Part 4: Overlapping Generations Models

• Basic Version: Each birth-cohort lives for two periods, young and old age.
  - Individuals are identical except for their date of birth => Each generation has a representative agent.
  - Young individuals earn labor income, consume, and save. Savings are invested in capital and earn a return.
  - Old individuals consume their investment earnings (extension: also work).

=> Key issue: **How much does the young generation save?**

• Comparison to representative agent models:
  - Technical shift: trade easier individual problem for aggregation issues.
  - New issues: Intergenerational redistribution, dynamic inefficiency…

• Flexible framework: OG can be extended to many periods, more choices, more heterogeneity.

• Agenda here:
  1. Basic model – dynamics of the capital stock.
  2. Government policy: Debt, taxation with income effects, social security.

    [Real spending withdraws resources as in representative agent models—nothing new, hence ignored.]
  3. Altruism and bequests: The dynastic model as limiting case.
  4. Dynamic inefficiency: excessive savings cannot be ruled out.

    Phenomena in inefficient economies: Bubbles, Ponzi schemes, fiat money as store of value.

• Graphical analysis with 3 diagrams: individual optimization; credit market; dynamics of capital.
A. The Basic Model (see Romer ch.2B)

• **Preferences:** Individuals born in period \( t \) maximize: \( u(C_{1t}) + \beta \cdot u(C_{2t+1}) \)
  - Preferences \( u \): increasing, declining marginal utility.
  - Romer’s notation: Time preference \( \rho \), discount factor \( \beta = 1/(1+\rho) \), power utility.

• **Constraints:** Labor income when young & working = \( W_t \). [Exogenous labor supply.]
  - Savings = Assets \( a_t \). Invested at an interest rate \( r_{t+1} \). [Exogenous to the individual.]
    Intuition: If one period ~ generation ~ 25-30 years, then \( 1+r \sim 300\% \).
  - Budget equation for workers: \( C_{1t} + a_t = W_t \)
  - Budget equation for retirees: \( C_{2t+1} = (1 + r_{t+1}) \cdot a_t \)

\( \Rightarrow \) **Intertemporal budget constraint:** \( W_t = C_{1t} + \frac{1}{1+r_{t+1}} \cdot C_{2t+1} \)

• **Optimality condition:** \( u'(C_{1t}) = \beta \cdot u'(C_{2t+1}) \cdot (1 + r_{t+1}) = \frac{1+r_{t+1}}{1+\rho} \cdot u'(C_{2t+1}) \)
  - **Diagram #1:** Indifference curves & budget line. (As in two-period model.)
  - Lower consumption in retirement if \( \rho > r_{t+1} \).
  - Implicit solution for assets: \( u'(W_t - a_t) = \beta \cdot (1 + r_{t+1}) \cdot u'((1 + r_{t+1}) \cdot a_t) \)

\( \Rightarrow \) **Savings function:** \( a_t = a(W_t, r_{t+1}) \). Savings rate \( s_t = a_t / W_t = s(W_t, r_{t+1}) \)
  - Savings function is increasing in \( W \). Interest rate has an ambiguous effect.
  - **Diagram #2** (Credit market): Savings = supply of credit = function of the interest rate.
Example #1 (Logarithmic utility): \[ u(c) = \ln(c) \]

FOC: \[ \frac{1}{C_{1t}} = (1 + r_{t+1}) \cdot \frac{\beta}{C_{2t+1}} = (1 + r_{t+1}) \cdot \frac{\beta}{(1 + r_{t+1})a_t} = \frac{\beta}{a_t} \]

and \[ W_t - a_t = C_{1t} = a_t / \beta \quad \Rightarrow \quad a_t = \frac{\beta}{1 + \beta} \cdot W_t \]

• Conclude: Constant savings rate \[ a_t / W_t = \frac{\beta}{1 + \beta} \]

Example #2 (Power utility) \[ u(C) = \frac{C^{1-\theta}}{1-\theta} \]

FOC: \[ C_{1t}^{-\theta} = \beta \cdot (1 + r_{t+1}) \cdot C_{2t+1}^{-\theta} \quad \Rightarrow \quad C_{2t+1} = [\beta \cdot (1 + r_{t+1})]^{1/\theta} \cdot C_{1t} \]

\[ \Rightarrow W_t = C_{1t} + \frac{1}{1 + r_{t+1}} C_{2t+1} = C_{1t} + \beta^{1/\theta} \cdot (1 + r_{t+1})^{1/\theta-1} \cdot C_{1t} \]

\[ \Rightarrow C_{1t} = \frac{1}{1 + \beta^{1/\theta} \cdot (1 + r_{t+1})^{1/\theta-1}} \cdot W_t, \quad a_t = (1 - \frac{1}{1 + \beta^{1/\theta} \cdot (1 + r_{t+1})^{1/\theta-1}}) \cdot W_t \]

and \[ s_t = s(r_{t+1}) = 1 - \frac{1}{1 + \beta^{1/\theta} \cdot (1 + r_{t+1})^{1/\theta-1}}. \text{ So } a_t = s(r_{t+1}) \cdot W_t. \]

• Conclude:

1. For log and power utility, consumption and assets are proportional to income.

2. The sign of the interest rate effect depends on the elasticity of intertemporal substitution \(1/\theta\):

   If \(1/\theta > 1\), then \(s(r)\) is increasing. If \(1/\theta < 1\), then \(s(r)\) is decreasing.

3. Savings functions can be increasing or decreasing; position depends on \(W\).
Production:
- Labor supply = Young generation:  \( L_t = L_{t-1} \cdot (1 + n) \)
- Output:  \( Y_t = F(K_t, A_t L_t) - \delta K_t \) or  \( y_t = f(k_t) = F(k_t, 1) - \delta k_t \)
  
  [Romer assumes \( \delta = 0 \). Here subsume depreciation into \( f() \) to keep the notation comparable.]

- Firms maximize profits \( \Rightarrow r_t = f'(k_t) \) and  \( w_t = W_t / A_t = f(k_t) - k_t \cdot f'(k_t) \)
  
  where \( w_t \) = wage in efficiency units. Each worker supplies \( A_t \) efficiency unit
- **Credit market diagram:** Demand for credit \( k_t = (f'')^{-1}(r_t) \) is negatively sloped.

**Equilibrium Condition:** Savings by the young = credit demand by firms.
- Note: Multiple solutions possible if savings function has negative slope.

- Condition:
  \[
  K_{t+1} = L_t \cdot a(W_t, r_{t+1}) = L_t \cdot W_t \cdot s(W_t, r_{t+1}) \\
  \Rightarrow k_{t+1} = \frac{L_t}{L_{t+1}} \frac{A_t}{A_{t+1}} \frac{W_t}{A_t} s(W_t, r_{t+1}) = \frac{s(w_t A_t, r_{t+1})}{(1+n)(1+g)} W_t
  \]
- Note: No balanced growth if the savings rate depends on the level of wages.
  
  \( \Rightarrow \) From now on assume either no growth (A constant) or log/power utility.

- Verify Walras’ law: Capital market equilibrium implies that demand for goods equal supply:

  Demand for goods:  \( L_t \cdot C_{1t} + L_{t-1} \cdot C_{2t} + (K_{t+1} - K_t) = \) consumption + gross investment

  \( L_t \cdot C_{1t} + L_{t-1} \cdot C_{2t} + K_{t+1} - K_t = L_t \cdot (W_t - a_t) + L_{t-1} \cdot (1 + r_t) \cdot a_{t-1} + K_{t+1} - K_t \\
  = [L_t \cdot W_t + L_{t-1} \cdot a_{t-1} \cdot r_t] + [K_{t+1} - L_t \cdot a_t] + [L_{t-1} \cdot a_{t-1} - K_t] = Y_t \)
**Capital Accumulation**

- Evaluate the savings function at the equilibrium factor prices: defines a correspondence between $k_t$ and $k_{t+1}$:

$$k_{t+1} = \frac{s(A_t) w_t + r_{t+1})}{(1+n)(1+g)} W_t = \frac{s[A_t(f(k_t) - k_t f'(k_t)), f''(k_{t+1})]}{(1+n)(1+g)} [f(k_t) - k_t f'(k_t)]$$

(*)

- Note that the savings rate depends on $k_{t+1}$; relationship is not necessarily a function. If (*) has a unique solution for $k_{t+1}$ for any $k_t$, the relationship defines an implicit function, denoted $k_{t+1} = K(k_t)$.

- Interpret the wage as (labor share) * (output):

$$w(k_t) = f(k_t) - k_t f'(k_t) = \alpha_L(k_t) \cdot f(k_t)$$

where

$$\alpha_L(k_t) = \frac{f(k_t) - k_t f'(k_t)}{f(k_t)} = 1 - \frac{k_t f'(k_t)}{f(k_t)}$$

is the labor share in output.

- Interpret $K_{t+1}/Y_t = s(\cdot) \cdot \alpha_L(\cdot)$ as aggregate savings rate = (savings rate of workers) * (labor share).

- Conclude: the dynamics of capital depends on the shape of the savings function, on the labor share, the shape of the production function, and on the growth factors $n$ and $g$.

- **Diagram #3**: Graph of $k_{t+1}$ against $k_t$.

  - Steady states require $k_{t+1} = k_t = k^* = K(k^*)$. In the graph: intersections of $k_{t+1} = K(k_t)$ with $45^\circ$-line.

  - Intuition for multiple steady states: $k_{t+1}$ may be sensitive to changes in $k_t$, either because of backward-bending savings function or variable labor share.

- Graphical illustrations: *Romer Fig.2.13*
FIGURE 2.13 Various possibilities for the relationship between $k_t$ and $k_{t+1}$
Properties of the Capital-Accumulation Correspondence

- If \( K(k_t) \) a function, the derivative \( K'(k_t) \) follows from the implicit function theorem,

Given: \( k_{t+1} = K(k_t) = \frac{s[\Lambda_t w(k_t), f''(k_{t+1})]}{(1+n)(1+g)} \cdot w(k_t) \), where

\[
w(k_t) = \alpha_L(k_t) \cdot f(k_t) \text{ has derivative } w'(k_t) = \alpha_L'(k_t) \cdot f(k_t) + \alpha_L(k_t) \cdot f'(k_t).
\]

- Total differential: \((1+n)\cdot(1+g)dk_{t+1} = s(\cdot) \cdot w'(k_t)dk_t + w_t \cdot s_w \cdot A_t w'(k_t)dk_t + w_t \cdot s_r f''(k_{t+1})dk_{t+1}

\[
\Rightarrow \quad [(1+n)\cdot(1+g) - w_t s_r f''] dk_{t+1} = [s(\cdot) + A_t w_t s_w] \cdot w'(k_t)dk_t
\]

- Note: If \((1+n)\cdot(1+g) - w_t s_r f'' > 0\), then \( K(k_t) \) is single-valued \( \Rightarrow k_{t+1} = K(k_t) \) is a function.

- Intuition from credit market: Demand \( k = (f')^{-1}(r) \) has slope \( \frac{1}{f''} < 0 \). Supply has slope \( \frac{s_r w}{(1+n)(1+g)} \).

\[
\Rightarrow \text{For given } k_0, \quad \frac{1}{f''} < \frac{s_r w}{(1+n)(1+g)} \text{ ensures unique market clearing interest rate.}
\]

- Sufficient condition: \( s_r \geq 0 \Rightarrow \text{credit supply has non-negative slope, not backward-bending.}\)

- ASSUME in the following that \((1+n)\cdot(1+g) - w_t s_r f'' > 0\), so \( K(k_t) \) is a function.

- Function \( k_{t+1} = K(k_t) \) defines a difference equation.

- Difference equation has a unique steady state if \( |K'(k_t)| < 1 \) at the 45°-line. Equivalent to \( \left| \frac{d \ln(k_{t+1})}{d \ln(k_t)} \right| < 1 \).

- Difference equation is \textbf{stable}, if \( |K'(k_t)| < 1 \) for all \( k_t = \text{Implies convergence to a steady state from any starting value } k_0 \). Convergence is \textbf{monotone} if \( K'(k_t) \geq 0 \).
Analysis

- Slope of $K(k_t)$, assuming $(1 + n) \cdot (1 + g) - w_t s_r f'' > 0$: write

\[
\frac{d k_{t+1}}{d k_t} = \frac{1}{(1+n)(1+g)+(-f'')w_t s_r} \left(1 + s_w \frac{A_t w_t}{s'}\right) \cdot s'w'(k_t)
\]

- Note that

\[
\begin{align*}
\frac{s'(k_t)}{s(k_t)} &= k_t w'(k_t) \\
&= k_t \left[ \frac{f'(k_t)}{f(k_t)} + \frac{\alpha_L'(k_t)}{\alpha_L(k_t)} \right]
\end{align*}
\]

\[
\Rightarrow \quad \frac{d k_{t+1}}{d k_t} = \frac{1}{1 + \frac{(-f'')w_t}{(1+n)(1+g)} s_r} \left[1 + s_w \frac{A_t w_t}{s'} \right] \left[ \frac{f'(k_t)}{f(k_t)} + \frac{\alpha_L'(k_t) k_t}{\alpha_L(k_t)} \right] \frac{k_{t+1}}{k_t}
\]

- Intuition: $f'(k)$ is steep at low $k \Rightarrow$ segment with $d k_{t+1}/d k_t > 1$ difficult to rule out. But $K(k_t)$ must “bend down” to approach 45°-line $\Rightarrow$ cuts 45°-line from above $\Rightarrow$ suggests $|K'(k_t)| < 1$ at the 45°-line.

- Formal argument: Show stability of $\ln(k)$, which is a monotone transformation. Write

\[
\frac{d \ln(k_{t+1})}{d \ln(k_t)} = \frac{k_t}{k_{t+1}} \frac{d k_{t+1}}{d k_t} = \frac{1}{1 + \frac{(-f'')w_t}{(1+n)(1+g)} s_r} \left(1 + \frac{A_t w_t s_w}{s'} (\alpha_K + \frac{k_t \alpha_L'}{\alpha_L}) \right)
\]

- Sufficient condition for stability and monotonicity: Cobb Douglas production (so $\alpha_L' = 0$) and either log-utility or power utility with $\theta \leq 1$ (so $s_r \geq 0, s_w = 0$). Then $0 < \frac{d \ln(k_{t+1})}{d \ln(k_t)} < \alpha_K$.

- Sufficient condition with homothetic preferences (if $s_w = 0$): $0 \leq \alpha_K + \frac{k_t \alpha_L'}{\alpha_L} < 1 + \frac{(-f'')w_t}{(1+n)(1+g)} s_r$. 
Example: Log-utility & Cobb-Douglas production

- Log-utility implies constant \( s = \beta / (1 + \beta) \)
- Cobb-Douglas implies constant \( \alpha_L = 1 - \alpha \) and \( w(k_t) = (1 - \alpha) \cdot k^\alpha \)

- Dynamics:
  \[
  k_{t+1} = \frac{\beta / (1 + \beta)}{(1 + n)(1 + g)} (1 - \alpha) \cdot k_t^\alpha
  \]

  \[
  => \quad \frac{dk_{t+1}}{dk_t} = \frac{\beta / (1 + \beta)}{(1 + n)(1 + g)} (1 - \alpha) \cdot \alpha \cdot k_t^{\alpha-1}
  \]

  - Note that \( dk_{t+1} / dk_t > 1 \) for small \( k \). But \( 0 < \frac{d\ln(k_{t+1})}{d\ln(k_t)} = \alpha < 1 \)

  \[
  => \text{Monotone convergence from any } k > 0.
  \]

- Steady state:
  \[
  k^* = \frac{\beta / (1 + \beta)}{(1 + n)(1 + g)} (1 - \alpha) \cdot (k^*)^\alpha
  \]

  \[
  => \quad k^* = \left[ \frac{\beta / (1 + \beta)}{(1 + n)(1 + g)} (1 - \alpha) \right]^{\frac{1}{1-\alpha}}
  \]

- Compare to Solow model, where \( k^* = \left[ \frac{s_{agg}}{n + g + \delta} \right]^{\frac{1}{1-\alpha}} \)

  - Matches for \( s_{agg} = s \cdot (1 - \alpha) \), \( \delta = 1 \), and \( (1 + n) \cdot (1 + g) = 1 + n + g \)

  \[
  => \text{Another interpretation of the Solow model}
  \]

- Question to think about: What goes wrong with the analogy if \( \delta < 1 \)?