Digression: Discrete-Time Optimization

[For now: As motivation for continuous time. For later: Preview of discrete-time macro.]

- Consider optimal consumption and capital accumulation problem over T periods:
  - Preferences: $$U = \sum_{t=1}^{T} \beta^{t-1} u(c_t) = u(c_1) + \beta u(c_2) + \ldots + \beta^{T-1} u(c_T)$$
  - Budget equations: $y_t = f(k_t) = c_t + [k_{t+1} - (1-\delta) k_t]$ [Simplify the example: A=L=1]
  - Initial condition: Take $k_1 > 0$ as given.
  - Finite horizon T => Capital is useless after period T => Terminal condition $k_{T+1} = 0$.
  - Choice variables: $k_2, \ldots, k_T$ and $c_1, \ldots, c_T$. Finite number.
- Optimization: Maximize utility subject to the budget equations.
  - Define T Lagrange multipliers λ_t, t=1,…,T, for the T budget equations.
  - Standard Lagrangian expression:
  $$L = \sum_{t=1}^{T} \beta^{t-1} u(c_t) + \sum_{t=1}^{T} \lambda_t \{ f(k_t) - c_t + (1-\delta) \cdot k_t - k_{t+1} \}$$
    where $k_{t+1} = 0$ for t=T; and $k_1 > 0$ is given for t=1.
  - Optimality conditions:
    (i) $\partial L / \partial c_t = 0$ for t=1,…,T; (ii) $\partial L / \partial k_t = 0$ for t=2,…,T; (iii) satisfy the budget equations.
  - Claim: These conditions are analogous to the conditions of the Maximum principle.

Discrete-time Optimality Conditions

- Repeat: $L = \sum_{t=1}^{T} \beta^{t-1} u(c_t) + \sum_{t=1}^{T} \lambda_t \{ f(k_t) - c_t + (1-\delta) \cdot k_t - k_{t+1} \}$

  i. Differentiate with respect to c_t (for t=1,…,T):

  $$\frac{\partial L}{\partial c_t} = \beta^{t-1} u'(c_t) - \lambda_t = 0 \quad \Rightarrow \quad \lambda_t = \beta^{t-1} u'(c_t)$$

  ii. Differentiate with respect to k_t (for t=2,…,T):

  - Note that k_t appears twice: in the period-t constraint and in the period-(t-1) constraint.

  $$\frac{\partial L}{\partial k_t} = \frac{\partial}{\partial k_t} [\lambda_t \{ f(k_t) - c_t + (1-\delta) \cdot k_t - k_{t+1} \}] + \frac{\partial}{\partial k_t} [\lambda_{t-1} \{ f(k_{t-1}) - c_{t-1} + (1-\delta) \cdot k_{t-1} - k_t \}]$$

  $$= \lambda_t \{ f'(k_t) + (1-\delta) \} - \lambda_{t-1} = 0$$

  - Write as: $\lambda_t - \lambda_{t-1} = -\lambda_t \cdot [f'(k_t) - \delta]$

  iii. Differentiate with respect to the multipliers λ_t (for t=1,…,T):

  $$\frac{\partial L}{\partial \lambda_t} = f(k_t) - c_t + (1-\delta) \cdot k_t - k_{t+1} = 0 \quad \Rightarrow \quad k_{t+1} - k_t = f(k_t) - \delta \cdot k_t - c_t$$

- Three parts – each analogous to the Maximum Principle:

  (i) Optimal consumption => Marginal utility = Shadow value.

  (ii) Optimal capital => Change in multiplier proportional to return on capital.

  (iii) Optimal multiplier => Recovers the budget equations

Important insight: Discrete-time Euler Equations

[Result worth remembering for later.]

- Derivation: combine $\lambda_t = \beta^{t-1} u'(c_t) \Rightarrow u'(c_t) = \lambda_t / \beta^{t-1}$

  and $\lambda_{t-1} = \lambda_t \{f'(k_t) + (1-\delta)\} = \lambda_t \{1 + f'(k_t) - \delta\} = \lambda_t \cdot (1 + r_t)$

  - Apply to any periods t and t+1:

$$u'(c_t) = \lambda_t / \beta^{t-1} = (1 + r_{t+1}) \cdot \lambda_{t+1} / \beta^{t-1} = (1 + r_{t+1}) \cdot \beta \cdot \lambda_{t+1} / \beta^{t} = (1 + r_{t+1}) \cdot \beta u'(c_{t+1})$$

- Result: $u'(c_t) = (1 + r_{t+1}) \cdot \beta u'(c_{t+1})$

  or $u'(c_t) = [1 + f'(k_{t+1}) - \delta] \cdot \beta u'(c_{t+1})$

=> Marginal utility now = Discounted marginal utility next period * (1 + interest rate)

where (1 + interest rate) = gross return on capital = 1+ MPK – depreciation.

- Equivalent: $\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_{t+1} = 1 + f'(k_{t+1}) - \delta$

=> Marginal rate of substitution now vs. next period = Gross return on capital.

Intuition: Slope of indifference curve = Slope of budget line.

- Equivalent: $\frac{u'(c_t)}{u'(c_{t+1})} = \beta (1 + r_{t+1})$

Marginal utilities declining over time iff $\beta(1 + r_{t+1}) > 1$ ⇔ Consumption increasing.

=> Consumption growth is positive if and only if $(1 + r_{t+1}) > 1/\beta$.