Note on Linear Differential Equations

Econ 204A - Prof. Bohn[1]

We will have to work with differential equations throughout this course. Differential equations – and their discrete-time analogs: difference equations – are economically interesting because they link levels to changes; we are often interested in linking the current situation or status of an economy to changes that we are trying to predict or understand.

This note is about *linear* differential equations, linear relationships between a variable and its time-derivative. The general specification is

(1) $$\frac{dy(t)}{dt} = \gamma(t) \cdot y(t) + x(t)$$

The variable $y = y(t)$ is a function of time, to be determined. The derivative dy/dt is also a function of time, variously denoted $\frac{d}{dt} y(t) = \frac{dy}{dt}(t) = y'(t) = \dot{y}(t)$. The terms γ(t) and x(t) are known functions of time, called the coefficients or forcing variables. Solving a differential equation means writing y(t) as function of time that does not involve the derivative.

Categorizations:

1. Fixed and variable coefficients: The solutions simplify if γ and x are constants, also called fixed coefficients. *Caution*: Writers often suppress time-dependence when working with differential equations. That is, equation (1) is often written more compactly as

(1’) $$\dot{y} = \gamma \cdot y + x$$

Readers are expected to determine from the context if γ and/or x are constant or if they should be treated as variables.

2. Homogenous and non-homogenous equations: A differential equation is homogenous if there is no additive part, i.e., if $x(t) \equiv 0$ for all t. Otherwise it is non-homogenous. *Useful fact:* The solution to a non-homogenous equation is always the solution to the homogeneous part—omitting the x-part—plus a function of time. In some applications, the non-homogenous part is not economically interesting and one can simply examine the homogenous part.

3. General and special solutions: Equation (1) is typically solved by a parametric class of functions, which is called the *general solution* to (1). To pin down a unique y-function—a *specific solution*—we

[1] Disclaimer and request: The Note is meant as a concise summary and reference, not a self-contained text. You should also read Barro/Sala-i-Martin’s appendix and (if you need it) consult a suitable math-for-economists text (e.g., Chiang). This is a new Note that may still contain typos. I hope it’s useful nonetheless. For the benefit of other students, *please* let me know if you find errors! Also, please let me know about items that are unclear or unhelpful.

need additional pieces of information called *boundary conditions*. For linear differential equations, the general solution is indexed by a single parameter (denoted A below). A single boundary condition is sufficient to determine this parameter.

For this note, I will assume that the boundary condition is the y-value at a particular time $t=t_0$. That is, $y(t_0) = y_0$ is assumed known and provides the boundary condition. In applications, we often normalize $t_0=0$. I show general solutions if they are more compact than specific solutions.

4. Stable solutions, explosive solutions, and degenerate cases: Depending on the parameters, solutions will display different types of asymptotic behavior. Important characteristics are stability—convergence to a finite limit value as $t \to \infty$, and explosive behavior—divergence to infinity. Sometimes special cases deserve attention.

Main Cases:

Four cases are worth distinguishing. I will state all solutions and then explain how they are obtained.

1a. Homogenous equation with fixed coefficient γ: $\dot{y}(t) = \gamma \cdot y(t)$.

General solution: $y(t) = A \cdot e^{\gamma t}$

Specific solution: $y(t) = y(t_0) \cdot e^{\gamma(t-t_0)}$

Alternate notations are $y(t) = y_0 \cdot e^{\gamma(t-t_0)}$ or $y(t) = y(t_0) \cdot \exp\{\gamma(t-t_0)\}$.

The general solution highlights the exponential shape of the function, which is the key feature. If $\gamma<0$. the solution converges to zero from any starting value. If $\gamma>0$, the solution diverges to plus infinity from any positive starting value and to minus infinity from any negative starting value. If $\gamma=0$, $y(t)=y(t_0)$ is constant; and if $y(t_0)=0$, $y(t)=0$ for all γ.

1b. Homogenous equation with variable coefficient γ(t): $\dot{y}(t) = \gamma(t) \cdot y(t)$.

General solution: $y(t) = A \cdot e^{\int \gamma(s) ds}$

Specific solution: $y(t) = y(t_0) \cdot e^{\int_{t_0}^{t} \gamma(s) ds}$

The indefinite integrate in the general solution highlights the key feature, the integral over the forcing function in the exponent. In the specific solution, the γ-function is integrated over time from the boundary time t_0 to time t. (The integration index s has no significance.)

2a. Non-homogenous equation with fixed coefficients γ and x: $\dot{y}(t) = \gamma \cdot y(t) + x$.

General solution: $y(t) = A \cdot e^{\gamma t} - \frac{x}{\gamma}$ for $\gamma \neq 0$;

Specific solution: $y(t) = y(t_0) \cdot e^{\gamma(t-t_0)} - \frac{x}{\gamma} \cdot [1 - e^{\gamma(t-t_0)}]$ for $\gamma \neq 0$;

The solutions are explosive if $\gamma>0$ and stable if $\gamma<0$, as in the homogenous case. If $\gamma<0$, a nonzero x-value implies a "displacement" of the limit value away from zero. Note that $(-x/\gamma)>0$ if $x>0$ and $\gamma<0$, so the limit has the same sign as x. For the intuition, note that setting $dy/dt=0$ yields $0=\gamma y+x$, which is solved by y=–x/γ. Thus, the limit value is the only value at which y can stabilize. If γ>0, the constant x/γ is negligible in the limit. If $\gamma = 0$, the equation reduces to $\dot{y}(t) = x$. Because x defines a constant slope, y must be linear: $y(t) = A + x \cdot t$ in general, and $y(t) = y(t_0) + x \cdot (t - t_0)$ specifically.

2b. <u>Non-homogenous equation with variable coefficients</u> γ(t) and x(t): $\dot{y}(t) = \gamma(t) \cdot y(t) + x(t)$.

Solution: $$y(t) = y_0 \cdot e^{\int_{t_0}^{t} \gamma(s)ds} + \int_{t_0}^{t} x(v) \cdot e^{\int_{v}^{t} \gamma(s)ds} dv$$

The solution is specific for specific values $y_0 = y(t_0)$, and general if one treats $y_0 = A$ as undetermined parameter (equally lengthy, hence not written separately). Note the double integration over the forcing variables: The x-value at time v is first multiplied by an exponential that involves an integral over γ. Then the "weighted" x-values are integrated over time.

For an intuition, think about y as an account balance, γ as the interest rate, and x as a series of deposits. The starting balance y(t_0) earns compound interest at the rate γ. If γ is constant and nothing is deposited, the balance grows exponentially, as case 1a. If γ is constant, the growth rate varies, as in 1b. If deposits are added, each deposit earns interest from the time of deposit until the final time t; this explains why x(v) is multiplied by an integral from time v to t in case 2b. Because the deposits add up over time, one must further integrate over the x-values.

Proofs:

<u>Proof #1</u>: An elementary way to prove solutions to a differential equation is to differentiate the proposed solution with respect to time. For example, consider 2b: One finds

$$\begin{aligned}
\tfrac{d}{dt} y(t) &= y_0 \cdot \tfrac{d}{dt}[e^{\int_{t_0}^{t} \gamma(s)ds}] + \tfrac{d}{dt}[\int_{t_0}^{t} x(v) \cdot e^{\int_{v}^{t} \gamma(s)ds} dv] \\
&= y_0 \cdot \gamma(t) e^{\int_{t_0}^{t} \gamma(s)ds} + \int_{t_0}^{t} x(v) \cdot \tfrac{d}{dt}[e^{\int_{v}^{t} \gamma(s)ds}] dv + \left[x(v) e^{\int_{v}^{t} \gamma(s)ds} \right]_{v=t} \\
&= y_0 \cdot \gamma(t) e^{\int_{t_0}^{t} \gamma(s)ds} + \int_{t_0}^{t} x(v) \cdot \gamma(t) e^{\int_{v}^{t} \gamma(s)ds} dv + x(t) \\
&= \gamma(t) \cdot [y_0 \cdot e^{\int_{t_0}^{t} \gamma(s)ds} + \int_{t_0}^{t} x(v) e^{\int_{v}^{t} \gamma(s)ds} dv] + x(t) = \gamma(t) \cdot y(t) + x(t)
\end{aligned}$$

Analogous arguments apply to cases 1a, 1b, and 2a, which are specials cases of 2b.

A proof by verification does not provide much intuition. One may ask therefore how the solutions can be derived more constructively.

Proof #2: For a more intuitive proof, note that a *change* divided by a *level* has the dimension of a *growth rate*. In the homogenous fixed-coefficient case, the differential equation (1) can be written as

$$\tfrac{dy(t)}{dt}/y(t)=\gamma$$

It simply says that y(t) has a constant growth rate. Also note that growth rates are log-differences:

$$\tfrac{d}{dt}\ln y(t)=\tfrac{1}{y(t)}\tfrac{d}{dt}y(t),$$

using the chain rule of differentiation. Because logs and exponentials are inverse operations, it should not surprise that the solutions involve exponentials. To prove case 1a, define $z(t)=\ln(y(t))$ and write (1) as $\gamma=\tfrac{d}{dt}\ln y(t)=dz/dt$. A function with a constant derivative γ must be linear with slope γ. (Or more generally, integrating over a derivative recovers the function. Integrating over a constant yields a linear function.) Thus, z has the form $z(t)=a+\gamma\cdot t$ for some constant a. Taking exponentials, $y(t)=e^{z(t)}=e^{a+\gamma\cdot t}=A\cdot e^{\gamma\cdot t}$ with $A=e^{a}$. This provides a constructive proof for case 1a. The other cases are generalizations, as follows.

Proof #3: An intuition for a general proof derives from the product rule of differentiation applied to functions of the form $z(t)=y(t)\cdot e^{\Lambda(t)}$. The derivative of z has the format

(2) $$z'(t)=y'(t)\cdot e^{\Lambda(t)}+y(t)\cdot\Lambda'(t)\cdot e^{\Lambda(t)}=[y'(t)+\Lambda'(t)\cdot y(t)]\cdot e^{\Lambda(t)}.$$

The key insight is that the derivative of z involves a linear combination of the function y and its derivative y'. The fundamental theorem of calculus says that integrating over a derivative recovers the function. The term $e^{\Lambda(t)}$ is called the *integration factor*. The key idea is that, if one can regroup the y and y' terms from the differential equation (1) in the form (2), with z' equal to something known, one can recover the z-function through integration. That's the main intuition—everything else is tedious algebra.

Turning to the algebra, compare (1) and (2): To obtain the same linear combination of y and y', we need $\Lambda'(t)=-\gamma(t)$. Integrating over Λ', one finds that the integration factor must have the form $e^{\Lambda(t)}=\exp\{-\int\gamma(v)dv\}$. Because we have a t_0-dated boundary information, this is written most conveniently as $e^{\Lambda(t)}=\exp\{-\int_{t_0}^{t}\gamma(v)dv+a_0\}$ with some integration constant a_0 and integration starting at time t_0. Writing (1) as $y'(t)-\gamma(t)\cdot y(t)=x(t)$ and multiplying by $e^{\Lambda(t)}$, we find

$$z'(t)=[y'(t)-\gamma(t)\cdot y(t)]\cdot e^{\Lambda(t)}=x(t)\cdot e^{\Lambda(t)}.$$

This can be integrated to obtain $z(t)=y(t)\cdot e^{\Lambda(t)}=\int_{t_0}^{t}x(v)\cdot e^{\Lambda(v)}dv+a_1$ for some integration constant a_1. Then divide by $e^{\Lambda(t)}$ to obtain $y(t)=a_1\cdot e^{-\Lambda(t)}+\int_0^t x(v)\cdot e^{\Lambda(v)}dv\cdot e^{-\Lambda(t)}$. Finally, note that $e^{-\Lambda(t)}=\exp\{\int_{t_0}^{t}\gamma(v)dv-a_0\}$ and $e^{\Lambda(v)-\Lambda(t)}=\exp\{\int_v^t\gamma(s)ds\}$ are identical to the corresponding terms in 2b. Comparing like terms, one may interpret $y_0=a_1\cdot e^{-a_0}$ as combination of integration constants. This proves case 2b. The others case are specializations.

Optional Exercise: Prove cases 1a, 1b, and 2a directly by defining suitable integration factors and going through the same steps as above.

Final note on variable coefficients: γ and x are both fixed in case 2a and both variable in case 2b. An intermediate case would be a *constant* γ *and variable x*: $\dot{y}(t) = \gamma \cdot y(t) + x(t)$. In this case, the solution to 2b would simplify to $y(t) = y_0 \cdot e^{\gamma(t-t_0)} + \int_{t_0}^{t} x(v) \cdot e^{\gamma(t-v)} dv$. If x is also constant, it can be taken out of the integral. Then for $\gamma \neq 0$, the second integral reduces to $\int_{t_0}^{t} x(v) e^{\gamma(t-v)} dv = x \cdot \int_{t_0}^{t} e^{\gamma(t-v)} dv = \frac{x}{\gamma}[e^{\gamma(t-t_0)} - 1]$, This integration explains why the ratio x/γ appears in case 2a. The 'A' in the general solution 2a corresponds to $y_0 + x/\gamma$. For $\gamma = 0$, the intermediate case reduces to $y(t) = y_0 + \int_{t_0}^{t} x(v) dv$, or a linear function for constant x.