Dynamic Properties of the Optimal Growth Model

I. Graphical Analysis

• Restate the key differential equations (in effective units for convenience):
  1. Euler equation: \[ \frac{\dot{c}}{c} = \frac{1}{\theta} (r - n - g - \beta) = \frac{1}{\theta} (f'(k) - \delta - \rho - \theta g) \]
  2. Dynamics of capital: \[ \dot{k} = f(k) - c - (n + g + \delta)k \]

• Boundary conditions: (i) given \( k(0) > 0 \); (ii) \( \lambda(T) \cdot k(T) \to 0 \).
  [Note: Analysis relies only on these equations/conditions – derives as in Romer or with optimal control.]

• Graphical solution with a Phase Diagram in the \((c, k)\)-space:
  1. Line of constant consumption: Vertical line at \( f'(k^*) = \rho + \delta + \theta g \)
  2. Line of constant capital: Steady state relation \( c(k) = f(k) - (n + g + \delta)k \)

  Line goes through \( c(0) = 0 \). Declining MPK \( \Rightarrow \) There is some \( k_{\max} \) such that \( c(k_{\max}) = 0 \).
  Note that \( c'(k^*) = f'(k^*) - (n + g + \delta) = \beta > 0 \).

• Graphs in Romer:
  - Fig.2.1-2.5: Step-by step derivation.
  - Fig.2.6: Application = Sudden fall in the discount rate (rate of time preference)
The Phase Diagram

Main elements:
1. Lines of constant c and constant k. Intersection: Steady state E.
2. Arrows indicating movement: c rising for k<k*; c falling for k>k*; k rising below c(k) line; k falling above the c(k) line.
3. Dynamics from a generic starting point k(0):
   - From points like A-B-C: Trajectory leads to k=0 => Path not feasible.
   - From points like D: Trajectory leads to into (k=k_{max},c=0). Claim: Violates terminal condition.

• Conclude: Optimal c(0) must be at Point F = On the saddle path leading to point E.
Why Point D violates the Transversality Condition:

• Claim: Trajectories leading to \( (k=k_{\text{max}},c=0) \) violate the transversality condition.

• Proof:
  - Consider \( c(k) = f(k) - (n + g + \delta)k \)
    - Peak of \( c(k) \) is reached at \( c'(k)=f'(k)-(n+g+\delta)=0 \) \( \Rightarrow \) Golden Rule level \( k_{Golden} \)
    - Trajectories to \( k_{\text{max}} \) must cross \( k_{Golden} \) in at some finite date \( T_0 \). For \( t>T_0, r(t)<n+g \).
  - Consider \( \dot{\lambda}(t) = -[r(t) - (n + g)] \cdot \lambda(t) \).
    - Linear differential equation with variable coefficients.
    - Solution: \( \lambda(T) = \lambda(0) \cdot e^{-\int_0^T [r(t)-n-g]dv} \)
    - Finding \( r(t)<n+g \) for \( t>T_0 \) implies \( \lambda(T) \rightarrow \infty \)
  - Find lower bound: \( \lambda(T)k(T) \geq \lambda(T_0)k_{Golden} > 0 \ \forall t > T_0 \).

• Additional insights:
  - All trajectories ending with over-accumulation violate the transversality condition.
  - The peak of \( c(k) \) is to the right of Point E: \( k^* < k_{Golden} \)
Applications/Examples

- Destruction of initial capital at t=0: Jump down in k(0). Return to k* over time.

  Impact on c(0)? [To prove: Down]

  Question of logic: What do “unexpected” changes mean in a deterministic model?

  Least-flawed answer: One time disturbance with near-zero probability.

- Increase in the rate of time preference ρ: Shift left in k* (see Romer Fig.2.6).

  Impact on c(0)? [To prove: Up]

- Increase in population growth: Shift down in c(k). No change in k*.

  Impact on c(0)? [To prove: Down]

  General point: Demographic changes have no long-run impact on the capital-labor ratio and no long-run impact on returns on capital.

- Increase in the depreciation rate: Shift down in c(k). Shift left in k*.

  Impact on c(0)? [Depends: Is the new saddle path above or below the old steady state?]

- Increase in productivity growth: Shift down in c(k). Shift left in k*.

  Impact on c(0)? [Depends!]
**Dynamics Properties II: Approximating the Convergence Process**

- **Questions:** How quickly does the economy approach the steady state? What do we know about the saddle path?

- **Approach:**
  - Linearize the differential equations for (c,k) around the steady state.
  - Compute the rate of convergence.

- **Two methods of linearization:**
  1. Linearization in levels = Romer’s approach.
  2. Linearization in logarithms = Barro/Sala-i-Martin’s approach. Advantages:
     - Deviation from the steady state are percentages (unit-free)
     - Responses of endogenous variables to disturbances interpretable as elasticities.

- Both useful to practice linearization and for solving differential equations.
Linearization in Levels

- Steady state: $f'(k^*) = \delta + \rho + \theta g$ and $c^* = f(k^*) - (n + g + \delta)k^*$

- Nonlinear model:
  \[
  \dot{k} = f(k) - c - (n + g + \delta)k \\
  \dot{c} = \frac{1}{\theta} [f'(k) - \delta - \rho - \theta g] \cdot c
  \]

- Taylor series approximation at $(c,k) = (c^*,k^*)$
  \[
  \dot{k} \approx [f'(k^*) - (n + g + \delta)] \cdot (k - k^*) - (c - c^*) = \beta \cdot (k - k^*) - (c - c^*) \\
  \dot{c} \approx c^* f''(k^*) \cdot (k - k^*)
  \]
  - Zero coefficient on $c - c^*$ because $\frac{dc}{dc} = \frac{1}{\theta} [f'(k) - \delta - \rho - \theta g] = 0$ at $k = k^*$.

- Matrix representation of the linearized dynamics:
  \[
  \begin{pmatrix}
  \dot{k} \\
  \dot{c}
  \end{pmatrix} =
  \begin{pmatrix}
  \beta & -1 \\
  \frac{c^*}{\theta} f''(k^*) & 0
  \end{pmatrix}
  \begin{pmatrix}
  k - k^* \\
  c - c^*
  \end{pmatrix}
  = A \cdot \begin{pmatrix}
  k - k^* \\
  c - c^*
  \end{pmatrix}
  \\
  \\
  \text{with } A = \begin{pmatrix}
  \beta & -1 \\
  \frac{c^*}{\theta} f''(k^*) & 0
  \end{pmatrix}
  \]

- System of homogenous linear differential equations: $\dot{x} = A \cdot x$ with $x = (k - k^*, c - c^*)$.
  - General Solution: Exponential functions with eigenvalues of the characteristic matrix $A$ in the exponents.


- Main qualitative findings: “A” has one negative eigenvalue $\mu_1$. Rate of convergence = $|\mu_1|$.
  
  [Second eigenvalue is positive. Boundary conditions imply zero weight on positive eigenvalues.]
Linearization in Logarithms

- Another example of doing linearizations:

- Steady state: \( f'(k^*) = \delta + \rho + \theta g \) and \( c^* = f(k^*) = (n + g + \delta)k^* \)

- Nonlinear model in logarithms:

\[
\frac{\dot{k}}{k} = \frac{d\ln(k)}{dt} = f(k)/k - c/k - (n + g + \delta) = f(e^{\ln(k)})e^{-\ln(k)} - e^{\ln(c)-\ln(k)} - (n + g + \delta)
\]

\[
\frac{\dot{c}}{c} = \frac{d\ln(c)}{dt} = \frac{1}{\theta} f'(e^{\ln(k)}) - \frac{1}{\theta} (\delta - \rho - \theta g)
\]

- Taylor series approximation at \((c,k) = (c^*,k^*)\) [Skipping steps: Exercise to verify on your own.]

\[
\frac{k}{k} \approx \left[ f' - \frac{f''}{k^*} + \frac{c^*}{k^*} \right] \ln\left(\frac{k}{k^*}\right) - \frac{c^*}{k^*} \ln\left(\frac{c}{c^*}\right) = \beta \cdot \ln\left(\frac{k}{k^*}\right) - \frac{c^*}{k^*} \ln\left(\frac{c}{c^*}\right)
\]

\[
\frac{\dot{c}}{c} \approx \frac{f''(k^*)k^*}{\theta} \cdot \ln\left(\frac{k}{k^*}\right)
\]

- Define \( \kappa = -\frac{1}{\theta} f''(k^*)k^* \) \( > 0 \)

- Matrix representation of the log-linearized dynamics:

\[
\begin{pmatrix}
\frac{\dot{k}}{k} \\
\frac{\dot{c}}{c}
\end{pmatrix}
= \begin{pmatrix}
\beta & -\frac{c^*}{k^*} \\
-\kappa & 0
\end{pmatrix}
\begin{pmatrix}
\ln k - \ln k^* \\
\ln c - \ln c^*
\end{pmatrix}
= \tilde{A} \cdot \begin{pmatrix}
\ln(k/k^*) \\
\ln(c/c^*)
\end{pmatrix}
\text{for } \tilde{A} = \begin{pmatrix}
\beta & -\frac{c^*}{k^*} \\
-\kappa & 0
\end{pmatrix}
\]

Solving the Log-linearized System

- Eigenvalues \( \mu = \mu_i \) are the solutions to the characteristic equation \( \text{Det}(\tilde{A} - \mu I) = 0 \)

- For matrix \( \tilde{A} = \begin{pmatrix} \beta & -c^* \\ -\kappa & 0 \end{pmatrix} \), compute \( \text{Det} \begin{pmatrix} \beta - \mu & -c^*/k^* \\ -\kappa & -\mu \end{pmatrix} = 0 \)

\[ \Rightarrow \text{Solutions: } \mu_{1,2} = \frac{\beta}{2} \pm \sqrt{\left(\frac{\beta}{2}\right)^2 + \frac{c^*}{k^*} \kappa} \]

- Observation: \( \mu_1 < 0 < \mu_2 \). One positive and one negative root.

- General solution to the system of differential equations (undetermined coefficients \( \varphi \))

- For log-capital:

\[ \ln(k(t)) = \ln(k^*) + \varphi_1 e^{\mu_1 t} + \varphi_2 e^{\mu_2 t} \]

  - If \( \varphi_2 \) were positive, capital would diverge to infinity \( \Rightarrow \) Boundary conditions imply \( \varphi_2 = 0 \).

\[ \Rightarrow \ln(k(t)) = \ln(k^*) + \varphi_1 e^{\mu_1 t} \]

  - Initial condition determines \( \varphi_1 \):

\[ \ln(k_0) - \ln(k^*) = \varphi_1 e^{\mu_1 0} = \varphi_1 \]

\[ \Rightarrow \ln(k(t)) = \ln(k^*) + [\ln(k_0) - \ln(k^*)] e^{\mu_1 t} \]

- Analogous arguments for log-consumption:

\[ \ln(c(t)) = \ln(c^*) + [\ln(c_0) - \ln(c^*)] \cdot e^{\mu_1 t} \]

  - Loose end: Initial consumption is endogenous, must be determined (unlike initial capital).

  Defer question about the starting point – first consider speed of convergence.
The Speed of Convergence

• Speed of convergence in logarithms
  - Differentiate the solutions to for capital and consumption:
    \[ \frac{\dot{k}}{k} = \frac{d}{dt} \ln(k(t)) = \ln(k_0) - \ln(k^*) \cdot \mu_1 e^{\mu_1 t} = \mu_1 \cdot \ln\left(\frac{k(t)}{k^*}\right), \text{ and similarly} \]
    \[ \frac{\dot{c}}{c} \approx \mu_1 \cdot \ln\left(\frac{c(t)}{c^*}\right) \]
  - Result: Common convergence rate for \( \ln(c) \) and \( \ln(k) = \mu_1 \), the negative eigenvalue.

• Speed of convergence in levels:
  - Approximation around the steady state:
    \[ \dot{k} \approx k^* \cdot \frac{\dot{k}}{k} = k^* \cdot \mu_1 \ln\left(\frac{k(t)}{k^*}\right) \approx k^* \cdot \mu_1 \frac{k(t) - k^*}{k^*} = \mu_1 \cdot \left[k(t) - k^*\right], \text{ and similarly} \]
    \[ \dot{c} \approx \mu_1 \cdot \left[c(t) - c^*\right]. \]
  - Result: \( \mu_1 \) is also the convergence rate for \( c \) and \( k \) in levels.

• Observation: We have two equations for the growth rate of capital
  - From the original linearized dynamics:
    \[ \frac{\dot{k}}{k} = \beta \cdot \ln\left(\frac{k}{k^*}\right) - \frac{c^*}{k^*} \ln\left(\frac{c}{c^*}\right) \]
  - From the solution:
    \[ \frac{\dot{k}}{k} = \mu_1 \cdot \ln\left(\frac{k}{k^*}\right) \]
    \[ \Rightarrow \quad \ln\left(\frac{c}{c^*}\right) = \frac{k^*}{c^*} (\beta - \mu_1) \cdot \ln\left(\frac{k}{k^*}\right) \]
  - Result: Linear relationship between capital- and consumption-deviations from the steady state.
  - Valid for all \( t \). Apply for \( t=0 \) \( \Rightarrow \) Obtain optimal initial consumption.
Optimal Initial Consumption

- Finding: \[ \ln \left( \frac{c(0)}{c^*} \right) = \frac{k^*}{c^*} (\beta - \mu_1) \cdot \ln \left( \frac{k(0)}{k^*} \right) \]

  - Note that \( \beta - \mu_1 = \mu_2 > \beta > 0 \) is the positive eigenvalue of the characteristic matrix.

  \[ \Rightarrow \ln \left( \frac{c}{c^*} \right) = \frac{k^*}{c^*} \mu_2 \cdot \ln \left( \frac{k}{k^*} \right). \]

- Implied relation in levels = Slope of the saddle path:

  \[ c(0) - c^* \approx c^* \cdot \ln \left( \frac{c(0)}{c^*} \right) = c^* \cdot \left( \frac{k^*}{c^*} \mu_2 \cdot \ln \left( \frac{k(0)}{k^*} \right) \right) = \mu_2 \cdot [k(0) - k^*] \]

- Implications for the Phase Diagram:

  \[ \mu_2 = \text{Slope of the saddle path}; \text{always positive. [Different in logs: scaled by } k^*/c^*] \]

  \[ \beta = \text{Slope of the c(k) function}; \text{also positive.} \]

  \[ \mu_2 > \beta \text{ proves that the saddle path is steeper than the c(k) function.} \]

- One key determinant of the roots: Elasticity of intertemporal substitution \( \frac{1}{\theta} \) (given \( \beta \))

  \[ \mu_1 = \frac{\beta}{2} - \sqrt{\left( \frac{\beta}{2} \right)^2 + \frac{1}{\theta} (-f''(k^*)c^*)} \text{ is decreasing in } \frac{1}{\theta} \Rightarrow | \mu_1 | \text{ is increasing in } \frac{1}{\theta} \]

  \[ \mu_2 = \frac{\beta}{2} + \sqrt{\left( \frac{\beta}{2} \right)^2 + \frac{1}{\theta} (-f''(k^*)c^*)} \text{ is increasing in } \frac{1}{\theta} \]

- Result: Linkage between substitution and speed of convergence: High EIS means

  1. Willingness to set initial consumption far from the steady state value, steep c(k)
  2. Fast convergence: Capital stock and consumption adjust quickly.
The Role of the Production Function

- The roots also depend on the curvature of the production function \( f'' \):

\[
\mu_{1,2} = \frac{\beta}{2} \pm \sqrt{\left(\frac{\beta}{2}\right)^2 + \frac{1}{\theta} \left( -f''(k^*) c^* \right)}
\]

- Write

\[
(-f'' c^*) = \left( \frac{c^*}{k^*} \right) \cdot \left( \frac{-f'' k^*}{f'} \right) \cdot f'(k^*)
\]

1. Fact from microeconomics: \((-f'' k^*) / f' = [1 - \alpha_k(k^*)] / \varepsilon(k^*)\)

   is related to the capital share \( \alpha \) and to the

   - Elasticity of Factor Substitution \( \varepsilon \), which is defined as the elasticity of the capital labor ratio \( k \) with respect to the factor price ratio \( (F_L/F_k) \):
     \[ \varepsilon = \varepsilon(k) = \frac{d \ln(k)}{d \ln[F_L(k,1)/F_K(k,1)]]}. \]

   - Benchmark value: Cobb-Douglas has \( \varepsilon=1 \). Then \((-f'' k^*) / f' = 1 - \alpha \).

2. Note that \[ \frac{c^*}{k^*} = \frac{y^*}{k^*} - (n + g + \delta) \] where \[ \frac{y}{k} = \frac{f(k)}{k} = \frac{f'(k)}{\alpha_k(k)} \Rightarrow \frac{c^*}{k^*} = \frac{f'(k)}{\alpha_k(k)} - (n + g + \delta). \]

3. Note that \[ r^* = f'(k^*) - \delta = \rho + \theta g = n + g + \beta \Rightarrow f'(k^*) = n + g + \delta + \beta \]

- Combine:

\[
\mu_{1,2} = \frac{\beta}{2} \pm \sqrt{\left(\frac{\beta}{2}\right)^2 + \frac{1}{\theta} \cdot \frac{1}{\varepsilon} \cdot (1 - \alpha_k) \cdot f'(k^*) \cdot \frac{c^*}{k^*}}
\]

- Main qualitative insight: \( \mu_2 \) and \( |\mu_1| \) are also increasing functions of \( 1/\varepsilon \).

A low elasticity of factor substitution implies fast adjustment and a steep saddle path.

- Examples:
  \( \theta=2 \); \( \rho=6\% \); \( g=1\% \); \( n+g+\delta=6\% \); \( \alpha=1/3 \); \( \varepsilon=1 \) \( \Rightarrow \beta=6\% \); \( f^*=12\% \); \( c^*/k^*=0.3 \)

  \( \Rightarrow \mu_{1,2} = 6\%/2 \pm \sqrt{1.3\%} = [-0.0836, 0.1436] \). With \( \theta=4 \), \( \rho=4\% \):
  \( \mu_{1,2} \approx 6\%/2 \pm \sqrt{69\%} = [-0.051, 0.1131] \)
Comparison to Solow: How Do Savings Rates Vary?

- Savings rate: \( s(t) = I / Y = 1 - \frac{c(t)}{f(k(t))} \)

Simplify the analysis: Assume Cobb-Douglas production: \( f(k) = k^\alpha \)

- Steady state conditions with Cobb-Douglas production:

\[
f'(k^*) = \rho + \delta + \theta g = \alpha (k^*)^{\alpha-1} \Rightarrow k^* = \left( \frac{\alpha}{\rho + \delta + \theta g} \right)^{1/(1-\alpha)}
\]

\[
c^* = f(k^*) - (n + g + \delta)k^* = (k^*)^\alpha - (n + g + \delta)k^*
\]

\[
= (k^*)^\alpha \cdot [1 - (n + g + \delta)(k^*)^{1-\alpha}] = (k^*)^\alpha \cdot (1 - s^*)
\]

\[
\Rightarrow \text{Steady state savings rate} \quad s^* = (n + \delta + g)(k^*)^{1-\alpha} = \alpha \cdot \left( \frac{n + \delta + g}{\rho + \delta + \theta g} \right) = \alpha \cdot \left( \frac{n + \delta + g}{\beta + n + \delta + g} \right)
\]

- Demonstrates that \( s^* < \alpha \) if and only if \( \beta > 0 \)

   Example: \( \theta = 2; \rho = 6\%; \ g = 1\%; \ n + g + \delta = 6\%; \ \alpha = 1/3 \Rightarrow \beta = 6\%; \ s^* = 1/6 \)

- Phase diagram: \( k^* \) is to the left of the “peak” in the \( c(k) \) relationship: Below the Golden Rule level.

- Savings rates during the convergence process:

  - High \( 1/\theta \) => Negatively sloped \( s(k) \) relationship. Strong response to interest rates => high savings at low capital stocks, declining to \( s^* \); low savings at high capital stocks, rising during convergence.

  - Low \( 1/\theta \) => Positively sloped \( s(k) \): Consumption smoothing motive to set \( c(0) \) near \( c^* \) => near \( c(k) \).
Graphical Analysis of savings rates
[From Barro/Sala-i-Martin, Fig. 2.2-2.3]

- Final Note: Because savings rates converge, economies along a balanced growth path are well described by the Solow model. Likely to fit empirical data.
- But optimal savings rates change in response to disturbances => Different responses than in the Solow model. Again: Empirical fit does not imply correct predictions.
Learning Objectives for Optimal Growth

• Applied: Ability to solve problems
  - Set up a model from a description of the assumptions. Define the optimization problem.
  - Solve optimal control problems by applying a Hamiltonian and the Maximum principle.
  - Derive graphical solutions in the phase diagram. Derive linearized dynamics.
    Problem sets for practice.

• Conceptual:
  - Know the model’s main assumptions, theoretical properties.
  - Know the conditions for balanced growth.
  - Explain why solutions lies on the saddle path (role of boundary conditions).

• Technical skills:
  - Graphing and interpreting phase diagrams.
  - Solving systems of linear differential equations.
  - Linearizing around steady states.