, u \geq d} (u_1 - d_1)(u_2 - d_2)
  \]
- Asymmetric NBS (Kalai, 1977): Having dropped SYM,
  \[
  f(S, d) = \arg\max_{u \in S, u \geq d} (u_1 - d_1)^{\alpha_1} (u_2 - d_2)^{\alpha_2}
  \]
  \[
  (\alpha_1, \alpha_2) = f \left( \{ u \in \mathbb{R}^2_+: u_1 + u_2 \leq 1 \}, 0 \right)
  \]
In Nash’s theory, the payoff set $S$ is required to be convex.

In reality, $S$ is usually non-convex, e.g., duopolies with asymmetric MC (Bishop (1960), Schmalensee (1987), Tirole (1988)), employer-employee bargaining (Aoki (1980), McDonald and Solow (1981), Miyazaki (1984))...

Usual justification is allowing randomization and assuming expected utility maximizers.

But randomization may not be realistic in practice (e.g., conflicts of interest at post-realization stage).

Convexity and INV go together?

I don’t agree because

preferences over lotteries doesn’t imply uncertainty (The fact that I’m able to compare two choices doesn’t imply that the two choices are available.)

INV can be interpreted in context of time preferences, rather than preferences over lotteries.
Single-valued bargaining solutions

- Conley and Wilkie (1996): extend NBS, violate IIA
- Zhou (1997): allowing non-convexity, any single-valued bargaining solution that satisfies the axioms is a selection of Nash product maximizers (This isn’t a full characterization.)
- Qin, Shi, and Tan (2014): relaxing convexity to log-convexity, NBS doesn’t change (including that Nash product maximizer is still unique)

Multi-valued bargaining solutions

- Kaneko (1980): extend NBS, require upper-hemicontinuity, set of all Nash product maximizers
- Peters and Vermeulen (2012): multi-valued bargaining solutions include iteratively maximizing asymmetric Nash products, and uncountably many more.
This paper

- fully characterizes single-valued bargaining solutions that satisfy Nash’s axioms (IIA in particular)
- provides a simple game form that implements our solutions as SPE outcome
A bargaining problem (in utility form) is a tuple \((S, d)\) where

\[d \in S \subseteq \mathbb{R}^2,\]

\(S\) is compact,

\[\exists u \in S \text{ s.t. } u \gg d,\]

and \(S\) is star-shaped about \(d\), i.e.,

if \(u \in S\) and \(p \in [0, 1]\), then \(pu + (1 - p)d \in S\).

Let \(\mathcal{B}\) denote the collection of all bargaining problems. (\(S\) in a bargaining problem in \(\mathcal{B}\) may not be convex.)
**Definition:** A (single-valued) bargaining solution on $\mathcal{B}$ is a function $f : \mathcal{B} \to \mathbb{R}^2$ such that $f(S, d) \in S$ for all $(S, d) \in \mathcal{B}$.

**INV** (Invariance to positive affine transformation): For any $(S, d) \in \mathcal{B}$, if $(S', d')$ is obtained from $(S, d)$ by a positive affine transformation $u_i \mapsto \beta_i u_i + \gamma_i$ with $\beta_i > 0$ for $i = 1, 2$, then $f_i(S', d') = \beta_i f_i(S, d) + \gamma_i$ for $i = 1, 2$.

**SIR** (Strict individual rationality): For any $(S, d) \in \mathcal{B}$, $f(S, d) \succ d$.

**IIA** (Independence of irrelevant alternatives): For any $(S, d), (T, d) \in \mathcal{B}$ with $S \subset T$, if $f(T, d) \in S$, then $f(S, d) = f(T, d)$.

**WS** (Weak symmetry): If $f\left(\{u \in \mathbb{R}_+^2 : u_1 + u_2 \leq 1\}, 0\right) = u$, then $u_1 = u_2$. 
Following Roth’s (1977) logic, INV, SIR, and IIA together imply

**Pareto optimality (PO):** For any $(S, d) \in \mathcal{B}$, $f(S, d)$ belongs to the strict Pareto frontier of $S$. 
Theorem (1)

A bargaining solution $f$ on $\mathcal{B}$ satisfies INV, SIR, IIA, and WS iff there exists some $i \in \{1, 2\}$ such that, for any $(S, d) \in \mathcal{B}$,

$$\{f(S, d)\} = \arg\max_{u \in \Sigma(S, d)} u_i$$

where

$$\Sigma(S, d) \equiv \arg\max_{u \in S, u \geq d} (u_1 - d_1) (u_2 - d_2).$$

- Zhou’s (1997) result holds here: any single-valued bargaining solution that satisfies the axioms is a selection of Nash product maximizers
- How to select? Have to either always favor player 1, or always favor player 2
If you have to pick one, which one do you pick?
Example

Suppose $f(S, 0) = v$

IIA: $f(com(u, v), 0) = v$ and $f(com(v, w), 0) = v$

The first and INV imply $f(com(v, w), 0) = w$, contradicting the second
Idea of the general proof

- Given $f$ that satisfies INV, SIR, IIA, let $u \succ v$ denote $f(\text{com}(u, v), 0) = u$
- $\succ$ is a strict linear order, whose maximizer over $S$ is $f(S, 0)$
- Need to know $\{u : u \succ v\}$ and $\{u : v \succ u\}$ for every $v$
- INV and SIR: only need $\{u \in R_{++}^2 : u \succ 1\}$ and $\{u \in R_{++}^2 : 1 \succ u\}$
- INV and IIA: after log-transformation, they are convex, and are negatives of each other (See graphs)
- Should maximize $\alpha_1 \log u_1 + \alpha_2 \log u_2$, then break any tie by always favoring 1, or 2
- SIR: $\alpha_1, \alpha_2 > 0$; WS: $\alpha_1 = \alpha_2$
Bargaining environment

- Two players, 1 and 2. A set \( X \) of alternatives. A status quo \( q \in X \)
- Lotteries are not feasible to be an agreement
- Delay is feasible; time is continuous
- A feasible bargaining outcome is some \( \langle x, t \rangle \) with \( x \in X \) and \( t \in [0, \infty] \)
- Player \( i \)'s utility for \( \langle x, 0 \rangle \) is represented by \( U_i(x) \), with \( U_i(q) = 0 \)
- Player \( i \)'s utility for \( \langle x, t \rangle \) is \( e^{-rt} U_i(x) \), where \( r > 0 \) is the common instantaneous discount rate

**Alternative interpretation:** Every \( x \in X \) is a state that yields a flow utility \( rU_i(x) \) at every instant of time for each player \( i \). A feasible outcome \( \langle x, t \rangle \) is interpreted as staying at the status quo \( q \) until switching to alternative \( x \) after waiting for time \( t \).
Assumptions:

- $X$ is compact (in a topological space)
- $U(\cdot) \equiv (U_1(\cdot), U_2(\cdot))$ is continuous on $X$
- There exists $x \in X$ s.t. $U(x) \geq 0$

$(X, q, r, U)$ is bargaining environment, or bargaining problem in physical form

Given $(X, q, r, U)$, the set of feasible payoff pairs is

$$S \equiv \{ e^{-rt} U(x) : x \in X \text{ and } t \in [0, \infty] \}$$

$$= \{ pU(x) : x \in X \text{ and } p \in [0, 1] \}.$$
Bargaining problems in utility form

- $S$ and $d$ (normalized to be 0) satisfy:

$$d \in S \subset \mathbb{R}^2,$$  

(1)

$S$ is compact,

(2)

$$\exists u \in S \text{ s.t. } u \gg d,$$  

(3)

and $S$ is *star-shaped about* $d$, i.e.,

$$\text{if } u \in S \text{ and } p \in [0, 1], \text{ then } pu + (1 - p) d \in S.$$  

(4)

- Conversely, $(S, d)$ with (1) – (4) is a bargaining problem in utility form for some $(X, q, r, U)$.  

• Binmore, Rubinstein, and Wolinsky’s (1986) approximate implementation result extends to non-convex problems?
  
  No! See Herrero (1989) for SSPE; Binmore (1987) for SPE

• Nash’s (1953) approximate implementation result extends to non-convex problems?
  
  Again, no! All local maximizers of Nash product

Given non-convexity and the axioms, is single-valuedness consistent with game-theoretic equilibrium concepts?

• Yes! Each of the above generalized NBSs can be SPE-implemented by the "nomination and trial game" below.
Let $\Gamma^2(i, t, Y)$ denote the two-period alternating offer game (as in Stahl (1972)) where

- $i \in \{1, 2\}$ is first mover, $t \in [0, \infty]$ is length of time delay between two offers
- $Y \subset X$ with $q \in Y$ is the set of alternatives on the table

Two phases:

**Nomination phase**: Player 1 (2) nominates a candidate alternative $x$ (y) from $X$, sequentially or simultaneously.

**Trial phase**:

1. Player 1 (2) specifies a delay time $t_1$ ($t_2$) in $[0, \infty]$.
2. If $t_1 < t_2$, then the two players play $\Gamma^2(1, t_1, \{q, x, y\})$.
3. If $t_2 < t_1$, then the two players play $\Gamma^2(2, t_2, \{q, x, y\})$. 
This trial phase is equivalent to Moulin’s (1984) game if \( \{q, x, y\} \) is replaced by \( X \), implementing Kalai-Smorodinsky (1975) solution.

**Theorem (2)**

Given any \((X, q, r, U)\), the nomination and trial game version 1 has at least one SPE, and the set of SPE payoffs is the set of all Nash product maximizers. That is, the game SPE-implements Kaneko (1980) solution.

In the above game, the delay time chosen was the smaller one in \( \{t_1, t_2\} \). The result would be the same if we take the larger one, or any weighted average of \( t_1 \) and \( t_2 \).
Idea of partial proof: SPE payoff must maximize Nash product

Consider $\Gamma^2(i, t, \{q, x, y\})$

- Let $u \equiv U(x)$ and $v \equiv U(y)$
- Only need to consider $u, v \geq 0$, $u_1 \geq v_1$ and $v_2 \geq u_2$
- If $u_1 u_2 > v_1 v_2$, then $u_2/v_2 > v_1/u_1$
  - $\exists t$ s.t. $e^{-rt} u_1 > v_1$ and $e^{-rt} v_2 < u_2$ so that $x$ is the unique SPE outcome of $\Gamma^2(i, t, \{q, x, y\})$ for $i = 1, 2$

Consider the whole game

- In any SPE in which payoffs do not maximize Nash product
- Either 1 or 2 has a profitable deviation in the following form
  - nominate a Nash product maximizer
  - specify some $t$ s.t. this Nash product maximizer must win in the $\Gamma^2$ subgame
Two phases:

**Nomination phase:** Player 1 (2) nominates a candidate alternative \( x \) (\( y \)) from \( X \), sequentially or simultaneously. Let \( u \equiv U(x) \) and \( v \equiv U(y) \).

**Trial phase:**

1. Player 1 specifies a delay time \( t_1 \in [0, \infty] \).
2. Then player 2 specifies another delay time \( t_2 \in [0, \infty] \) with \( t_2 \neq t_1 \).
3. If \( t_1 < t_2 \), then the two players play \( \Gamma^2(1, \frac{t_1 + t_2}{2}, \{q, x, y\}) \).
4. If \( t_2 < t_1 \), then the two players play \( \Gamma^2(2, \frac{t_1 + t_2}{2}, \{q, x, y\}) \).
Theorem (3)

Given any \((X, q, r, U)\), the nomination and trial game version 2 has at least one SPE, and all the SPE payoffs are given by the generalized Nash bargaining solution described in Theorem 1 with \(i = 1\).

Idea of the proof of the second part:

- Let \(f(S, d)\) be the payoff pair given by the bargaining solution in Theorem 1 with \(i = 1\)
- Each player \(i\) has a strategy to guarantee him the payoff \(f_i(S, d)\)
Generalizations: Dropping weak symmetry

Theorem (4)

A bargaining solution $f$ on $\mathcal{B}$ satisfies INV, SIR and IIA iff there exists some $i \in \{1, 2\}$ such that, for any $(S, d) \in \mathcal{B}$,

$$\{f(S, d)\} = \arg\max_{u \in \Sigma(S, d)} u_i$$

where

$$\Sigma(S, d) \equiv \arg\max_{u \in S, u \geq d} (u_1 - d_1)^{\alpha_1} (u_2 - d_2)^{\alpha_2},$$

$$\alpha \equiv f \left( \{u \in \mathbb{R}^2_+ : u_1 + u_2 \leq 1\}, 0 \right) \gg 0.$$
Generalizations: Dropping weak symmetry

Theorem (5)

Given \((X, q, r, U)\) with \(r \in R^2_{++}\) (so that \(r_1 \neq r_2\) is allowed), the nomination and trial game version 2 has at least one SPE, and all the SPE payoffs are given by the generalized asymmetric NBS described in Theorem 4 with \(i = 1\), \(\alpha_1 = r_2 / (r_1 + r_2)\), and \(\alpha_2 = r_1 / (r_1 + r_2)\).
Generalizations: More than 2 players

Theorem (6)

An n-player bargaining solution \( f \) on \( B^n \) satisfies INV, SIR, and IIA iff there exist \( n \) pairwise orthogonal non-zero vectors \( \alpha_1, \ldots, \alpha_n \in \mathbb{R}^n \) s.t.

\[
\{ f(S, d) \} = \Sigma^n(S, d),
\]

where

\[
\Sigma^0(S, d) \equiv S \cap \{ u \in \mathbb{R}^n : u \geq d \},
\]

\[
\Sigma^j(S, d) \equiv \arg\max_{u \in \Sigma^{j-1}(S, d)} \prod_{i=1}^{n} (u_i - d_i)^{\alpha^j_i}, \quad j = 1, \ldots, n,
\]

\[
\alpha^1 = f \left( \left\{ u \in \mathbb{R}^n : \sum_{i=1}^{n} u_i \leq 1 \right\}, 0 \right) \gg 0.
\]

Each \( \alpha^j \) above is unique up to positive-scaler-multiplication (i.e., the direction of \( \alpha^j \) is unique).
develop a natural model that easily generates non-convexity in 2-person bargaining problems

fully characterize the set of bargaining solutions that satisfy INV, SIR, IIA, and WS
  - there are only two: selecting a Nash product maximizer by always favoring 1, or 2

provide a simple game form that implements our solutions as SPE outcome

provide generalizations to more than 2 asymmetric players