Digression: Discrete-Time Optimization

[For now: As motivation for continuous time. For later: Preview of discrete-time macro.]

- Consider optimal consumption and capital accumulation problem over T periods:
  - Preferences: \( U = \sum_{t=1}^{T} \beta^{t-1} u(c_t) = u(c_1) + \beta u(c_2) + \ldots + \beta^{T-1} u(c_T) \)
  - Budget equations: \( y_t = f(k_t) = c_t + [k_{t+1} - (1-\delta)k_t] \) [Simplify: A=L=1]
  - Initial condition: Take \( k_1 > 0 \) as given.
  - Finite horizon T => Capital is useless after period T => Terminal condition \( k_{T+1} = 0 \).
  - Choice variables: \( k_2, \ldots, k_T \) and \( c_1, \ldots, c_T \). Finite number.

- Optimization: Maximize utility subject to the budget equations.
  - Define T Lagrange multipliers \( \lambda_t, t=1,\ldots,T \), for the T budget equations.
  - Standard Lagrangian expression:
    \[
    L = \sum_{t=1}^{T} \beta^{t-1} u(c_t) + \sum_{t=1}^{T} \lambda_t \{ f(k_t) - c_t + (1-\delta) \cdot k_t - k_{t+1} \}
    \]
    where \( k_{t+1} = 0 \) for \( t=T \); and \( k_1 > 0 \) is given for \( t=1 \).
  - Optimality conditions:
    (i) \( \partial L / \partial c_t = 0 \) for \( t=1,\ldots,T \); (ii) \( \partial L / \partial k_t = 0 \) for \( t=2,\ldots,T \); (iii) satisfy the budget equations.
  - Claim: These conditions are analogous to the conditions of the Maximum principle.
Discrete-time Optimality Conditions

• Repeat:  \[ L = \sum_{t=1}^{T} \beta^{t-1} u(c_t) + \sum_{t=1}^{T} \lambda_t \{ f(k_t) - c_t + (1 - \delta) \cdot k_t - k_{t+1} \} \]

  i. Differentiate with respect to \( c_t \) (for \( t=1,\ldots,T \)):
  \[ \frac{\partial L}{\partial c_t} = \beta^{t-1} u'(c_t) - \lambda_t = 0 \Rightarrow \lambda_t = \beta^{t-1} u'(c_t) \]

  ii. Differentiate with respect to \( k_t \) (for \( t=2,\ldots,T \)):
  - Note that \( k_t \) appears twice: in the period-\( t \) constraint and in the period-(t-1) constraint.
  \[ \frac{\partial L}{\partial k_t} = \frac{\partial}{\partial \beta^t \{ \lambda_t \{ f(k_t) - c_t + (1 - \delta) \cdot k_t - k_{t+1} \} \} \] + \frac{\partial}{\partial \beta^{t-1} \{ \lambda_{t-1} \{ f(k_{t-1}) - c_{t-1} + (1 - \delta) \cdot k_{t-1} - k_t \} \] \]
  \[ = \lambda_t \{ f'(k_t) + (1 - \delta) \} - \lambda_{t-1} = 0 \]
  - Write as: \( \lambda_t - \lambda_{t-1} = -\lambda_t \cdot [f'(k_t) - \delta] \)

  iii. Differentiate with respect to the multipliers \( \lambda_t \) (for \( t=1,\ldots,T \)):
  \[ \frac{\partial L}{\partial \lambda_t} = f(k_t) - c_t + (1 - \delta) \cdot k_t - k_{t+1} = 0 \Rightarrow k_{t+1} - k_t = f(k_t) - \delta \cdot k_t - c_t \]

  • Three parts – each analogous to the Maximum Principle:
    (i) Optimal consumption ⇒ Marginal utility = Shadow value.
    (ii) Optimal capital ⇒ Change in multiplier proportional to return on capital.
    (iii) Optimal multiplier ⇒ Recovers the budget equations

• Limiting case \( T \rightarrow \infty \) is analogous; terminal condition is \( \lambda_T k_T \rightarrow 0 \).
**Important insight: Discrete-time Euler Equations**

<Result worth remembering for later.>

- Derivation: combine \( \lambda_t = \beta^{t-1} u'(c_t) \Rightarrow u'(c_t) = \frac{\lambda_t}{\beta^{t-1}} \)
  
  and \( \lambda_{t-1} = \lambda_t \{ f'(k_t) + (1- \delta) \} = \lambda_t \{ 1 + f'(k_t) - \delta \} = \lambda_t \cdot (1 + r_t) \)
  
- Apply to any periods \( t \) and \( t+1 \):
  
  \[
  u'(c_t) = \frac{\lambda_t}{\beta^{t-1}} = (1 + r_{t+1}) \cdot \frac{\lambda_{t+1}}{\beta^{t-1}} = (1 + r_{t+1}) \cdot \beta \cdot \frac{\lambda_{t+1}}{\beta^t} = (1 + r_{t+1}) \cdot \beta u'(c_{t+1})
  \]

- Result:
  
  \[
  u'(c_t) = (1 + r_{t+1}) \cdot \beta u'(c_{t+1})
  \]
  
  or \( u'(c_t) = [1 + f'(k_{t+1}) - \delta] \cdot \beta u'(c_{t+1}) \)

=> Marginal utility now = Discounted marginal utility next period \( \times (1 + \text{interest rate}) \)

where \( (1 + \text{interest rate}) = \text{gross return on capital} = 1 + \text{MPK} - \text{depreciation}. \)

- Equivalent:
  
  \[
  \frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_{t+1} = 1 + f'(k_{t+1}) - \delta
  \]

=> Marginal rate of substitution now vs. next period = Gross return on capital.

   Intuition: Slope of indifference curve = Slope of budget line.

- Equivalent:
  
  \[
  \frac{u'(c_t)}{u'(c_{t+1})} = \beta (1 + r_{t+1})
  \]

  Marginal utilities declining over time iff \( \beta(1 + r_{t+1}) > 1 \Leftrightarrow \text{Consumption increasing}. \)

=> Consumption growth is positive if and only if \( (1 + r_{t+1}) > \frac{1}{\beta} \).
Dynamic Programming Approach

- Approach is convenient in discrete-time infinite horizon problems
- Idea of a value function: Maximum value of $U = \sum_{t=1}^{\infty} \beta^{t-1} u(c_t)$ depends on initial capital $k = k_1$.
  
  \[ V(k) = \max \{ \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \mid k_1 = k \} = \text{maximum utility for any given } k. \]

  \[ V(k_1) = \max \{ u(c_1) + \beta \sum_{t=2}^{\infty} \beta^{t-2} u(c_t) \} = \max \{ u(c_1) + \beta V(k_2) \} \]

  Defines the Bellman equation: task is to find a function that satisfies this recursion.

- Theory: under general conditions, iterating on the Bellman equation (starting with an arbitrary function) yields a sequence of functions that converges to $V$.

- Note: optimal capital investment reduces to a univariate problem: for given $k_t$ at any time $t$, maximize $u[f(k_t) + (1 - \delta) - k_{t+1}] + \beta V(k_{t+1})$ by choice of $k_{t+1}$
  
  \[ V(k_t) = \max \{ u(c_t) + \beta \sum_{t=2}^{\infty} \beta^{t-2} u(c_t) \} = \max \{ u(c_t) + \beta V(k_2) \} \]

  => Solutions define an optimal policy function $k_{t+1} = K(k_t)$; optimal $c_t$ follows.