

Collection of Practice Problems Econ 204A

Henning Bohn *

In previous years, students have often asked me about practice problems in addition to the problem sets. Here is a collection. A subset will be assigned for the weekly problem sets. I hope the others are useful for practice.

Request: Please tell me about errors or ambiguities. Many of the questions below are old exam problems. As I am updating the course over the years, the notation, references, and sequencing has changed, which means that some old problems may have lost educational value without me noticing immediately. So if a problem seems obscure to you, please let me know. (Your incentive: your problem set might shrink if you convince me a problem is unclear.)

Part 1:

1. Consider the two-period consumption model: Individuals have initial assets A , earn interest r on assets, and earn wage income (w_1, w_2) . They maximize utility $U = u(c_1) + \beta u(c_2)$.
 - a. Assume $u(c) = \ln(c)$.
 - i. Solve for optimal consumption and period-1 asset holdings as functions of wage income, the interest rate, and the time-discount factor β . Discuss under what conditions higher interest rates reduce consumption. [Discuss means: Interpret the solution. Conditions may be exact, or necessary, or sufficient. Hint: Distinguish cases with $w_2 = 0$ vs. $w_2 > 0$.]
 - ii. Show that the dependence of period-1 consumption on (y_1^d, y_2^d) can be expressed in terms of permanent income.
 - b. Assume $u(c) = \frac{1}{1-\gamma} c^{1-\gamma}$ where $\gamma > 0, \gamma \neq 1$. Do the same as in (a). In the discussion, identify which results apply for all γ , and which ones only for γ greater or less than one.
2. Consider the permanent income model with n periods. Assume $\beta = 1/(1+r)$. Assume initial assets are zero ($A=0$). To answer questions about taxes, interpret w as wage income net of taxes. This question is intended to give you a quantitative perspective. (Hint: Use a spreadsheet.)
 - a. Assume the real interest rate is 3% per year. Assume the time horizon is $n=50$ years (about the life expectancy of males age 25 or females age 30).

Determine the marginal propensity to consume ($MPC=dc/dw$) from (i) a one-year wage increase; (ii) an increase in wages that lasts 5 years; (iii) and a wage increase that last for 35 years

* Provided online for use by UCSB students. (C) Copyright 2009 Henning Bohn

[intuition: until about retirement]; (iv) a tax reduction for one year followed by a tax increase of the same size in the next year.

Discuss: How do the results compare? Do you find them surprising? Realistic? Why or why not?

- b. Assume the real interest rate is again 3% per year. Determine the impact of a one-year tax cut for consumers with alternative planning horizons of, respectively, $n=1$ year; $n=2$ years; $n=10$ years; $n=50$ years; the limiting case of $n=\infty$.

Discuss: How do the results compare? Do you find them surprising? Realistic? Why or why not?

Part 2:

1. Romer, Advanced Macroeconomics ch.1, problems 1.3, 1.4, 1.6.

[Note: Most problems in Romer are instructive. Do more if you have time.]

2. Suppose an economy has a production function $y_t = k_t^\alpha$ (in efficiency units), a savings rate $s > 0$, a population growth rate n , and a depreciation rate of δ .

- a. Suppose $\alpha=1/3$, $s=0.2$, $n=1\%$, $g=1\%$, $\delta=4\%$. What are the steady state value of the capital-labor ratio, output per efficiency unit, and consumption per efficiency unit?

For parts b-e, assume the economy starts in the steady state derived in (a).

- b. Suppose an earthquake destroys 10% of the capital stock. Sketch the time paths of the capital-labor ratio and of per-capita consumption. Compare to (a).

[To clarify: Per-capita means per actual worker, not in efficiency units.]

- c. Suppose savings are increased to $s=0.22$. What is the impact effect on consumption? What are the new steady state values of the capital-labor ratio, output per efficiency unit, and consumption per efficiency unit? Sketch the time paths of the capital-labor ratio and of per-capita consumption. Compare to (a).

- d. Suppose population growth is increased to $n=2\%$. What are the new steady state values of the capital labor ratio, output per efficiency unit, and consumption per efficiency unit? Sketch the time paths of the capital-labor ratio and of per-capita consumption. Compare to (a).

- e. Suppose productivity growth is increased to $g=2\%$. What are the new steady state values of the capital labor ratio, output per efficiency unit, and consumption per efficiency unit? Sketch the time paths of the capital-labor ratio and of per-capita consumption. Compare to (a).

3. Suppose an economy has a production function $y_t = 3 \cdot k_t^{0.5}$ and a savings rate of 30%, a population growth rate of 5%, and a depreciation rate of 10%. Productivity is constant.

- a. What are the steady state value of the capital-labor ratio, output per worker, and consumption per worker?

- b. How do the values in (a) change if the savings rate is 40%?

c. How do the values in (a) change with 8% population growth (still 30% savings)?

4. This question is about the continuous-time Solow growth model with human capital. Assume production is Cobb-Douglas:

$$Y = K^\alpha H^\beta (AL)^{1-\alpha-\beta}$$

Capital K has share $\alpha > 0$, human capital H has share β . A is total factor productivity (TFP). L is raw labor. A fixed share $s_k > 0$ of output is invested in physical capital, a share s_h is invested in human capital. Assume zero depreciation. The raw labor force grows at a fixed rate $n > 0$. TFP grows at a fixed rate $g > 0$.

For parts (a) and (b), assume $\beta = s_h = 0$, as in the standard Solow model.

- a. Derive the steady state capital stock per efficiency unit of labor. Derive steady state output per efficiency unit of labor. Derive a formula for per-capita output along the steady state growth path. [Derive means: Write down the model and show your work; no credit for memorized formulas.]
- b. Suppose the saving rate s_k is increased marginally. Derive the quantitative impact on steady state output per efficiency unit of labor. Describe the transition process to the new steady state. What can you say about the growth rate of per-capita output during the transition?

For parts (c)-(e), assume $\beta > 0$ and $s_h > 0$.

- c. Derive the steady state capital stock, steady state human capital stock, and steady state output, all per efficiency unit of labor.
- d. Derive the quantitative impact on steady state output per efficiency unit of labor of a marginal increase in changes (i) s_k and (ii) s_h . Describe under what conditions s_k has a greater, smaller or equal output effect as s_h . [Hint: You are seeking a simple condition relating savings rates to production parameters.]
- e. Consider the limiting case $\alpha + \beta \rightarrow 1$. Explain why and in what sense growth becomes endogenous in the limit. Derive the economy's growth rate and show that it is positive even if $g=0$.

5. This question is about economic growth with an exogenous savings rate (s); notation as in Romer unless noted. Assume production is Cobb-Douglas:

$$Y = K^\alpha (AL_Y)^{1-\alpha}$$

with capital share $0 < \alpha < 1$ and depreciation δ . Population L grows at rate n.

For parts (a)-(c): Assume total factor productivity A grows at an exogenous rate $g_A = g$; the entire population works in production, $L_Y = L$. At time zero, the economy is on the balanced growth path.

- a. Derive the steady state capital stock per efficiency unit of labor. Derive steady state output per efficiency unit. Derive a formula for per-capita output along the steady state growth path. [Derive means: Show your work; no credit for memorized formulas.]

- b. Suppose at time $t=0$, a genial discovery makes productivity A jump up by 100%. Productivity growth then continues at the original rate g .
- Determine how the change affects the steady state output and capital stock per efficiency unit of labor. Graph the time path.
 - Determine how the change affects the output and capital stock per worker. Graph the time path.
- c. Suppose at time $t=0$, the productivity growth rate increases from $g_A = g$ to $g_A = \hat{g}$. [But there is no jump in productivity levels.]
- Determine how the change affects the steady state capital stock per efficiency unit of labor. Graph the time path.
 - Determine how the change affects the output and capital stock per worker. Graph the time path. Is everyone better off with higher productivity growth?
- For parts (d)-(e): Assume productivity growth depends on research labor $L_A = s_R \cdot L$ and on existing productivity: $\dot{A} = \gamma \cdot L_A \cdot A^\phi$, where $\gamma > 0$, $0 < \phi < 1$, and $0 < s_R < 1$ are parameters. Production labor is $L_Y = (1 - s_R) \cdot L$.
- Derive the steady state growth rate of productivity. Show that per-capita output grows at the same rate.
 - Suppose at time $t=0$, the share of research labor is increased to $\hat{s}_R > s_R$. Graph the time paths of productivity and of per-capita output. Is everyone better off with more research labor? Explain.

Part 3:

1. Romer, Advanced Macroeconomics, ch.2, problems 2.1, 2.2, 2.4.

2. Consider an economy with utility-maximizing, infinitely-lived households; notation as in Romer unless noted. Assume population (L) and the productivity index (A=1) are constant. Preferences are

$$\max \int_{t=0}^T e^{-\beta t} u(c(t)) dt,$$

where u is increasing and concave and $0 < \beta < 1$. Capital accumulation is described by

$$\dot{k} = f(k) - \delta k - c.$$

- Set up the Hamiltonian, apply the Maximum Principle, and derive a pair of equations that describe the dynamics of capital and consumption.
- Suppose a hurricane destroys half the capital stock. Describe graphically how consumption and the capital stock will adjust over time.
- Starting in a steady state with positive depreciation, suppose depreciation is suddenly eliminated, $\delta = 0$. Describe graphically how consumption and the capital stock will adjust over time. Is there a finite steady state?

3. Consider an individual with constant rate of time preference τ who faces a given path of future real wages $w(t)$ and a constant interest rate r . The individual supplies one unit of work and maximizes the utility function

$$\max \int_{t=0}^T e^{-\tau t} u(c(t)) dt$$

where $u(c) = -\exp\{-\alpha \cdot c\}$ and $\alpha > 0$ is a constant parameter (This is known as the exponential utility function with constant absolute risk aversion α .) Initial asset holdings a_0 are given. Asset holdings must be non-negative at time T.

- Set up the Hamiltonian problem, apply the Maximum Principle, and derive an optimality condition for dc/dt .
- Derive the intertemporal budget constraint and solve for the optimal consumption path $c(t)$.
[Hint: A differential equation of the form “ $dx/dt = \text{constant}$ ” has the linear solution $x(t) = x(0) + t \cdot \text{constant}$.]

4. Romer, Advanced Macroeconomics ch.2, problems 2.6, 2.9.

[Most problems in Romer are instructive. Do more if you have time.]

5. Consider a continuous-time representative agent economy with constant population $L=1$ and constant productivity. Individuals have preferences

$$U = \int_{t=0}^{\infty} [e^{-\rho t} u(C(t))] dt,$$

where u is increasing and concave. The stock of capital for the aggregate economy evolves according to the equation $dK/dt = I - \delta K$. Government spending is a function of time $G(t)$. The National Income identity is given by $Y = F(K, L) = C + I + G$, where F satisfies the Inada conditions. Taxes are lump sum and equal to government expenditure in every period. The representative agent expects that government expenditure will be constant over time.

- a. Set up the representative agent's problem. Apply the maximum principle.
- b. Derive the phase diagram and show that K converges to a steady state value K^* from any initial value K_0 . Explain why C is an increasing function of K during the convergence process.
- c. Suppose at time $t=0$, the government announces a tax-financed increase in government spending starting at $t=1$ and ending at $t=2$. Assuming the economy was at the steady state, show the dynamics of C , K and the interest rate r . Illustrate these dynamics in the phase diagram and by sketching time-series charts for C , K , and r .

6. Consider an economy with population N that grows at the rate n . Output Y is produced with physical capital K , labor N (each person supplying one unit), and human capital H according to the production function $Y = K^\alpha \cdot H^\gamma \cdot N^{1-\alpha-\gamma}$. Physical and human capital both depreciate at a common rate $\delta > 0$.

- a. Suppose individuals invest fixed fractions s_K and s_H of output in physical and human capital, respectively; $s_K, s_H > 0$, $s_K + s_H < 1$. Initial values K_0 and H_0 are given. Does the economy have a steady state in level and/or in per-capita values? Compute the steady state output and explain how it depends on the model parameters.
- b. Suppose $n=0$. Individuals maximize $\int_{t=0}^{\infty} e^{-\rho t} u(c(t)) dt$ where $u(c)$ is increasing and concave.

Consumption equals output minus the investment in K and H . Derive the first order conditions for optimality. (For simplicity, ignore non-negativity constraints on gross investment.) Compute the steady state output level and comment on how its determinants differ from the determinants in (a). Can you determine the optimal steady state savings rates? What can you say about the relationship between $K(t)$ and $H(t)$?

7. Assume you are the dictator of a small country that just lost a war. Since you lost, you are required to pay war reparations of a given amount X . Also, there is a trade embargo so that imports and exports (other than reparations) are zero. Assume production $F(K, L)$ has constant returns, satisfies the Inada conditions, and that depreciation is zero. The continuous-time GDP identity is

$$Y = F(K, L) = C \cdot L + X + dK/dt$$

- a. Assume the population L_t grows at rate $n > 0$. Derive the budget equation linking per-capita consumption C and the per-capita capital stock $k = K/L$.

- b. Assume that the reparations and the trade embargo are in place forever. You want to maximize the utility of the representative agent, $\int_{t=0}^{\infty} e^{-\theta t} u(C(t)) dt$ subject to the constraint derived in (a). K_0 is given.
- (i) Set up the Hamiltonian and state the necessary conditions for optimality. What are the conditions for a steady state?
- (ii) Use a phase diagram to explain the dynamics of the model.
- c. Assume now that the war reparations are known to end at a finite date T ($X=0$ for $t>T$). The trade embargo remains in place forever. Use the phase diagram to explain how the path of C and k differs from (b).

8. Consider an individual facing the following utility maximization problem:

$$\max \int_{t=0}^{\infty} e^{-\theta t} u(c(t), l(t)) dt$$

where c is consumption and l is leisure; $U_c > 0$, $U_l > 0$. The individual is endowed with 1 unit of time and works: $n(t) = 1 - l(t)$. The per-worker capital stock evolves according to

$$\frac{dk}{dt} = k^\alpha n^{1-\alpha} - c - g,$$

where g is a constant level of per-capita government spending. There is no depreciation and the population is constant.

- a. Assume initially that n and l are constant at some level, $0 < n < 1$, $0 < l < 1$. Set up the Hamiltonian and find the optimality conditions.
- b. Solve for the steady state values c^* and k^* . How do changes in g affect c^* and k^* ? How do exogenous changes in l affect c^* and k^* ?
- c. Now let $n(t)$ be another choice variable. Set up the Hamiltonian and find the optimality conditions.
- d. Assume $U(c, l) = \ln(c) + \ln(l)$. Solve for the steady state values c^* , k^* , and n^* . How do changes in g affect c^* , k^* , and n^* ?