Calculation of a Population Externality

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Abstract

Harford (1998) shows that when people generate externalities, a birth also generates an externality and efficiency requires a Pigovian tax/subsidy on having children. The size of the population externality is important for studying policy. We calculate the size of the externality in a specific case. We consider the possibility that greenhouse gas emissions turn out to be a serious problem and assume government reacts by restricting emissions. This converts the population externality to a form that can be evaluated from existing data without relying on detailed assumptions about the environment. Calculated population externalities are large enough to matter for policy.

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Harford (1998) shows that when people generate externalities and fertility is endogenous, a birth also generates an externality and efficiency requires a child tax or subsidy equal to the externality. Because it equals the corrective tax/subsidy, the size of the population externality is useful for studying population policy. We consider a simple case: if greenhouse gas emissions turn out to be a problem[1] and more people means more emissions, then population would also be a problem. The size of the population externality measures the size of the induced population problem.

To evaluate the population externality under the maintained hypothesis that greenhouse gas emissions are a problem without having to model the physical environment, we assume government responds by imposing a cap that holds emissions constant and prevents environmental damage.[2] This converts the underlying emissions-damage externality into a fixed-common-property externality that can be calculated from existing data: under the cap, a birth means more people split the same total amount of emissions so living standards are lower. The approach is policy-relevant in that government would likely respond with restrictions if emissions turn out to be a serious problem, and allows numerical evaluation of how the population externality changes with the restrictiveness of the cap.[3]

We use a balanced-growth setting in which output is produced from labor and greenhouse gas emissions. To avoid assuming away an important part of the emis-

[1]see e.g. IPCC (2008).

[2]Modelling the physical environment would be necessary to compute an optimal cap but is unnecessary here, would distract from the focus on fertility, and would introduce sensitivity to assumptions about environmental science. For these reasons and because governments may not be able to set optimal policies, we do not require the cap to be optimal. The optimal cap is treated in an appendix. Note that if the cap does not eliminate damage, the induced population externality may be greater than we calculate.

[3]If a cap is imposed but emissions cause *no* external damage, then the analysis here implies that a birth still reduces living standards and induces a population externality. That is, it would be a bad idea to restrict emissions if emissions are not a problem.

sions problem, factor productivities grow exogenously at constant rates as in Solow-type models.[4] Population and hence labor are determined endogenously by dynastic households with Barro-Becker (1988, 1989) preferences.

We divide time into two stylized eras and assume the transition between eras occurs at an instant. First is an "no-cap era" in which the possibility of global warming is unrecognized, emissions are unrestricted, and population and emissions grow exponentially. Second is a "cap era" in which people are concerned about emissions and government imposes a cap that eventually binds. The cap era is Malthusian in that population growth reduces incomes which in turn restrains population growth. Living standards may still rise over time in the cap era, however, because of exogenous productivity growth.

We calibrate the model to the no-cap era and compute for the cap era both the population sequence households would choose without any population policy to balance the population externality (the *natural population*) and the population sequence that maximizes the utility of the representative household taking account of the externality (the *optimal population*).[5] From these, we calculate the sequence of Pigovian taxes on having a child (*optimal child taxes*).

The negative population externality here does not imply that the aggregate of all population externalities would necessarily be negative if emissions are restricted, be-

[4]Kelly and Kolstad (2001), argue that the assumptions in integrated assessment models that productivity and population growth fall exogenously to zero are empirically unrealistic and reduce the emissions problem. Consistent with this, they find the welfare cost of a marginal birth is small if there is no cap and productivity and population growth fall exogenously to zero.

[5]Golosov et al (2007) study the broad question of how to treat children who may never be born in Paretian welfare analysis when fertility is endogenous. Here, we are concerned with the positive actions of currently living agents who determine actual fertility and who may collectively act to control externalities that arise among themselves through fertility decisions. Parents here care about progeny and progeny's consumptions; these underlie actual fertility choice.

cause positive population externalities arise from other sources that we do not model. Two examples are scale or spillover effects in human capital formation (e.g. Kremer, 1993; Jones, 1999), and the pay-as-you-go externality under which the lifetime social security taxes paid by an additional person benefit all elderly in the previous generation instead of just the parent (Schoonbroodt and Tertilt, 2011). Evaluating the aggregate of all positive and negative population externalities would be a substantial task.

Nonetheless, many current tax, welfare, and schooling policies subsidize children. The results here suggest that if emissions turn out to be a serious enough problem so restrictions are imposed, then there may also be pressure to make population policy less pronatalist.

We model the essential details of a population externality under a cap in section I. Section II briefly describes population in the no-cap era. Section III describes the natural population in the cap era. Section IV describes the optimal population and child taxes in the cap era. In section V, we extend the model by adding time costs of children, a more general technology, and exogenously-given productivity growth, which are important for calibrations. Section VI contains calibrations. Proofs are in the appendix.

I. SETTING

Denote the time-t adult population by N_t and aggregate labor by $L_t = l_t N_t$ where l_t is per-capita labor. (Per-capita means per-adult.) A representative firm produces output Y_t under perfect competition from labor and E_t units of greenhouse gas emissions according to $Y_t = F(L_t, E_t)(1 - \delta_t)$, where F captures the productivities of labor and emissions as inputs and δ_t is the share of output lost (damage) from global warming at t. We add exogenously-growing factor productivity in section V.

We assume F has constant returns so $F(L, E) = Lf(e)$, where $e_t \equiv E_t/L_t$ is the *emissions ratio* and f is output per unit of labor. We also assume $f(0) = 0$ for now but relax this in section V. The marginal product of emissions must be driven to zero if emissions are unrestricted so we assume there is a positive value $e^+ < \infty$ at which $f'(e^+) = 0$ with $f'(e_t) > 0$ and $f''(e_t) < 0$ for $0 \leq e_t < e^+$, as in figure 1. Thus the emissions ratio is constant at e^+ in the no-cap era and input growth is balanced: $E = e^+L$ grows at the same rate as L. In the cap era with a binding cap, on the other hand, total emissions are constant so growth in population and hence labor causes e to decline, driving down output per unit of labor as indicated by the arrows in the figure. This is how a binding cap introduces a Malthusian force.

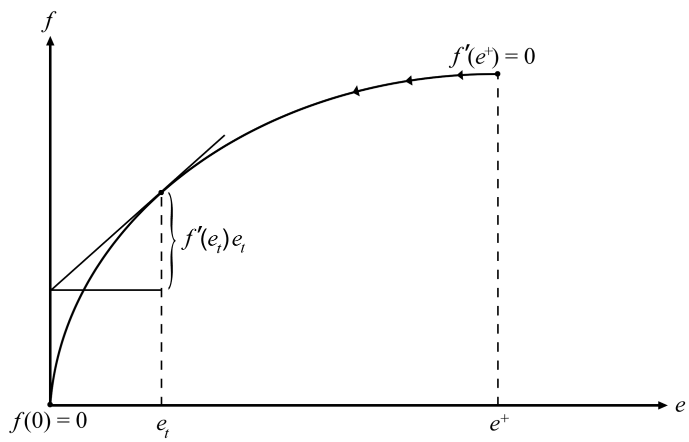

Figure 1. Input Distortion, and Geometry of the Real Population Externality

As explained in the introduction, we assume government imposes a constant emissions cap that prevents environmental damage, so $\delta_t = 0$ for all t.[6] Then

$$Y_t = F(L_t, E_t) = L_t f(e_t) \tag{1}$$

[6]In the appendix, we consider an expanded model in which damage results from a known environmental process and government sets the optimal emission policy in each period. We show that the steady-state externality in the expanded setting equals the steady-state externality in the setting here with a fixed cap and zero damage, if the cap here is set equal to the steady-state cap in the expanded model and damage δ is subsumed in f.

in both eras.

The cap era begins in a period indexed $t = 0$. In the cap era, government creates and auctions $\hat{E}$ permits each period, each permit allowing one unit of emissions in the period so $E_t \leq \hat{E}$ for $t \geq 0$. Treating permits as valid for a single period fits U.S. legislative proposals that state that permits are not property rights and that nothing restricts future government from terminating or limiting an emission allowance.[7]

In the market for emissions permits, the government is the supplier and the representative firm is the demander. The firm maximizes profits $L_t f(e_t) - p_t E_t - w_t L_t$, where p_t is the price of permits and w_t is the wage. The first-order conditions are $p_t = f'(e_t)$ and $w_t = w(e_t) \equiv f(e_t) - e_t f'(e_t)$.

The quantity of permits demanded at $p_t = 0$ is $e^+ L_t$. If $e^+ L_t < \hat{E}$, the cap does not bind, $p_t = 0$, and $e_t = e^+$. If $e^+ L_t > \hat{E}$, the cap binds, $p_t > 0$, and $e_t = \hat{E}/L_t < e^+$. (If $e^+ L_t = \hat{E}$, then $p_t = 0$ and $e_t = e^+$.) Compactly, the emissions ratio is $e(L_t) \equiv \min(e^+, \hat{E}/L_t)$ for any L_t.

Because $L_t = l_t N_t$, a cap means the emissions ratio depends on population:

$$e_t = e(l_t N_t) = \min(e^+, \frac{\hat{E}}{l_t N_t}). \tag{2}$$

To highlight the dependence, we assume until section V that per-capita labor is fixed and normalized to one. Then labor equals population, per-capita emissions equal $e(N_t)$, and per-capita output equals $f(e(N_t))$.

Population in turn depends on fertility $n_t \geq 0$. A large number of representative dynastic households each contain a single adult who chooses n_t continuously to maximize utility. When all households choose n_t, the population growth factor is also n_t,

[7] We show in the appendix that the population externality would be internalized by parents if current government can and does establish iron-clad permanent *private* property rights to the *public* revenue stream from all permits to $t = \infty$. This is an interesting idea but is difficult to achieve in practice. Permanence fails if government later changes policy to expropriate the "permanent" rights. The U.S. legislative proposals effectively acknowledge that this cannot be prevented.

that is, $N_{t+1} = n_t N_t$.

Adults trade off own consumption and number of children. An adult's consumption $c_t \geq 0$ is per-capita income y_t less the output cost of having and raising children to adulthood, so $c_t = y_t - \chi n_t$ where χ is the output cost of a child. Maximum feasible fertility is y_t/χ.

We follow Barro and Becker's (1988, 1989) specification of household preferences. A period-t adult's utility U_t is utility u from own consumption plus utility from children:

$$U_t = u(c_t) + \beta(n_t)U_{t+1}, \tag{3}$$

where children are identical and utility from children is the utility of a child times a weight β that depends on the number of children.

We assume power utility with parameter $\theta > 0$:

$$u(c) = \frac{1}{1-\theta}c^{1-\theta}; \tag{4}$$

the power form is needed later to allow for balanced growth. We also assume β is a power function with parameters $b_0 > 0$ and $b > 0$:[8]

$$\beta(n) = b_0 n^{1-b}. \tag{5}$$

In Barro and Becker's original specification, u is positive so $\theta < 1$, and β is increasing and concave, so $b < 1$. These ensure that parent's utility rises at a decreasing rate with the number of children. Jones and Schoonbroodt (2007) and Jones et al (2008) show that parent's utility also rises at a decreasing rate with the number of children if utility is negative so $\theta > 1$, as long as β is decreasing and convex so $b > 1$; they argue this case may better explain historical fertility trends.[9]

[8]Power β has a sensible property: it is equivalent to assuming the utility an adult derives from grandchildren, $\beta(n_t)\beta(n_{t+1})U_{t+2}$, is independent of the number of children—see appendix.

[9]Utility (3) is infinite if discount factors given by β are too great. When $\theta < 1$, we assume $\beta < 1$

From (3) - (5), the contribution of children's consumption to parent's utility is $\beta(n_t)u(c_{t+1}) = \frac{b_0}{\omega(1-b)}(n_t c_{t+1}^{\omega})^{1-b}$ where $\omega \equiv \frac{1-\theta}{1-b}$ is the weight a parent places on per-child consumption relative to the number of children. Equal curvatures ($\theta = b$, so $\omega = 1$) mean the contribution depends on children's aggregate consumption, $n_t c_{t+1}$. We do not rule out ω greater or less than one, but $\omega \approx 1$ may be reasonable if parents view children's aggregate consumption as an economic resource. Values far less than one (parents care little about their children's consumption relative to the number of children), on the other hand, may be difficult to square with small families in which parents devote substantial resources to ensuring children's consumptions.

In choosing fertility, a household takes its income as well as the incomes and fertilities of future generations as given. The latter determine the utility of children. Generically (dropping time subscripts), the household chooses $n \in [0, y/\chi]$ to maximize

$$V(n, y, U) \equiv u(y - \chi n) + \beta(n)U$$

given $y > 0$ and finite U, where $U > 0$ when $\theta < 1$ and $U < 0$ when $\theta > 1$.

The first-order condition balances the costs and benefits of children:[10]

$$V_n(n, y, U) = -u'\chi + \beta' U = 0. \tag{6}$$

The marginal value of children ($\beta' U$) becomes infinite as $n \to 0$ and marginal cost of children ($u'\chi$) becomes infinite as $n \to y/\chi$ (so $c \to 0$). Therefore $V_n \to \infty$ as $n \to 0$ and $V_n(n, y, U) \to -\infty$ as $n \to y/\chi$. Continuity of V_n then implies that for any finite $y > 0$ and finite U, there is a unique optimal fertility strictly between zero and y/χ.

at maximum feasible fertility to ensure $\beta(n) < 1$ for all feasible n, so utility is finite. When $\theta > 1$ and β is decreasing, we assume $b_0 < 1$ to ensure there is an $n^\circ < 1$ at which $\beta(n^\circ) = 1$ with $\beta(n) < 1$ for all $n > n^\circ$. This implies finite utility on paths with constant population.

[10]The second-order condition, $V_{nn} = u''\chi^2 + \beta'' U < 0$, holds by assumptions on primitives.

Income and children's utility drive fertility. The partial elasticity of fertility with respect to income is

$$\varepsilon_{n,y} \equiv \frac{y}{n}\frac{\partial n}{\partial y} = -\frac{yV_{ny}}{nV_{nn}} = \left[\frac{b}{\theta} \cdot \frac{c}{y} + \frac{\chi n}{y}\right]^{-1}, \tag{7}$$

which is positive by assumptions on primitives. This elasticity holds children's utility constant, which makes it conceptually different from many empirically estimated income elasticities.

The sign of β' determines how fertility changes with children's utility, because $\frac{\partial n}{\partial U} = \frac{\beta'}{-V_{nn}}$. When $\theta < 1$ so $\beta' > 0$, fertility rises with U. When $\theta > 1$ so $\beta' < 0$, fertility falls with U.

We close the model by assuming government redistributes revenue from emissions auctions as equal lump sums to households. Per-capita income is therefore wages plus transfers, $TR_t = p_t E_t / N_t = p_t e_t$. From the firm's first-order conditions, per-capita income equals output per unit of labor: $y_t = w_t + p_t e_t = f(e_t)$.

The combination of a permit auction with revenue redistribution to households can represent a range of policies that restrict emissions. The combination is equivalent here to: (*i*) issuing and giving permits to households who then sell them at price p_t; (*ii*) issuing and giving permits to firms owned by households; and (*iii*) imposing a tax on emissions at rate f', which would just hold total emissions to $\hat{E}$, and redistributing the revenue to households.

In sum, the model describes an endogenous population-growth process. In the no-cap era, output per adult is $f(e^+)$, which leads to time paths of fertility and population. When a cap is imposed and population growth is positive, e_t and $f(e_t)$ eventually fall as depicted by the arrows in figure 1. This alters the time path of population. The exact path after a cap is imposed depends on how fertility responds to induced changes in income and children's utility.

II. POPULATION IN THE NO-CAP ERA

A perfect-foresight solution in the no-cap era is a steady state with constant fertility and utility that solves the household's first-order condition. In any steady state, (3) implies

$$U = \frac{u(f(e) - \chi n)}{1 - \beta(n)}. \tag{8}$$

The steady-state relationship between emissions and natural fertility is found by substituting (8) into the first-order condition (6) to eliminate U:

$$S(n, e) \equiv -u'(f(e) - \chi n)\chi + \frac{\beta'(n)}{1 - \beta(n)} u(f(e) - \chi n) = 0. \tag{9}$$

With no cap, the emissions ratio is e^+. Steady-state fertility n^+ is the fertility that solves $S(n^+, e^+) = 0$. Such an n^+ exists and is unique.[11] Steady-state utility U^+ is the value of (8) at (n^+, e^+).

In general, n^+ may be greater or less than one. It is greater than one as long as child costs are not too great a fraction of output. To focus on equilibria in which population and emissions ($E_t = e^+ L_t$) grow so emissions can be a problem, we assume

$$\chi < \phi f(e^+) \text{ where } \phi \equiv 1 \Big/ \left(1 + \frac{(1-\theta)(1-b_0)}{(1-b)b_0}\right) < 1. \tag{10}$$

Equation (10) rearranges to $S(1, e^+) > 0$, which ensures $n^+ > 1$. Population and emissions then grow without bound at constant rate $n^+ - 1 > 0$.

[11] As $n \to f(e^+)/\chi$, $u' \to \infty$, so $S(n, e^+) \to -\infty$. When $\theta < 1$, $\beta' \to \infty$ as $n \to 0$ so $S(n, e^+) \to \infty$. Because S is continuous, $S(n^+, e^+) = 0$ for some $n^+ \in (0, f(e^+)/\chi)$. When $\theta > 1$, $1/(1-\beta(n)) \to \infty$ as $n \to n^\circ$ from above (where $\beta(n^\circ) = 1$) so $S(n, e^+) \to \infty$. Because S is continuous, $S(n^+, e^+) = 0$ for some $n^+ \in (n^\circ, f(e^+)/\chi)$. (For $n < n^\circ$, $\beta(n) > 1$, so $S < 0$.) From (9), $\frac{\partial S}{\partial n} = u''\chi^2 - \frac{\beta' u' \chi}{1-\beta} + \frac{\beta''(1-\beta) - (\beta')^2}{1-\beta} u$, which reduces to $u''\chi^2 + \beta'' u$ at n such that $S = 0$. Because $\beta'' < 0$ and $u > 0$ when $\theta < 1$, and $\beta'' > 0$ and $u < 0$ when $\theta > 1$, $\frac{\partial S}{\partial n} < 0$. Hence S crosses zero only once.

III. NATURAL POPULATION IN THE CAP ERA

Population cannot grow without bound after a cap is imposed because this would eventually drive output per unit of labor $f(e)$ below χ so fertility would fall below replacement. Under a regularity condition described below, the natural population instead converges monotonically after a cap is imposed to a unique steady-state level N_{ss}. (Subscripts ss denote a variable's steady-state value.)

A perfect-foresight path in the cap era satisfies (3) and the household's first-order condition (6) for all t, which can be written as pair of first-order difference equations in $\{U_t, N_t\}_{t\geq 0}$:[12]

$$U_t \;=\; u(f(e(N_t)) - \chi\frac{N_{t+1}}{N_t}) + \beta(\frac{N_{t+1}}{N_t}))U_{t+1}, \tag{11}$$

$$V_n(t) \;\equiv\; \beta'\left(\frac{N_{t+1}}{N_t}\right)U_{t+1} - u'\left(f(e(N_t)) - \frac{N_{t+1}}{N_t}\chi\right)\chi = 0. \tag{12}$$

Steady state is a pair (U_{ss}, N_{ss}) that satisfies (11) and (12) with $U_t = U_{t+1} = U_{ss}$ and $N_t = N_{t+1} = N_{ss}$. The latter implies that steady-state fertility equals replacement: $n_{ss} = 1$. In the cap era with $n_{ss} = 1$ and $e_{ss} = e(N_{ss})$, the steady-state condition is $S(1, e_{ss}) = 0$. We show in the appendix that $S(1, e)$ crosses zero exactly once on $[f^{-1}(\chi), e^+]$ so e_{ss} exists and is unique. Thus $N_{ss} = \hat{E}/e_{ss}$ exists and is unique. Because $S(1, e^+) > 0$, it must be that $e_{ss} < e^+$ so the cap binds in steady state.

If fertility is too sensitive to changes in population then a population increase from t to $t+1$ can reduce fertility so much that population decreases from $t+1$ to $t+2$. To rule out such "overshooting," we restrict the sensitivity of fertility to changes in population by assuming

$$\varepsilon_{n_t, y_t}\left(\frac{f'(e_t)e_t}{f(e_t)}\right) < 1, \tag{13}$$

[12]The system depends on $\hat{E}$ through e, but we suppress the dependence notationally except when considering how alternative values of $\hat{E}$ affect the economy.

at the steady state and at all t.[13] We show in the appendix that the system (11) and (12) is then saddle-path stable and that population converges monotonically to N_{ss} from any initial population $N_0 > 0$.

Fertility along the perfect-foresight natural path (*natural fertility,* η) is a function of population and the level of the cap, $n_t = \eta(N_t \mid \hat{E})$. Because $\hat{E}$ enters the model only through (2) as a determinant of e_t, natural fertility is homogeneous of degree zero: $\eta(N_t \mid \hat{E}) = \eta(\xi N_t \mid \xi \hat{E})$ where $\xi > 0$ is a constant. In words: natural fertility at population N_t under cap $\hat{E}$ equals fertility at population ξN_t under cap $\xi \hat{E}$, because both have the same e_t.

When $\theta < 1$, natural fertility lies below n^+ and falls with N_t to the steady state at N_{ss}, as in figure 2a. There are two forces. A cap has no effect on income as long as

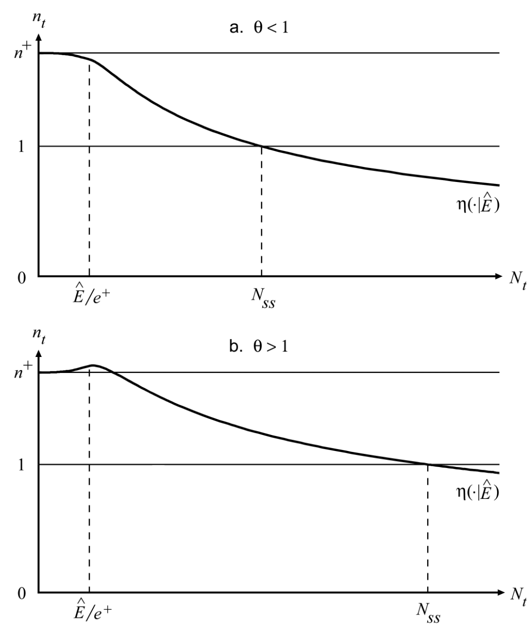

Figure 2. Natural Fertility

[13] In (13), ε_{n_t,y_t} is the partial elasticity of fertility with respect to income (7) and the factor share $f'(e_t)e_t/f(e_t)$ is also the elasticity of income with respect to population. Note that (13) is not very strong. Because $f'(e)e < f(e)$, (13) holds if $\varepsilon_{n,y} \leq 1$, which holds in turn if $b \geq \theta$. On the other hand, if $\varepsilon_{n,y} > 1$ then fertility tends to fall off sharply as a declining emissions ratio reduces income so the factor share remains small, and (13) can still easily hold.

$N_t \leq \hat{E}/e^+$ so the cap does not bind, but reduces income once it binds. Reduced income in turn reduces fertility increasingly as N_t rises above $\hat{E}/e^+$. Because the cap eventually binds and utility is determined recursively, utility is less than U^+ as soon as the cap is imposed, which also acts to reduce fertility for all N_t.

When $\theta > 1$ (figure 2b), natural fertility lies above n^+ and rises as long as the cap does not bind, then peaks and falls below n^+ as N_t becomes closer to N_{ss}. This reflects opposing income and utility effects. As when $\theta < 1$, income declines as N_t rises above $\hat{E}/e^+$ and utility is less than U^+. When $\theta > 1$, however, reduced utility raises fertility. This lifts fertility above n^+ when a cap is imposed and causes fertility to rise with N_t for $N_t \leq \hat{E}/e^+$ and also slightly above $\hat{E}/e^+$. For N_t sufficiently close to N_{ss}, the income effect dominates and fertility lies below n^+. Because fertility first rises when $\theta > 1$, steady-state population tends to be greater than when $\theta < 1$.

Population dynamics after a cap is imposed follow from the natural fertility function. Figure 3 illustrates when $\theta < 1$. To minimize clutter, we choose units so $e^+ = 1$

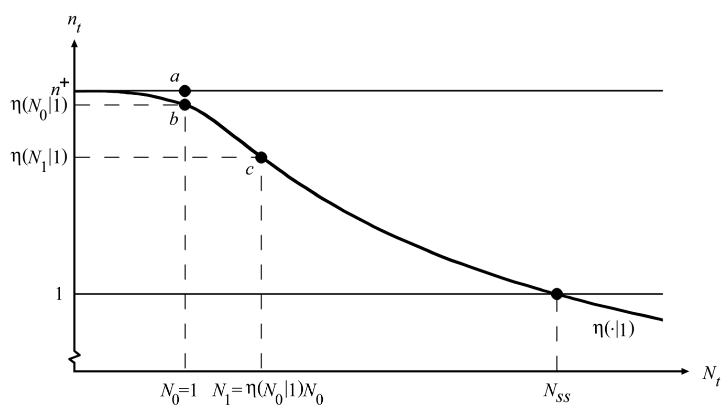

Figure 3. Population Dynamics Under an Emissions Freeze

and $N_0 = 1$. Consider a freeze, meaning a cap at the emissions level that would otherwise occur in the no-cap era at $t = 0$, which is $E_0 = e^+N_0 = 1$. Without a cap, fertility would be n^+ in period 0, at a. When the cap is imposed, fertility instead jumps down to $\eta(N_0 \mid 1) = \eta(1 \mid 1)$, at b. In period 1, the economy is therefore at c

with population $N_1 = \eta(N_0 \mid 1)N_0 = \eta(N_0 \mid 1)$ and fertility $\eta(N_1 \mid 1)$. The economy then iterates down the fertility function and converges to population N_{ss} with fertility $\eta(N_{ss} \mid 1) = 1$.

Natural fertility functions for caps other than a freeze can be derived from fertility for a freeze. A 25-percent cut, for instance, is $\hat{E} = 0.75$. Because η is homogeneous of degree zero, $n_t = \eta(N_t \mid 0.75) = \eta(N_t/0.75 \mid 1)$, that is, fertility at population N_t and cap $\hat{E} = 0.75$ equals fertility at population $N_t/0.75$ and $\hat{E} = 1$. Graphically, this means the fertility function for a 25-percent cut, $\eta(\cdot \mid 0.75)$, is the function for a freeze, $\eta(\cdot \mid 1)$, shifted 25 percent of the distance to the vertical axis, as in figure 4.

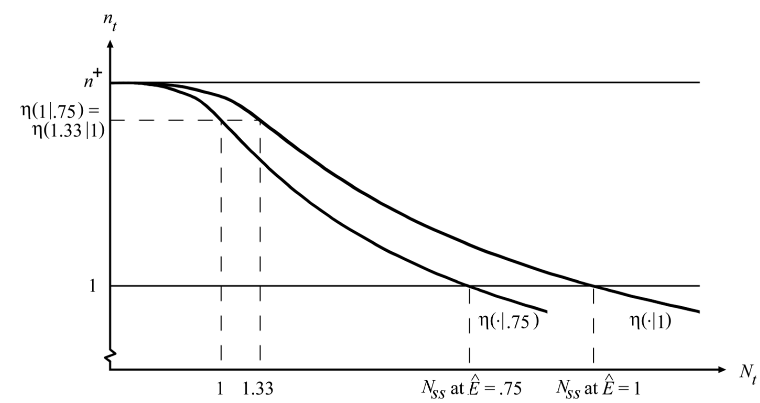

Figure 4. An Emissions Cut Versus a Freeze

(Because $N_{ss} = \hat{E}/e_{ss}$ is proportional to $\hat{E}$, steady-state population given a 25-percent cut is also 0.75 times steady-state population under a freeze.) Similarly, fertility functions for caps that do not immediately bind lie to the right of $\eta(\cdot \mid 1)$.

Any cap eventually binds and leads to the same steady-state emission ratio e_{ss}. The impact effect of imposing a binding cap is to reduce e_0 from e^+ to $\hat{E}/N_0$; subsequent dynamics take e_t the rest of the way to e_{ss}. The greater the value of $\hat{E}$ and hence the lower the impact reduction in e_0, the greater is the adjustment of e_t after period 0.

IV. OPTIMAL POPULATION IN THE CAP ERA

A household is small compared with total population so in maximizing utility, it ignores the external reduction in the emissions ratio and hence in everyone's future income caused by its having a child. We characterize the population sequence that is optimal for living adults; this sequence maximizes the utility of the representative household taking account of the population externality.

The optimal population for arbitrary t and hence given $N_t > 0$ maximizes U_t by choice of future populations $N_{t+1}, N_{t+2}, ...$ The maximization can be written as a dynamic programming problem with Bellman equation

$$V^*(N_t) = \max_{n_{t+1}}\{u(f(e(N_t)) - \chi n_{t+1}) + \beta(n_{t+1})V^*(N_{t+1})\}, \tag{14}$$

where $n_{t+1} = N_{t+1}/N_t$ and where the value function V^* captures the dependence of U_{t+1} on N_{t+1}. This is not a "standard" dynamic programming problem because the discount factor $\beta(n_{t+1})$ is endogenous. Alvarez (1999) shows that solutions can nonetheless be obtained by solving the transformed problem of maximizing $U_t^\circ \equiv \beta(N_t)U_t$, which has the same optimal policy.

In detail, $U_t^\circ \equiv \beta(N_t)[u(c_t)+\beta(n_{t+1})U_{t+1}] = \beta(N_t)u(c_t)+b_0\beta(N_{t+1})U_{t+1} = u^\circ(N_t, N_{t+1})+ b_0U_{t+1}^\circ$ where $u^\circ(N_t, N_{t+1}) \equiv \beta(N_t)u(f(e(N_t)) - \chi N_{t+1}/N_t)$, so the transformed problem has Bellman equation

$$V^\circ(N_t) = \max_{0\leq N_{t+1}\leq \frac{1}{\chi}N_t f(e(N_t))}\{u^\circ(N_t, N_{t+1}) + b_0V^\circ(N_{t+1})\}, \tag{15}$$

with $0 < b_0 < 1$.[14] Solutions to this problem exist for all (θ, b). (Technical claims in this section are proved in the appendix.)

To compare the optimal and natural steady states, let stars denote optimal values. For any (θ, b), steady-state optimal population N_{ss}^* satisfies the optimal steady-state

[14]In footnote 9, we assumed $b_0 < 1$ when $\theta > 1$ and $\beta(n) < 1$ for all feasible n when $\theta < 1$. Using (10), the latter implies $b_0 < 1$.

condition $S^*(\hat{E}/N^*_{ss}) = 0$, where

$$S^*(e) \equiv (1 - b_0)S(1, e) - b_0 u'(f(e) - \chi)f'(e)e.$$

The optimal and natural steady-state conditions thus differ by a term that reflects the population externality. A root $e^*_{ss} = \hat{E}/N^*_{ss}$ that solves $S^*(\hat{E}/N^*_{ss}) = 0$ exists and lies strictly between e_{ss} and e^+.[15] Because $e_{ss} < e^*_{ss}$, the steady-state optimal population N^*_{ss} is less than the steady-state natural population N_{ss}.

If $\omega = \frac{1-\theta}{1-b} \leq 1$, the value function V° is unique, strictly concave, and differentiable, and optimal population is a single-valued continuous function $N_{t+1} \equiv H(N_t \mid \hat{E})$ that also maximizes $V^*(N_t) = V^\circ(N_t)/\beta(N_t)$.[16] The optimal population sequence $\{N^*_t\}_{t\geq 0}$ starting at $t = 0$ from given $N_0 > 0$ is obtained by iterating on H. Moreover, e^*_{ss} is unique and the elasticity condition (13) is sufficient for the optimal population to converge monotonically to N^*_{ss} from any initial $N_0 > 0$. If $\omega > 1$, matters are more complicated but similar results hold if ω is not too great.[17]

Fertility along the optimal path (*optimal fertility,* η^*) follows from the optimal population as $\eta^*(N_t \mid \hat{E}) \equiv H(N_t \mid \hat{E})/N_t$. As with natural fertility, optimal fertility: (i) is homogeneous of degree zero in the sense that $\eta^*(N_t \mid \hat{E}) = \eta^*(N_t/\hat{E} \mid 1)$; ($ii$) jumps when a cap is imposed, from n^+ to $\eta^*(N_0 \mid \hat{E})$; and (iii) approaches $n^*_{ss} = 1$ as population converges to N^*_{ss}.

[15] The details are that S^* is continuous, $S^*(e_{ss}) = -\beta u'f'e < 0$ (because $S(1, e_{ss}) = 0$ and $e_{ss} < 1$), and $S^*(e^+) > 0$ (because $S(1, e^+) > 0$ and $f'(e^+) = 0$).

[16] Applications of the Barro-Becker model commonly assume $\omega \leq 1$ (e.g. Jones and Schoonbroodt, 2007). This helps ensure strict concavity of V° by ensuring that u° is concave. We allow $\omega > 1$.

[17] For instance, suppose (13) holds and $1/\omega > 1 - \varepsilon_{n_t,y_t}\kappa(e)$ for $e \in (f^{-1}(\chi), e^+)$, where $\kappa(e_t) \equiv -\frac{f''(e)e^2}{f(e)} > 0$ measures the contribution from concavity of f. Then N^*_{ss} is unique, the optimal population sequence converges monotonically to N^*_{ss} from any $N_0 > 0$, and V° is strictly concave for $N_t \geq \hat{E}/e^+$ so H is single-valued and V° and V^* are differentiable. If the cap binds when it is imposed ($\hat{E} \leq e^+ N_0$), the optimal population path is also unique. (If $\hat{E} > e^+ N_0$, the optimization (15) may not be concave when $\omega > 1$ so it is difficult to rule out multiple optimal population paths.)

The Population Externality: Comparing Natural and Optimal Populations

We evaluate the population externality using the Bellman equation (14). Because the value function V^* in (14) is differentiable, optimal fertility satisfies:

$$\begin{aligned} V_n^* &\equiv -u'(c_t^*)\chi + \beta'(n_t^*)V^*(N_{t+1}^*) + \beta(n_t^*)N_t^*\frac{dV^*}{dN}(N_{t+1}^*) \\ &= V_n(n_t^*, f(e(N_t^*)), V^*(N_{t+1}^*)) + \beta(n_t^*)N_t^*\frac{dV^*}{dN}(N_{t+1}^*) = 0, \end{aligned} \tag{16}$$

using the definition of V_n in (6). For $(y, U) = (f(e(N_t^*)), V^*(N_{t+1}^*))$, a household would set $V_n(n_t, y, U) = 0$ so optimal and natural paths differ. The term $\beta(n_t^*)N_t^*\frac{dV^*}{dN}(N_{t+1}^*)$ is ignored by households; it measures the population externality in units of parent's utility.

To evaluate $\frac{dV^*}{dN}(N_{t+1}^*)$, the envelope theorem applied to (14) implies:

$$\frac{dV^*}{dN}(N_{t+1}^*) = -u(c_{t+1}^*)f'(e(N_{t+1}^*))\frac{e(N_{t+1}^*)}{N_{t+1}^*} + \beta(n_{t+1}^*)n_{t+1}^*\frac{dV^*}{dN}(N_{t+2}^*). \tag{17}$$

Reapplying (17) iteratively to eliminate successive future derivatives of V^*, the future terms collapse into the discounted sum

$$\frac{dV^*}{dN}(N_{t+1}^*) = -\frac{1}{N_{t+1}^*}\sum_{i=1}^{\infty}\left[\prod_{j=1}^{i-1}\beta(n_{t+j}^*)\right]u'(c_{t+i}^*)f'(e(N_{t+i}^*))e(N_{t+i}^*). \tag{18}$$

The infinite sum reflects the fact that a birth at t creates a new dynasty whose members increase populations after $t+1$ and also generate externalities.

The terms $f'(e_{t+i})e_{t+i}$ in (18) are aggregate real population externalities measured in units of output or equivalently in units of descendants' consumption at $t+i$. Remaining terms convert the real population externalities into units of parent's (period-t) utility. The negative sign shows the externality reduces utility.

There are three interpretations of a real term $f'(e_{t+i})e_{t+i}$. First, the market value at price f' of the emissions produced by a person born at $t+i-1$, which come at the expense of emissions by everyone else under a cap. Second, the loss of per-capita output caused by a person born at $t+i-1$: output $f(e(N_{t+i}))$ is lower by

$f'(e_{t+i})e_{t+i}/N_{t+i}$; summing over the population at $t+i$ gives an aggregate loss of $f'(e_{t+i})e_{t+i}$. Third, the dilution of rents from auction revenue. When government auctions $\hat{E}$ permits, it receives total revenue $p_{t+i}\hat{E} = f'(e_{t+i})\hat{E}$ that it redistributes as equal lump sums so each person indirectly receives emission revenue $f'(e_{t+i})e_{t+i}$. With an additional birth at $t+i-1$, the population at $t+i$ loses the revenue $f'(e_{t+i})e_{t+i}$ that goes to the additional person.

Because any cap eventually binds, the discounted externality sum $\frac{dV^*}{dN}(N^*_{t+1})$ is strictly negative for all N_t in the cap era, even if population is initially so low that the cap does not yet bind so $f'(e(N_t)) = 0$.[18] Thus starting from any population N_t, the optimal population at $t+1$ is always less than the natural population.[19]

Pigovian Taxes on Having Children

To compute the sequence of child taxes that would be needed to change fertility and population from natural to optimal levels, we assume child-tax revenue is redistributed to households as equal lump sums. Let τ_t denote a tax per child and let $\bar{n}_t$ denote the average over households of n_t in t, so each household pays child taxes $\tau_t n_t$ and receives lump-sum revenue $\tau_t \bar{n}_t$.[20]

With child taxes, overall child costs include taxes and overall transfers includes

[18]Formally: $e(N^*_t) \to e^*_{ss} < e^+$ implies $f'(e_t) > 0$ for some t so $\frac{dV^*}{dN}(N^*_{t+1}) < 0$.

[19]Formally: $\frac{dV^*}{dN}(N^*_{t+1}) < 0$ plus concavity of V (see footnote 10) imply that the fertility n^*_t that solves (16) is strictly less than the fertility n_t that satisfies the household first-order condition evaluated along the optimal path, $V_n(n_t, f(e(N^*_t)), V^*(N^*_{t+1})) = 0$.

[20]There may be practical obstacles to implementing a given sequence of child taxes. Extracting taxes from parents may be difficult, for instance, and some ways of preventing population growth may be ethically unacceptable. On the other hand, a range of policies that include filing-status differences, personal exemptions, public-school spending, and welfare programs currently subsidize children. Because the issue of pre-existing child subsidies is complex, we adopt the conservative approach of assuming that $\tau_t = 0$ before a cap is imposed.

lump-sum redistributions of child-tax revenue, so the household generically maximizes $u(w + TR - \chi n - \tau n) + \beta(n)U$ taking w, $TR = pE/N + \tau\bar{n}$, and τ as given. The first-order condition is

$$V_n(n, w + TR, U|\tau) \equiv -u'(w + TR - \chi n - \tau n)(\chi + \tau) + \beta' U = 0. \quad (19)$$

To implement the optimal population sequence, each optimal tax τ_t^* must be set so n_t^*, which solves (16), also solves (19). Setting V_n^* from (16) equal to V_n from (19) and noting that $w_t + TR_t - \tau_t^* n_t^* = f(e(N_t^*))$ and $U_{t+1} = V^*(N_{t+1}^*)$ along the optimal path, τ_t^* must satisfy $u'(f(e(N_t^*)) - \chi n_t^*)\tau_t^* = -\beta(n_t^*)N_t^* \frac{dV^*}{dN}(N_{t+1}^*)$. From (18),

$$\begin{aligned} \tau_t^* &= \frac{\beta(n_t^*)N_t^*}{u'(c_t^*)}\left(-\frac{dV^*}{dN}(N_{t+1}^*)\right) \\ &= \frac{\beta(n_t^*)}{n_t^*}\sum_{i=1}^{\infty}\left[\prod_{j=1}^{i-1}\beta(n_{t+j}^*)\right]\frac{u'(c_{t+i}^*)}{u'(c_t^*)}f'(e(N_{t+i}^*))e(N_{t+i}^*). \end{aligned} \quad (20)$$

Because the externality sum $\frac{dV^*}{dN}(N_{t+1}^*)$ is strictly negative, optimal taxes are strictly positive for all $t \geq 0$. Optimal child taxes are Pigovian, as in Harford (1998): the optimal tax equals the discounted present value of the externalities generated by a child and all descendants of the child. The terms in (20) other than the real externalities $f'(e(N_{t+1}^*))e(N_{t+1}^*)$ can be interpreted as the number of descendants in a future period times products of single-period discount factors.[21] Overall, the τ_t^* measure population externalities in units of parent's consumption.

[21] In detail,

$$\frac{\beta(n_t^*)}{n_t^*}\left[\prod_{j=1}^{i-1}\beta(n_{t+j}^*)\right]\frac{u'(c_{t+i}^*)}{u'(c_t^*)} = \left(\prod_{j=1}^{i-1}n_{t+j}^*\right)\left[\prod_{j=0}^{i-1}\frac{\beta(n_{t+j}^*)}{n_{t+j}^*}\frac{u'(c_{t+j+1}^*)}{u'(c_{t+j}^*)}\right],$$

where $\prod_{j=1}^{i-1} n_{t+j}^* = L_{t+i}^*/L_{t+1}^*$ is descendants at time $t+i$ per child born at time $t+1$. The terms $\frac{\beta(n_{t+j}^*)}{n_{t+j}^*}\frac{u'(c_{t+j+1}^*)}{u'(c_{t+j}^*)}$ can be interpreted as single-period discount factors. Specifically, if individuals could trade consumption loans that are settled by their children, $\frac{\beta(n_{t+j}^*)}{n_{t+j}^*}\frac{u'(c_{t+j+1}^*)}{u'(c_{t+j}^*)}$ would be the market-clearing price in period j of a loan that pays one consumption unit in period $j+1$.

V. EXTENSIONS

To make calibrations more meaningful, we extend the model in three ways:

Time costs of children

Parents devote substantial time to children. To include time costs, we assume having a child requires a constant amount of parental time, ψ, in addition to output χ. Time spent having a child reduces labor supply so $l_t = 1 - \psi n_t$ depends on fertility, total labor supply $L_t = (1 - \psi n_t)N_t$ differs from population N_t, and the emissions ratio depends on fertility: $e_t = \min\left(e^+, \hat{E}/[(1 - \psi n_t)N_t]\right)$.

With time costs, the cost of a child becomes $\chi + \psi w(e_t)$, the sum of the output cost and foregone wages. The household takes the wage and transfers of permit revenue $TR_t = p_t E_t/N_t$ as given in maximizing utility so these replace income as determinants of fertility. In equilibrium, household income is the sum of wage income and transfers, and equals labor times output per unit of labor, so $y_t = (1-\psi n_t)w(e_t) + f'(e_t)E_t/N_t = (1 - \psi n_t)f(e_t)$. The generic household first-order condition becomes $V_n(n, w, TR, U) \equiv -u'\left((1 - \psi n)w + TR - \chi n\right) \cdot (\chi + \psi w) + \beta' U = 0$.

These changes carry through the resulting dynamics and are easily incorporated into calibrations. Notably, the function S defined in (9) gains terms and becomes

$$S(n, e) \equiv -u'((1 - \psi n)f(e) - \chi n)(\chi + \psi w(e)) + \frac{\beta'(n)}{1 - \beta(n)} u((1 - \psi n)f(e) - \chi n). \quad (21)$$

Roots of the resulting steady-state conditions $S(n^+, e^+) = 0$ and $S(1, e_{ss}) = 0$ exist as before. The roots are the steady-state values of fertility in the no-cap era, n^+, and the emissions ratio in the cap era, e_{ss}. Condition (10), which ensures $n^+ > 1$, gains a time-cost term $\psi f(e^+)$ and becomes $\chi + \psi f(e^+) < \phi f(e^+)$. The steady-state natural population is $N_{ss} = \hat{E}/[(1 - \psi)e_{ss}]$. The optimal population with time costs similarly implies a steady-state optimal emissions ratio e^*_{ss} with $e_{ss} < e^*_{ss} < e^+$, and steady-state optimal population $N^*_{ss} = \hat{E}/[(1-\psi)e^*_{ss}] < N_{ss}$.[22]

Backstop Technology

A common assumption in integrated assessment models is that a "backstop" technology may permit output to be positive without emissions.[23] If there is a positive backstop output level $f(0)$, steady-state natural and optimal populations exist as above if $f(0)$ is low enough, specifically, if $f(0) < f^B \equiv \frac{\chi}{\phi-\psi}$.[24]

If $f(0) > f^B$, however, income loss from a cap is insufficient to reduce fertility to replacement so the natural population does not converge to a steady-state value. Instead, fertility converges to the unique root $n_{ss} > 1$ of $S(n_{ss}, 0) = 0$, population grows without bound, $e_t \to 0$, and $f(e_t) \to f(0)$. In the limit, concavity implies $f'(e)e \to 0$ so the population externality vanishes and optimal fertility converges to the same limit n_{ss} as natural fertility.[25] For all finite periods, however, the population externality exists so the optimal child tax is positive, and natural fertility exceeds optimal fertility so the natural population exceeds the optimal population at all t and in the limit.[26]

[22]The optimal steady-state condition is now $S^*(e^*_{ss}) = 0$, where $S^*(e) \equiv (1-b_0)S(1,e) - b_0(1-\psi)u'((1-\psi)f(e)-\chi)f'(e)e$. Note that uniqueness of e_{ss} and e^*_{ss} requires regularity conditions that are detailed in the appendix and are satisfied in calibrations below.

[23]e.g. Nordhaus and Boyer (2000), Kelly and Kolstad (2001).

[24]If $f(0) < \chi/(1-\psi)$, then there is an $e = f^{-1}(\chi/(1-\psi)) > 0$ at which the marginal utility of consumption is infinite so $S(1, f^{-1}(\chi/(1-\psi))) < 0$ with S as defined in (21), and the steady-state analysis in section III holds, mutatis mutandis. Moreover if $f(0) \geq \chi/(1-\psi)$ but $f(0) < f^B$,then $S(1,0) < 0$, and again the reasoning of section III implies there is a unique steady-state natural emissions ratio $e_{ss} > 0$ defined by $S(1, e_{ss}) = 0$.

[25]Because $e_t l_t N_t = \hat{E}$, a value $e_t = 0$ is inconsistent with $\hat{E} > 0$. Thus there is no meaningful $e_{ss} = 0$, but allocations with $e_t > 0$ in which $e_t \to 0$ are meaningful.

[26]In the non-generic case with $f(0) = f^B$, $e_t \to 0$ and $n_t \to 1$ so all natural and optimal limit

The backstop output level $f(0)$ is key to knowing the economy's fate under a cap. As long as $f(0) < f^B$, a cap ultimately leads to a steady state with output low enough to choke off population growth. This is true even if the cost of eliminating *almost all* emissions is small: fertility then would remain high so e would continue to drop, until output is low enough so $n_{ss} = 1$.

Exogenous Growth in Factor Productivity

To add exogenous productivity growth, we assume production is

$$Y_t = F(L_t \lambda^t, E_t \alpha^t), \tag{22}$$

where $\lambda \geq 1$ is an exogenously given growth factor for labor productivity and $\alpha \geq 1$ is an exogenously given growth factor for emissions productivity. Greater emissions productivity α^t means fewer emissions are needed to produce a given output from a given amount of labor.

We assume the output cost of a child grows with labor productivity λ^t, so the household budget becomes $c_t = y_t - \lambda^t \chi n_t$. The idea is that greater productivity means more human capital, which requires that more resources be put into each child. Put differently, λ^t captures an exogenous trend in child quality that raises the cost of a child and tends to make fertility fall as productivity rises.[27]

An economy with growing productivity is equivalent to an economy with stationary values of productivity-adjusted variables, marked with tildes. The key state variable is growth-adjusted population, $\tilde{N}_t \equiv N_t \lambda^t / \alpha^t$. Also define $\tilde{n}_t \equiv n_t \lambda / \alpha$; this is the

conditions reduce to $S(1, 0) = S^*(0) = 0$, where S^* was defined in footnote 22.

[27]A demographic transition toward lower fertility would arise if the cost of children were to have risen in a transitional (non-steady-state) period. It is beyond the scope of the current analysis to also model this. The assumed proportionality of the output cost of a child to λ^t simply ensures that child costs do not vanish or explode as a fraction of income merely because productivity grows, which ensures balanced growth.

growth factor for productivity-adjusted population and for total emissions, which rise because of population and labor-productivity growth, and fall because of emissions-productivity growth.

With variables and parameters defined in growth-adjusted terms[28]:

$$u((1-\psi n_t)w_t + TR_t - (\lambda^t\chi + \tau_t)n_t) + \beta(n_t)U_{t+1}$$
$$= \lambda^{(1-\theta)t}\left[u((1-\tilde{\psi}\tilde{n}_t)\tilde{w}_t + \widetilde{TR}_t - (\tilde{\chi}+\tilde{\tau}_t)\tilde{n}_t) + \tilde{\beta}(\tilde{n})\tilde{U}_{t+1}\right],$$

so choosing n to maximize $u((1-\psi n)w + TR - (\chi+\tau)n) + \beta(n)U$ given (w, TR, U) is equivalent to choosing $\tilde{n}$ to maximize $u((1-\tilde{\psi}\tilde{n})\tilde{w} + \widetilde{TR} - (\tilde{\chi}+\tilde{\tau})\tilde{n}) + \tilde{\beta}(\tilde{n})\tilde{U}$ given $(\tilde{w}, \widetilde{TR}, \tilde{U})$. The equivalent problem thus has the same form as the problem without productivity growth except that growth-adjusted (tilde) variables replace regular variables. All analysis from previous sections goes through with growth-adjusted variables and parameters replacing regular variables and parameters:

In the no-cap era, the emissions ratio is $\tilde{e}_t = e^+$ and household income follows $y_t^+ \equiv (1-\tilde{\psi}\tilde{n}_t)f(e^+)\lambda^t$. A perfect-foresight solution is pair $(\tilde{n}^+, \tilde{U}^+)$ with $\tilde{U}^+ = u((1-\tilde{\psi}\tilde{n}^+)f(e^+) - \tilde{\chi}\tilde{n}^+)/(1-\tilde{\beta}(\tilde{n}^+))$, where $\tilde{n}^+$ is optimal given $\tilde{U}^+$. In any solution, growth-adjusted population grows at rate $\tilde{n}^+ - 1$. Because $E_t = \tilde{e}_t(1-\tilde{\psi}\tilde{n}^+)\tilde{N}_t$ and $\tilde{e}_t = e^+$, emissions also grow at rate $\tilde{n}^+ - 1$. We assume $\tilde{\chi} + \tilde{\psi}f(e^+) < \tilde{\phi}f(e^+)$ where $\tilde{\phi} = 1/\left(1 + \frac{(1-\theta)(1-\tilde{\beta}(1))}{(1-b)\tilde{\beta}(1)}\right) < 1$, so $\tilde{n}^+ > 1$.[29]

In the cap era, growth-adjusted population converges to steady-state value $\tilde{N}_{ss}$. Unless $\alpha = \lambda$, actual population N_t therefore changes over time. Specifically, $\tilde{n}_{ss} = n_{ss}\lambda/\alpha = 1$ implies that actual fertility is $n_{ss} = \alpha/\lambda$. This is a balanced-growth

[28]Specifically: $\tilde{e}_t \equiv \min(e^+, \frac{\hat{E}}{(1-\tilde{\psi}\tilde{n}_t)\tilde{N}_t})$, $\tilde{y}_t \equiv (1-\tilde{\psi}\tilde{n}_t)f(\tilde{e}_t) = y_t/\lambda^t$, $\tilde{w}_t \equiv f(\tilde{e}_t) - f'(\tilde{e}_t)\tilde{e}_t$, $\tilde{\chi} \equiv \chi\alpha/\lambda$ (so $\tilde{\chi}\tilde{n}_t = \chi n_t$), $\tilde{\psi} \equiv \psi\alpha/\lambda$ (so $\tilde{\psi}\tilde{n}_t = \psi n_t$), $\tilde{\beta}(\tilde{n}_t) \equiv \lambda^{(1-\theta)}\beta(\tilde{n}_t\alpha/\lambda) = \lambda^{(1-\theta)}\beta(n_t)$, $\tilde{U}_t \equiv U_t/\lambda^{(1-\theta)t}$, $\tilde{\tau}_t \equiv \tau_t(\alpha/\lambda)/\lambda^t$, and $\widetilde{TR}_t = f'(\tilde{e}_t)E_t/\tilde{N}_t + \tilde{n}_t\tilde{\tau}_t$. (Note that $TR_t = f'(e_t)E_t/N_t + n_t\tau_t$.)

[29]Emissions have increased historically, consistent with $\tilde{n}^+ > 1$. If future fertility were to fall sufficiently due to changes in tastes or if α/λ were to fall sufficiently, then $\tilde{n}^+$ could fall below one. Then emissions and the emissions problem would eventually vanish.

condition.[30] Growth in labor productivity (λ) introduces an increasing trend in each person's emissions footprint and growth in emissions productivity (α) introduces a decreasing trend, so exogenous productivity growth overall introduces per-capita emissions growth with factor λ/α per period. To hold total emissions constant in steady state, this means population must grow with factor α/λ. Similarly, steady-state optimal fertility is $n^*_{ss} = \alpha/\lambda$.

Four growth factors describe steady state in the cap era. Natural and optimal populations grow with factor α/λ, as just noted. Outputs per person and living standards grow with factor λ because $\tilde{y}_{ss} = y_t/\lambda^t$ and $\tilde{y}^*_{ss} = y^*_t/\lambda^t$ are constant. Total output, the product of population and output per person, grows with factor α, the product of α/λ and λ. Finally total emissions are constant, as output grows with the emissions-productivity growth factor α. The outcome is Malthusian modified for productivity growth: living standards continue to grow as long as $\lambda > 1$; total output continues to grow as long as $\alpha > 1$; and population grows (or shrinks) unless $\alpha = \lambda$.

Taxes in the transformed economy, $\tilde{\tau}_t \equiv \tau_t(\alpha/\lambda)/\lambda^t$, are taxes per growth-adjusted child. To convert optimal taxes $\tilde{\tau}^*_t$ to taxes per actual child τ^*_t, we divide out the growth-adjustment correction $(\alpha/\lambda)/\lambda^t$. The actual tax grows with factor λ, as does actual income along the optimal path, $y^*_t \equiv (1 - \tilde{\psi}\tilde{n}^*_t)f(\tilde{e}^*_t)\lambda^t$. In the calibrations below we remove growth factors by reporting optimal taxes as shares of income[31]

$$\{\tau/y\}^*_t \equiv \frac{\tau^*_t}{y^*_t} = \frac{\tilde{\tau}^*_t\lambda/\alpha}{(1 - \tilde{\psi}\tilde{n}^*_t)f(\tilde{e}^*_t)}.$$

[30]The production function (22) implies that output growth arises from growth in the inputs $l_t N_t \lambda^t$ and $E_t\alpha^t$. In steady state with actual fertility constant at n_{ss}, effective labor $(1 - \psi n_{ss})N_t\lambda^t$ has growth factor $n_{ss}\lambda$. Because emissions are capped at $\hat{E}$, the input $E_t\alpha^t$ has growth factor α. Balanced growth requires $n_{ss}\lambda = \alpha$, or $n_{ss} = \alpha/\lambda$.

[31]To interpret $\{\tau/y\}^*_t$ in a real world with two-adult households, each parent can be seen as paying one-half of the tax so the tax on any single child as a share of household income is $\frac{1}{2}\{\tau/y\}^*_t$. For the couple to replace themselves takes two children, so each would pay $\{\tau/y\}^*_t$ for replacement.

VI. CALIBRATION

We calibrate the model to a growing economy with annual steady-state population growth of 1.4 percent, per-capita output growth of 1.7 percent, and aggregate emissions growth of 1.8 percent, which were actual world rates over 1990-2005.[32] A period equals 30 years, so $n^+ = 1.52$, $\tilde{n}^+ = n^+\lambda/\alpha = 1.72$, $\alpha = 1.48$, and $\lambda = 1.67$.[33] The value of b_0 is chosen so the household's first-order condition holds given these growth rates. We choose units so $e^+ = 1$, $f(1) = 1$, and $\tilde{N}_0 = 1$. Unless noted, all variables except child taxes are growth-adjusted.

We consider two production functions. Cobb-Douglas production is $f(\tilde{e}) = f_0\tilde{e}^{f_1}(f_2 - \tilde{e})^{1-f_1}$ where f_0, $f_1 \in (0,1)$, and f_2 are parameters.[34] With Cobb-Douglas production, the factor share of emissions rises monotonically from zero at e^+ to f_1 as $\tilde{e}$ falls to zero. For any f_1, units choices pin down f_0 and f_2: $e^+ = 1$ implies $f'(1) = 0$ so $f_2 = 1/f_1$, and $f(1) = 1$ implies $f_0 = [f_1/(1-f_1)]^{1-f_1}$. To set f_1, we assume it costs 3 percent of output to reduce emissions by 25 percent, so $f(0.75) = 0.97$. This implies $f_1 = 0.483$. A 3-percent cost is in the range of estimates in Stern (2007). We also evaluate a 2-percent cost below, with $f_1 = 0.371$.

Cobb-Douglas production does not allow a positive backstop. To study a backstop and get a sense of how sensitive results are to the form of production, we also consider

[32] see World Resources Institute (2008). Trend assumptions matter; see sensitivity analysis below.

[33] Specifically $n^+ = \exp(30 \cdot .014) = 1.52$. Per-capita income grows with factor λ so $\lambda = \exp(30 \cdot .017) = 1.67$. Total emissions grow at the same rate as productivity-adjusted population, so $\tilde{n}^+ = n^+\lambda/\alpha = \exp(30 \cdot .018) = 1.72$. This implies $\alpha = n^+\lambda/\tilde{n}^+ = 1.48$.

[34] The form may be derived from three primitive assumptions: (i) labor is used to produce two intermediate goods in amounts y_1 and y_2 according to the linear technology $y_1 + y_2 = f_2$; (ii) a unit of good 1 generates a unit of emissions so $\tilde{e} = y_1$, whereas good 2 generates no emissions; and (iii) output per unit of labor is a Cobb-Douglas function $f_0 y_1^{f_1} y_2^{1-f_1}$. (If output per unit of labor were a CES function of y_1 and y_2 with an elasticity other than one, the factor share of emissions would approach either zero or one as $\tilde{e} \to 0$, which may be undesirable to impose.)

the abatement-cost specification used in many integrated assessment models: $f(\tilde{e}) = 1-(1-g_0)(1-\tilde{e})^{g_1}$, where g_0 and g_1 are parameters and backstop output is $f(0) = g_0$.[35] When we assume no backstop ($g_0 = 0$), we set g_1 by again assuming it costs 3 percent of output to reduce emissions by 25 percent, so $g_1 = 3.32$. With a positive backstop, we leave the curvature g_1 unchanged and simply assume a positive g_0, which proportionately reduces abatement costs at any $\tilde{e}$.

We assume children have an output cost of $\chi = 0.138$ and a time cost of $\psi = 0.110$, which are based on U.S. data. The output cost is from the sum of expenditures by families on children plus expenditures on K-12 and college education. The time cost assumes the difference between male and female labor-force participation rates is due solely to time devoted to having children so that with zero children, the average participation rate would equal the current male rate (0.76) instead of the current average of male and female rates (0.685). Details are in the appendix.

The time cost implies that per-capita labor in the no-cap era is $1 - \tilde{\psi}\tilde{n}^+ = 0.833$ and per-capita income is $\tilde{y}^+ = (1 - \tilde{\psi}\tilde{n}^+)f(e^+) = 0.833$.

Base Case

We begin with a base case that assumes equal utility curvatures ($\theta = b$), Cobb-Douglas production, and a cap that freezes emissions. A reasonable range of estimated θ values is $0.5 - 5.0$ (see e.g. Ogaki and Reinhardt, 1998; Bansal and Yaron, 2004). We set $\theta = b = 0.8$ in the base case. Population is normalized so $\tilde{N}_0 = 1$, and

[35] An interpretation is that a unit of labor gives a unit of output and a unit of emissions if no resources are devoted to abatement, and the cost of abating $1-\tilde{e}$ units of emissions is $(1-g_0)(1-\tilde{e})^{g_1}$ units of output. The factor share of emissions in abatement-cost cases has a knife-edge, which partly motivates why we use Cobb-Douglas for most calibrations. Without a backstop, the factor share rises monotonically from zero at $\tilde{e} = 1$ to *one* at $\tilde{e} = 0$, but with any positive backstop, the factor share rises from zero at $\tilde{e} = 1$ to a peak, then falls to *zero* at $\tilde{e} = 0$.

$E_0 = 1 - \tilde{\psi}\tilde{n}^+ = 0.833$ as emissions at $t = 0$ would be 0.833 without a cap. Thus a freeze means $\hat{E} = 0.833$.

Steady-state results are in table 1. In the base case, the growth-adjusted natural population in steady-state is 9.16 times $\tilde{N}_0$, and the steady-state emissions ratio is 0.101 times the emissions ratio without a cap. A cap substantially lowers incomes: per-capita output falls from $\tilde{y}^+ = 0.833$ to $\tilde{y}_{ss} = 0.408$.

Table 1. Steady States

Case	Regime	$\tilde{N}_{ss}$	$\tilde{e}_{ss}$	$\tilde{y}_{ss}$	N_{peak}	$f'\tilde{e}/f$	$\{\tau/y\}^*_{ss}$
Base case: $\theta = b = .8$	natural	9.16	.101	.408	3.76	.457	
	optimal	2.39	.386	.721	1.28	.365	.211
$\theta = b = 2$	natural	44.1	.021	.195	13.9	.469	
	optimal	11.1	.083	.373	3.52	.478	.957
$\theta = b = .4$	natural	4.97	.186	.529	2.35	.432	
	optimal	1.43	.644	.835	1.03	.250	.106
$\theta = .95$, $b = .8$	natural	9.90	.093	.394	4.16	.459	
	optimal	7.32	.126	.452	3.12	.450	.068
$\theta = 1.25$, $b = 2$	natural	28.7	.032	.239	7.76	.475	
	optimal	21.7	.042	.273	5.92	.472	.155
2% cost of 25% emission cut	natural	15.6	.059	.418	4.96	.357	
	optimal	3.96	.233	.665	1.63	.311	.180
$n^+ = 1$	natural	3.57	.277	.633	1.17	.404	
	optimal	1.20	.825	.889	1.00	.141	.051

The growth-adjusted optimal population in steady state is 2.39 so the natural population is almost four (9.16/2.39) times the optimal population. The optimal emissions ratio of 0.386 is almost four times the natural emissions ratio. Per-capita optimal output, 0.721, exceeds per-capita natural output and is 13.5 percent below output at $t = 0$.

The natural fertility functions (figure 5) imply that the natural and optimal populations converge smoothly from $\tilde{N}_0 = 1$ to steady-state values.[36] After five generations, $\tilde{N}_5 = 6.84$, for instance, and after ten generations, $\tilde{N}_{10} = 8.97$, close to the steady-state value of 9.16.

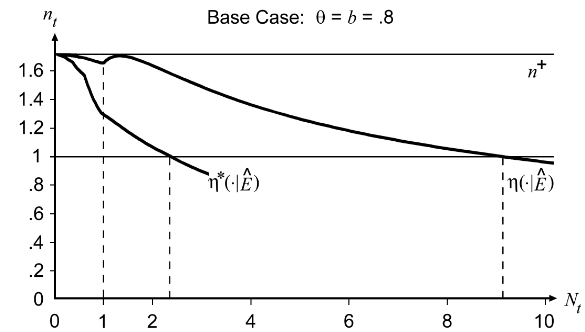

Figure 5. Base Case Fertilities

Figure 6 shows the actual (not adjusted) natural and optimal populations. In the no-cap era, population increases exponentially to infinity. With a cap, the natural and optimal populations grow at rate $(\alpha - \lambda)/\lambda = -0.113$ in steady state so both actual populations peak after a cap is imposed and then fall. World population under a cap would peak at 3.76 times its current level with no population policy and would peak at 1.28 times its current level under the optimal population policy, as reported in column N_{peak} in table 1.

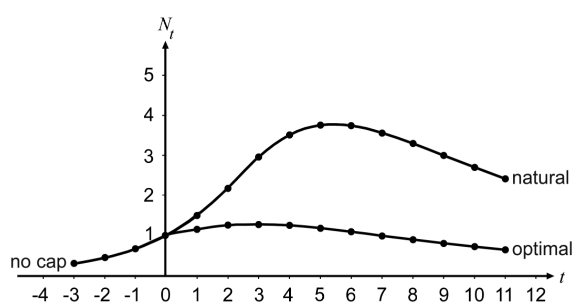

Figure 6. Populations

[36]The appendix describes the numerical procedures. In the figure, slopes of the natural and optimal fertility functions become more positive (or less negative) around $N = 1$ because the cap begins to bind so the wage and hence the time cost of children start to fall as N rises above one.

Permit revenue is 45.7 percent of output in the natural steady state and 36.5 percent of output in the optimal steady state, as population continues to grow and drive down e until e is low enough that revenue $f'(e)e$ is substantial.[37] These revenue numbers are large given that the Federal spending share over 1990-2005 averaged about 20 percent of output. Emissions revenue is small right after the cap is imposed but increases sharply as the emissions ratio falls below one. With the freeze, for instance, revenue jumps from 0.7 percent of output at $t = 0$ to 27.1 percent of output at $t = 1$.

The optimal child tax in steady state is 21.1 percent of per-capita income.[38] (From (20), the optimal steady-state child tax is the discounted sum of steady-state permit revenue $f'(e)e$.) To get a sense of this number in the U.S. context, U.S. personal income is roughly \$55,000 per adult per year, which may be interpreted as uncapped income ($\tilde{y}^+$) measured in dollars per year. Steady-state optimal income ($\tilde{y}^*_{ss}$), which is 13.5 percent less than uncapped income, would then be about \$48,000. Thus a child tax of 21.1 percent is equivalent to a tax of about \$10,000 each year for 30 years (the length of a generation in the model) for each child.[39] An alternative sense

[37] As can be seen from figure 1, maximum revenue as e declines would not be substantial in only two cases: (i) if the slope f' is always small and there is a high backstop output, so the entire emissions problem could be eliminated at little cost by simply restricting emissions to zero; or (ii) if f' is small until e is close to zero, at which point f bends sharply downward to a zero backstop, so almost all emissions could be eliminated at little cost. It may be difficult to distinguish the two cases from current economic observations around $e = e^+$, but neither case is interesting under the maintained hypothesis that emissions are problem. (In the second case, emissions are a problem only in the future when population has increased enough so incomes are low.)

[38] Income excludes redistributions of child-tax revenue, so a tax of 21.1 percent of income is equivalent to a tax of 17.4 (= 21.1/1.211) percent of income plus redistributions of child-tax revenue.

[39] Kelly and Kolstad (2001) calculate welfare costs from a marginal child in the range \$200-\$800. Such costs are tiny compared with costs of \$10,000 per year for 30 years. Kelly and Kolstad implicitly assume a backstop output of $f(0) = .93$ and also assume that population grows at an exogenously given rate that itself decreases at an exogenously given rate. With their production function in our model, the optimal policy would be to drive the emissions ratio to zero in steady state. This does

is that the annual cost of a child is about \$13,000, of which about \$6,000 is time costs. The optimal child tax therefore raises the full cost of a child in steady state by about three-quarters, from \$13,000 to \$23,000, to just under half (\$23,000/\$48,000) of income.

Optimal child taxes in steady state are independent of $\hat{E}$, but optimal child taxes along the path from $t = 0$ depend on $\hat{E}$ as shown in table 2. In the base case with a freeze (row one), the tax is 5.2 percent of income at $t = 0$, about a quarter of the steady-state value, and 11.6 percent of income at $t = 1$, a bit more than half the steady-state value. With a more restrictive cap, optimal child taxes are higher after imposition and with a less restrictive cap optimal child taxes are lower, as illustrated by a 25 percent cut in row two and a cap that is 25 percent slack in row three.

Table 2. Time Paths of Optimal Child Taxes

Case	$\{\tau/y\}_0^*$	$\{\tau/y\}_1^*$	$\{\tau/y\}_2^*$	$\{\tau/y\}_3^*$	$\{\tau/y\}_4^*$	$\{\tau/y\}_{ss}^*$
base case: $\hat{E} = E_0$	.052	.116	.152	.174	.188	.211
25% cut: $\hat{E} = .75E_0$	.122	.155	.176	.189	.197	.211
25% slack: $\hat{E} = 1.25E_0$	.028	.083	.133	.162	.180	.211
abatement cost, $f(0) = 0$	.055	.141	.195	.223	.237	.249
abatement cost, $f(0) = .4$	.037	.101	.142	.166	.179	.201
abatement cost, $f(0) = .6$	.026	.079	.093	.100	.098	0
$n^+ = 1$, $\hat{E} = .75E_0$	.071	.065	.060	.057	.055	.051

Sensitivity Analysis–Utility Curvature

Table 1 also reports sensitivity analysis of individual base-case assumptions. First is a case with $\theta = b = 2$. Steady-state population is substantially greater, income is lower, and optimal child taxes are higher than in the base case. In steady-state, the

not happen in their calculations because they assume growth slows enough so the backstop is never reached.

natural population is 44.1 times the transition population, income is $\tilde{y}_{ss} = 0.195$, and the child tax is 95.7 percent of income.[40] The implied ratio of the natural to optimal steady-state populations $\tilde{N}_{ss}/\tilde{N}^*_{ss}$ is still about four, however. In the transition, the optimal child tax is 10.3 percent of income at $t = 0$ and 18.1 percent of income at $t = 1$. Assuming high values of θ and b might make sense if one believes fertility is insensitive to economic incentives, but this is a pessimistic assumption here. The fertility path in figure 7 makes clear why. A cap reduces children's incomes, and when

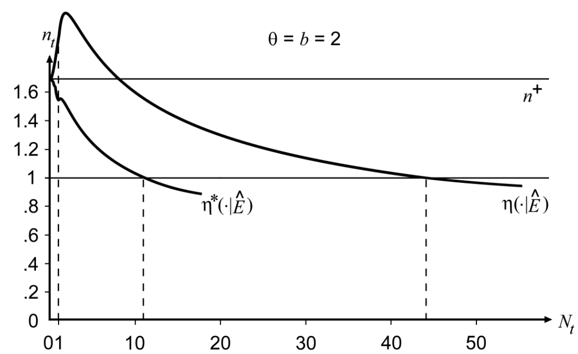

Figure 7. Dynamics when $\theta = b = 2$

$\theta > 1$, this causes a fertility burst. As a result, a greater income fall and hence a greater population are needed to reduce fertility to replacement.

With low enough θ, on the other hand, fertility would respond elastically to changes in population so a cap would act like a switch that turns off fertility, and the optimal child tax could be quite small. To see whether this happens with reasonable parameter values, we consider $\theta = b = 0.4$, at the low end of empirical estimates of θ. The resulting $\tilde{N}_{ss}$ is still 4.97 times the transition population (and $\tilde{N}^*_{ss} = 1.43$ so $\tilde{N}_{ss}/\tilde{N}^*_{ss} = 3.47$), and the optimal steady-state child tax is still 10.6 percent of income.

We also consider the equal-curvatures assumption, $\omega = 1$. Because ω measures how much parents care about per-child consumption relative to the number of children,

[40]This is equivalent to a tax of 48.9 (= 95.7/1.957) percent of income plus redistributions of child-tax revenue.

and the population externality is a loss from lower per-child consumption, a lower ω tends to reduce the utility value of the externality. Thus real reductions in children's utility and the real population externality might be substantial, but with low ω, parents would simply not care much about this so the gap between the natural and optimal populations would be small.

To judge this, we modify the first two cases, assuming $\omega = .25$. When $\theta = .95$ and $b = .8$, the ratio of natural to optimal steady-state populations is 1.4 and the steady-state child tax is 6.8 percent. When $\theta = 1.25$ and $b = 2$, the ratio of natural to optimal steady-state populations is 1.3 and the steady-state child tax is 15.5 percent. Even in these cases the population externality is substantial.

The population externality thus tends to be substantial because empirically reasonable assumptions about utility curvature imply that the desire to have children remains strong as e_t falls below e^+, so e_t ends up being driven a fair bit below e^+. Sensitivity to the utility parameter b_0, which is set by trend assumptions, is discussed below.

Sensitivity Analysis–Technology

The true cost of reducing emissions is uncertain. Table 1 reports steady-state results when f is parameterized assuming it costs 2 percent of output to reduce emissions by 25 percent ($f(0.75) = 0.98$), instead of 3 percent as in the base case. This reduces the population externality but not greatly: the optimal child tax falls from 21.1 percent in the base case to 18.0 percent. The reason is that when it is less costly to reduce emissions, incomes and hence fertility are higher at any given population, and steady state is reached only when population is so high and the emissions ratio so low that incomes are close to incomes in the base case. That is, the income reductions needed to choke off population growth doesn't change much when the output cost of reducing emissions falls.[41]

We also examine the abatement-cost specification. Under base-case utility assumptions, the critical backstop is $f^B = 0.507$ so we consider backstops of 0, 0.4, and 0.6. Comparing the first two rows of table 3 with the first two rows of table 1 shows the effects of changing from a Cobb-Douglas to an abatement-cost specification with no backstop. Steady-state natural and optimal populations fall by a bit less than half and factor shares of emissions rise, but optimal child taxes change little, rising from 21.1 percent to 24.9 percent of income.

Table 3. Steady States with Abatement-Cost Production at Different Backstops

Case	Regime	$\tilde{N}_{ss}$	$\tilde{e}_{ss}$	$\tilde{y}_{ss}$	N_{peak}	$f'\tilde{e}/f$	$\{\tau/y\}^*_{ss}$
$f(0) = 0$	natural	4.84	.190	.374	2.85	.842	
	optimal	1.58	.586	.805	1.09	.431	.249
$f(0) = .4$	natural	15.6	.059	.438	4.33	.168	
	optimal	2.84	.325	.702	1.36	.348	.201
$f(0) = .6$	natural	∞	0	.532	∞	0	
	optimal	∞	0	.532	∞	0	0

Comparing rows of table 3, a higher backstop means smaller income reductions as well as lower population externalities, but the effect is not great as long as $f(0) < 0.507$ (so adjusted population is constant in steady state). From table 3, the ratio of the natural to the optimal steady-state populations varies between about three and five when $f(0) < 0.507$.

[41]In detail, steady-state natural income is slightly higher than in the base case, 0.418 instead of 0.408, but steady-state optimal income is lower, 0.665 instead of 0.721. (The smaller difference between $\tilde{y}^*_{ss}$ and $\tilde{y}_{ss}$ means the real externality terms $f'(e)e$ in (18) are lower.) Steady-state emission ratios are much reduced, however: $\tilde{e}_{ss}$ is only 0.059 instead of 0.101 in the base case, and $\tilde{e}^*_{ss}$ is 0.233 instead of 0.386. Consequently, $\tilde{N}_{ss}$ and $\tilde{N}^*_{ss}$ are roughly two-thirds greater than in the base case, but their ratio remains about four.

If $f(0) > 0.507$, adjusted population grows forever and the population externality vanishes in the limit, quite a different long-run outcome than when $f(0) < 0.507$. The paths of the economy for the first few periods after transition, however, can be similar. Table 2 shows this. With $f(0) = 0.6$, abatement costs at any $\tilde{e}$ are reduced by 60 percent compared with abatement costs with $f(0) = 0$, and the optimal child tax in the transition period is similarly about half of the tax when $f(0) = 0$. With $f(0) = 0.6$, the tax peaks in the third period after transition at 10 percent, which is still about half the tax in the base case, then goes to zero in steady state.[42]

Sensitivity Analysis–Trend Assumptions

The maintained hypothesis that emissions are a problem requires $\tilde{n}^+ = n^+\lambda/\alpha > 1$ so emissions tend to rise on their own. Changes in trend assumptions that lower $\tilde{n}^+$ mean less pressure from emissions growth, which reduces the population externality.

The base case sets underlying trends in population and technology growth to actual values over 1990-2005, $n^+ = 1.52$ and $\lambda/\alpha = 1.13$, so $\tilde{n}^+ = 1.72$. To evaluate sensitivity to $n^+ = 1.52$, we recalibrate the model to an assumed value $n^+ = 1$, which assumes world population would remain constant absent policy.[43] Then $\tilde{n}^+ = 1.13$, as the only force driving emissions growth is technological change in a person's carbon

[42] In the effective-backstop case with $f(0) = 0.6$, both $\tilde{N}_{ss}$ and $\tilde{N}^*_{ss}$ go to infinity but their ratio converges to about 3.4. Also, population growth remains positive as the emissions ratio converges to zero: $\tilde{n}_t$ converges to 1.16 and actual (not adjusted) population growth $n_t = \tilde{n}_t(\alpha/\lambda)$ converges to 1.03. In the limit, n^*_t converges to the same limit of 1.03. This is substantially lower than actual population growth of $n^+ = 1.52$ in the uncapped economy.

[43] The United Nations (2010) median-fertility population projection assumes fertility in all countries is close to replacement ($n^+ \approx 1$), and finds that world population should nonetheless continue to grow until about 2100. The U.N.'s methodology generally is to take fertility trends as exogenously given and add assumptions about other factors such as mortality to derive population figures. Importantly, U.N. fertility trends are assumed and not determined from analysis of fundamentals.

footprint λ/α. In table 1, the implied population externality is 5.1 percent of parental income per child, lower than in the base case but not negligible.

This scenario effectively assumes poor countries reduce fertility to replacement without any increase in incomes and hence emissions. If an emissions increase is needed to bring fertility to replacement and total emissions are to be held constant, then the cap would have to be a cut equal in percentage to the necessary emissions increase. This would raise the population externality in the transition to steady state. For example, a 25 percent emissions cut given $n^+ = 1$ would result in the time path of taxes in the last row of table 2: the tax/externality in period zero would be 7.1 percent, which then falls as time passes to the steady-state value of 5.1 percent.

Because $\tilde{n}^+ = n^+\lambda/\alpha$, the same 5.1 percent steady-state externality would arise if assumed fertility (n^+) were left at the base case value of 1.52 but the environmental productivity-growth factor α were instead assumed to increase permanently from 1.48 to $2.25 = 1.48 \cdot 1.52$.

CONCLUSION

In thinking about population policy, it is useful to know the magnitudes of population externalities as these equal corrective Pigovian taxes. We consider a specific case: the size of the population externality induced by environmental limits on greenhouse gas emissions. We take as a maintained hypothesis that global warming turns out to be a serious problem so a cap is imposed, and that the problem does not go away because of a magic-bullet technology innovation, a permanent change in technology trends that makes emissions vanish asymptotically, or a permanent decline in fertility to less than replacement. The lowest population externality we calculate is 5 percent of a parent's lifetime income per child, which is large enough to matter in policy discussions.

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Appendix–not for publication, to be available online

This appendix provides additional derivations and details. Claims made in the text and footnotes are stated here as **Propositions**. Additional results are stated as **Remarks**. **Lemmas** are intermediate results. Throughout, parameters (θ, b) are admissible if they satisfy either $0 < \theta < 1$ and $0 < b < 1$, or $\theta > 1$ and $b > 1$.

Power β (Section I)

We claimed that power β is equivalent to assuming the utility an adult derives from grandchildren, $\beta(n_t)\beta(n_{t+1})U_{t+2}$, is independent of the number of children.

Proof: Independence implies $\beta(n_t)\beta(n_{t+1}) = \beta(1)\beta(n_t n_{t+1})$. Differentiating this with respect to n_t and n_{t+1} yields $n_t\beta'(n_t)/\beta(n_t) = n_{t+1}\beta'(n_{t+1})/\beta(n_{t+1})$ for any $n_t > 0$ and $n_{t+1} > 0$, so $n\beta'(n)/\beta(n)$ is a constant, denoted $1 - b$. The solution to the differential equation $\beta'(n)/\beta(n) = (1 - b)/n$ is $\beta(n) = b_0 n^{1-b}$. Conversely, $\beta(n_t) = b_0 n_t^{1-b}$ implies $\beta(n_t)\beta(n_{t+1}) = b_0 n_t^{1-b} b_0 n_{t+1}^{1-b} = \beta(1)\beta(n_t n_{t+1})$. QED.

Natural Population in the Cap Era (Section III)

Proposition 1 (Steady state exists and is unique): $S(1, e)$ crosses zero exactly once on $[f^{-1}(\chi), e^+]$.

Proof: From

$$S(n, e) = -u'(f(e) - \chi n)\chi + \frac{\beta'(n)}{1 - \beta(n)} u(f(e) - \chi n),$$

$S(1, e) \to -\infty$ as $e \to f^{-1}(\chi)$. Because S is continuous and $S(1, e^+) > 0$, there is at least one value on $(f^{-1}(\chi), e^+)$ at which $S(1, e) = 0$. The partial derivative of S is

$$\frac{\partial S(n, e)}{\partial e} = \left[-u''\chi + \frac{\beta'}{1 - \beta} u'\right] f' = \left[-\frac{u''\chi}{u'} + \frac{\beta'}{1 - \beta}\right] u' f'.$$

When $\theta < 1$ and $\beta' > 0$, we have $\partial S(1, e)/\partial e > 0$ for $e \in (f^{-1}(\chi), e^+)$, so $S(1, e)$

crosses zero only once. When $\theta > 1$, power utility and $n = 1$ imply that

$$-\frac{u''\chi}{u'} + \frac{\beta'}{1-\beta} = \frac{\theta\chi}{f(e)-\chi} + \frac{\beta'(1)}{1-\beta(1)}$$

approaches $+\infty$ as $e \to f^{-1}(\chi)$ and decreases strictly in e. Thus either $\partial S(1,e)/\partial e$ is strictly positive for all $e \in (f^{-1}(\chi), e^{+})$, in which case $S(1,e)$ crosses zero only once, or $\partial S(1,e)/\partial e$ changes sign once at a value $e^0 < e^+$, from positive for $e < e^0$ to negative for $e > e^0$. In the latter case, $S(1,e) \geq S(1,e^+) > 0$ for $e \geq e^0$ so $S(1,e)$ cannot cross zero on $[e^0, e^+]$, which means it crosses on $(f^{-1}(\chi), e^0)$. Because $\partial S(1,e)/\partial e > 0$ on $(f^{-1}(\chi), e^0)$, the crossing is unique. QED.

Proposition 2 (Convergence to steady state): Assume

$$\varepsilon_{n_t,y_t}\left(\frac{f'(e_t)e_t}{f(e_t)}\right) < 1 \text{ where } \varepsilon_{n_t,y_t} = \left(\frac{b}{\theta}\frac{c_t}{f(e_t)} + \frac{\chi n_t}{f(e_t)}\right)^{-1}. \tag{A.1}$$

(a) (A.1) at the steady state is necessary and sufficient for the system of difference equations in (U, N) to have two strictly positive real roots that straddle one; and (b) (A.1) for all t along the saddle path implies that population converges to N_{ss} and that convergence is monotone.

Proof: (a) The characteristic roots of the system

$$U_t = u(f(e(N_t)) - \chi\frac{N_{t+1}}{N_t}) + \beta(\frac{N_{t+1}}{N_t})U_{t+1}, \tag{A.2}$$

$$V_n(t) = \beta'\left(\frac{N_{t+1}}{N_t}\right)U_{t+1} - u'\left(f(e(N_t)) - \frac{N_{t+1}}{N_t}\chi\right)\chi = 0 \tag{A.3}$$

are obtained by differentiating the system at (N_{ss}, U_{ss}). Differentiate $V_n(t)$ to obtain

$$-\beta'(n_t)dU_{t+1} - V_{nn}n_t\frac{dN_{t+1}}{N_{t+1}} = (1 + \frac{V_{ny}}{V_{nn}n_t}f'(e_t)e_t)(-V_{nn})n_t\frac{dN_t}{N_t}. \tag{A.4}$$

Define $z_{0t} \equiv -\frac{V_{ny}}{V_{nn}n_t} f'(e_t)e_t = \epsilon_{n,y}\frac{f'(e_t)e_t}{f(e_t)}$ and $z_{1t} \equiv 1 - z_{0t}$, where $z_{0t} = 0$ for $N_t \leq \hat{E}/e^+$ and $z_{0t} > 0$ for $N_t > \hat{E}/e^+$. Define $z_{2t} \equiv \frac{\beta'(n_t)}{(-V_{nn})n_t}$, which is positive when $\theta < 1$ and negative when $\theta > 1$. Then (A.4) can be written

$$-z_{2t}dU_{t+1} + \frac{dN_{t+1}}{N_{t+1}} = z_{1t}\frac{dN_t}{N_t}. \quad \text{(A.5)}$$

Similarly differentiate $U_t = u(f(e(N_t)) - \chi\frac{N_{t+1}}{N_t}) + \beta(\frac{N_{t+1}}{N_t}))U_{t+1}$ to obtain

$$\begin{aligned} dU_t &= u'(c_t)\left[-f'(e_t)e_t\frac{dN_t}{N_t} - \chi n_t^2(\frac{dN_{t+1}}{N_{t+1}} - \frac{dN_t}{N_t})\right] \\ &\quad +\beta(n_t)dU_{t+1} + U_{t+1}\beta'(n_t)n_t(\frac{dN_{t+1}}{N_{t+1}} - \frac{dN_t}{N_t}). \end{aligned}$$

Using (A.3), this can be written

$$\beta(n_t)dU_{t+1} = dU_t + z_{3t}\frac{dN_t}{N_t}, \quad \text{(A.6)}$$

where $z_{3t} \equiv u'(c_t)f'(e_t)e_t \geq 0$.

The system consisting of (A.5) and (A.6):

$$\begin{pmatrix} -z_{2t} & 1 \\ \beta(n_t) & 0 \end{pmatrix} \begin{pmatrix} dU_{t+1} \\ \frac{dN_{t+1}}{N_{t+1}} \end{pmatrix} = \begin{pmatrix} 0 & z_{1t} \\ 1 & z_{3t} \end{pmatrix} \begin{pmatrix} dU_t \\ \frac{dN_t}{N_t} \end{pmatrix},$$

has roots μ that satisfy the characteristic equation

$$\left| \begin{pmatrix} -z_{2t}\mu & \mu - z_{1t}) \\ \beta(n_t)\mu - 1 & -z_{3t} \end{pmatrix} \right| = 0,$$

which implies

$$\mu_{1t,2t} = \frac{1}{2}(Z_t + 1) \pm \sqrt{D_t},$$

where $z_{4t} = \frac{z_{2t}z_{3t}}{\beta(n_t)}$, $Z_t = \frac{1}{\beta(n_t)} + z_{4t} - z_{0t}$, and $D_t = \frac{1}{4}(Z_t + 1)^2 - \frac{z_{1t}}{\beta(n_t)}$. Because $(Z_t + 1)^2 = (Z_t - 1)^2 + 4Z_t$, it follows that

$$D_t = \frac{1}{4}(Z_t - 1)^2 + z_{4t} + \frac{(1 - \beta(n_t))}{\beta(n_t)}z_{0t}. \quad \text{(A.7)}$$

Moreover,

$$z_{4t} = \frac{z_{2t}z_{3t}}{\beta(n_t)} = \frac{\beta'(n_t)u'f'e_t}{(-V_{nn}n_t)\beta(n_t)} = \frac{\beta'(n_t)u'}{V_{ny}\beta(n_t)}z_{0t} = \frac{1-b}{\theta}\frac{f(e_t)/n_t - \chi}{\chi}z_{0t},$$

using $V_{ny} = -u''\chi = \frac{u'\chi}{\theta c_t}$.

In steady state, $S(1, e_{ss}) = 0$ with power utility implies $(1-\theta)\chi = (1-b)\frac{\beta(1)}{1-\beta(1)}(f(e_{ss}) - \chi)$. Hence

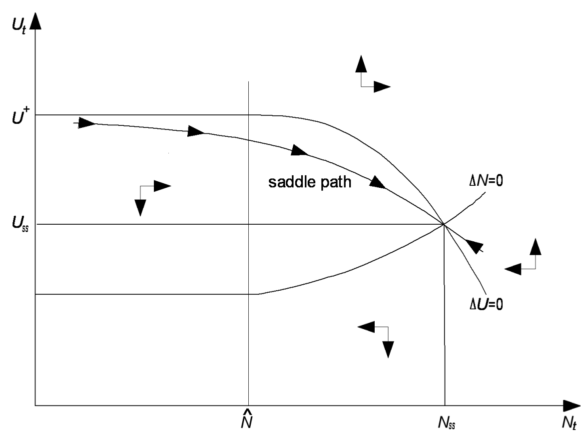
$$z_{4ss}\frac{\beta(1)}{1-\beta(1)} + z_{0ss} = \frac{z_{0ss}}{\theta\chi}[\chi\theta - (b-1)\frac{\beta(1)}{1-\beta(1)}(f(e_{ss}) - \chi)] = \frac{z_{0ss}}{\theta} > 0,$$

where $z_{0ss} > 0$ because $N_{ss} > \hat{E}/e^+$. Thus $D_{ss} = \frac{1}{4}(Z_{ss} - 1)^2 + \frac{(1-\beta(1))}{\beta(1)}\frac{z_{0ss}}{\theta} > 0$. This implies that both roots are real and that $\sqrt{D_{ss}} > \frac{1}{2}|Z_{ss} - 1|$. Hence $\mu_{1ss} = \frac{1}{2}(Z_{ss}+1) - \sqrt{D_{ss}} < \min(Z_{ss}, 1) \leq 1$ and $\mu_{2ss} = \frac{1}{2}(Z_{ss}+1) + \sqrt{D_{ss}} > \max(Z_{ss}, 1) \geq 1$. Also, $z_{4ss} = \frac{1-\beta(1)}{\beta(1)}(\frac{z_{0ss}}{\theta} - z_{0ss})$, which implies

$$Z_{ss} = \frac{1}{\beta(1)} + z_{4ss} - z_{0ss} = \frac{z_{1ss}}{\beta(1)} + \frac{z_{0ss}}{\theta}\frac{1-\beta(1)}{\beta(1)} \geq \frac{z_{1ss}}{\beta(1)}.$$

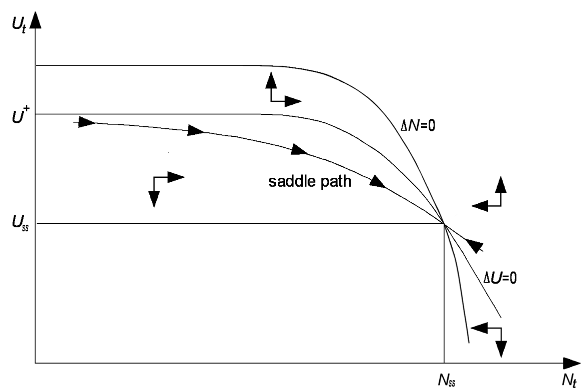
If (A.1) holds at the steady state, then $z_{1ss} > 0$ so $Z_{ss} > 0$, and $\sqrt{D_{ss}} < \frac{1}{2}(Z_{ss} + 1)$ so $\mu_{1ss} > 0$. Thus the roots μ_{1ss} and μ_{2ss} are real, strictly positive, and straddle one, proving sufficiency of (A.1). If (A.1) does not hold at the steady state, which means $\varepsilon_{n_t,y_t}\left(\frac{f'(e_t)e_t}{f(e_t)}\right) \geq 1$, then $z_{1ss} \leq 0$ and hence $\sqrt{D_{ss}} \geq \frac{1}{2}|Z_{ss} + 1|$, which implies $\mu_{1ss} \leq 0$. Thus (A.1) is also necessary for strictly positive roots.

(b) Global convergence is conveniently established using phase diagrams. The phase diagram when $\theta < 1$ is:



Saddle-path stability and a unique steady state follow from Proposition 1 and Proposition 2(a). Setting $U_{t+1} = U_t$ in (A.2) yields the phase arm labelled $\Delta U = 0$ that is horizontal at utility $U = U^+$ for $N \leq \hat{N} \equiv \hat{E}/e^+$ and slopes downward for $N > \hat{E}/e^+$. Setting $N_{t+1} = N_t$ in (A.3) and using (A.2) to substitute out U_{t+1} yields the phase arm labelled $\Delta N = 0$ that is horizontal at utility $U < U^+$ for $N \leq \hat{E}/e^+$ and cuts $\Delta U = 0$ from below at N_{ss}. (Note: for $N > \hat{E}/e^+$, the arm $\Delta N = 0$ can slope upward, as shown, or downward, but has less negative slope than $\Delta U = 0$, so the intersection is unique.) It is straightforward to show that the phase arrows are as shown, so there is a unique saddle path with strictly negative slope.

The phase diagram when $\theta > 1$ is:



The phase arm $\Delta U = 0$ has the same properties as when $\theta < 1$. However, because fertility declines in U, the arm $\Delta N = 0$ lies above $\Delta U = 0$ for $N \leq \hat{E}/e^+$, then declines and cuts $\Delta U = 0$ from above at N_{ss}. Inspecting the phase arrows, there is a unique saddle path with strictly negative slope that lies below $\Delta U = 0$ for $N < N_{ss}$ and above $\Delta U = 0$ for $N > N_{ss}$.

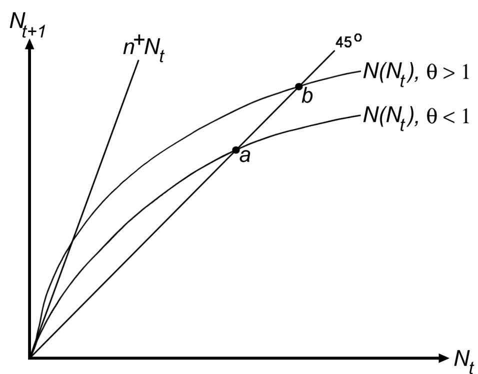
Let $U_t = U_{sp}(N_t)$ denote utilities on the saddle path and let $\nu(y, U)$ denote the fertility that solves the individual first-order condition $V_n(n, y, U) = 0$. Note that equilibrium fertility satisfies $\eta(N_t|\hat{E}) = \nu(f(e(N_t)), U_{sp}(N(N_t))$ for all N_t, where $N_{t+1} = N(N_t) = N_t\eta(N_t|\hat{E}))$. By the implicit function theorem,

$$\frac{dN_{t+1}}{dN_t} = \eta(N|\hat{E})(1 - \epsilon_{n,y}\frac{f'(e_t)e_t}{f(e_t)}) + \nu_U NU'(N)\frac{dN_{t+1}}{dN_t}, \text{ so}$$

$$\frac{dN_{t+1}}{dN_t} = \frac{\eta(N_t|\hat{E})}{1 - \nu_U N_t U'(N_t)}(1 - \epsilon_{n,y}\frac{f'(e_t)e_t}{f(e_t)}), \quad \text{(A.8)}$$

where saddle-path stability implies $1-\nu_U N_t U'(N_t) > 0$. Hence (A.1) implies $N'(N_t) = \frac{dN_{t+1}}{dN_t} > 0$. Because N_{ss} is stable and unique, $N(N_t)$ is upward sloping and crosses the 45-degree line at N_{ss} with slope $N'(N_{ss}) < 1$, which implies monotone convergence. QED.

Remark: The following diagram of N_t against N_{t+1} provides additional intuition about population dynamics and motivation for assumption (A.1):



Population dynamics in the no-cap era are on the ray with slope $n^+ > 1$. Steady states in the cap era are on the 45-degree line. The slope of $N(N_t)$ is given by (A.8).

When $\theta < 1$, $\nu_U > 0$ and $U'(N) < 0$ imply $0 < \frac{1}{1-N\nu_U U'(n)} < 1$. Hence $N(N_t)$ has an elasticity $\epsilon_{N_t N_{t+1}} \equiv N'(N_t)\frac{N_t}{N_{t+1}} < 1 - \epsilon_{n,y}\frac{f'(e_t)e_t}{f(e_t)} < 1$ for all N_t. The function starts out nearly parallel to the ray with slope n^+, then bends down until its meets the 45-degree line at point a. Assumption (A.1) implies $1 - \epsilon_{n,y}\frac{f'(e_t)e_t}{f(e_t)} > 0$, which ensures that $N'(N) > 0$ and hence the population dynamics obtained by iterating on $N_{t+1} = N(N_t)$ are monotone.

When $\theta > 1$, $\nu_U < 0$ and $U'(N) < 0$ imply $\frac{1}{1-N\nu_U U'(n)} > 1$. For $N_t \leq \hat{E}/e^+$, $\frac{f'e}{f} = 0$ implies $\epsilon_{N_t N_{t+1}} > 1$. Hence the $N(N_t)$ has a segment bending up above the uncapped path before bending down to meet the 45-degree line at b. Assumption (A.1) again implies $N'(N) > 0$ and hence monotone dynamics.

The Optimal Population (Section IV)

The claims in the text are proved as follows. Existence of a solution to the transformed problem is shown in Proposition 4(a). Necessity of $S^*(e^*_{ss}) = 0$ as the steady-state condition is in Proposition 6(a). For $\omega \leq 1$: properties of V° and H are in Proposition 4(b); uniqueness of N^*_{ss} and e^*_{ss} are in Proposition 6(b); and sufficiency of (A.1) for convergence of the optimal population to N^*_{ss} is in Proposition 7(b). When both $1/\omega > 1 - \varepsilon_{n_t,y_t}\kappa(e)$ and (A.1) hold: uniqueness of N^*_{ss} and e^*_{ss} is in Proposition 6(c); convergence of the optimal population to N^*_{ss} is in Proposition 7(b); and properties of V° and H are in Proposition 7(c).

The proofs below use Alvarez' (1999) transformation of the optimal population problem with the following notation: Define $N^{\max}(N_t) \equiv \frac{1}{\chi} N_t f((e(N_t))$ and $\Gamma(N_t) \equiv \{N_{t+1} \in [0, N^{\max}(N_t)]\} = \{N_{t+1} \in [0, \frac{1}{\chi} N_t f((e(N_t))]\}$. Because $c_t \geq 0$ and $N_t \geq 0$ imply $N_{t+1} \leq N^{\max}(N_t)$, the set of feasible choices for N_{t+1} is $\Gamma(N_t)$. Using, $u^\circ(N_t, N_{t+1}) \equiv \beta(N_t) u(f(e(N_t)) - \chi N_{t+1}/N_t)$, the Bellman equation for transformed problem can then be written

$$V^\circ(N_t) = \max_{N_t \in \Gamma(N_t)} \{u^\circ(N_t, N_{t+1}) + b_0 V^\circ(N_{t+1})\}. \tag{A.9}$$

The optimal policy is denoted $H(N_t)$; dependence of H on $\hat{E}$ is suppressed to lighten the notation. Also, we sometimes work with $\varpi \equiv \frac{\theta - b}{1-\theta} = \frac{1}{\omega} - 1$. Note that $\varpi \geq 0$ if and only if $\omega \leq 1$.

Remark: It would be technically convenient to restrict attention to (θ, b) such that $\omega = \frac{1-\theta}{1-b} \leq 1$ because, as shown below, this implies that V° is strictly concave everywhere. The assumption $\omega \leq 1$ is restrictive, however, so we do not impose it. Instead, we rely on conditions involving the curvature of f and on the elasticity condition (A.1); the latter is needed for convergence even if one assumes $\omega \leq 1$.

As preliminary step to deriving properties of V°, we show:

Lemma 3 (Properties of u°): (a) u° is twice continuously differentiable for all admissible (θ, b), except that $\frac{\partial^2 u^\circ}{\partial N_t^2}$ may have a discontinuity at $N_t = \hat{E}/e$.

(b) $u^\circ_{12}(N_t, N_{t+1}) = \frac{\partial^2 u^\circ(N_t, N_{t+1})}{\partial N_{t+1} \partial N_t} > 0$ if and only if (A.1) holds at (N_t, N_{t+1}), where $\varepsilon_{n_t, y_t} = \left(\frac{b}{\theta} \frac{f(e(N_t)) - \chi N_{t+1}/N_t}{f(e(N_t))} + \frac{\chi N_{t+1}/N_t}{f(e(N_t))} \right)^{-1}$.

(c) u° is strictly concave for $\omega < 1$ and u° is concave for $\omega = 1$.

(d) u° is strictly concave on

$$X_{con} \equiv \{(N_t, N_{t+1}) : \varpi > -\varepsilon_{n_t, y_t} \kappa(e(N_t))\},$$

where $\kappa(e) \equiv -\frac{f''(e)e^2}{f(e)} > 0$ for $e \in (0, e^+)$ and $\kappa(e^+) \equiv 0$.

Proof: (a) Taking derivatives of u°:

$$\begin{aligned} \frac{\partial u^\circ}{\partial N_t} &= [\varpi c_t + w(e(N_t))]\, B(N_t, N_{t+1}), \\ \frac{\partial u^\circ}{\partial N_{t+1}} &= -\chi B(N_t, N_{t+1}), \end{aligned}$$

where $B = B(N_t, N_{t+1}) \equiv \frac{\beta(N_t)}{N_t} u'(c_t) > 0$, $c_t = f(e(N_t)) - \chi N_{t+1}/N_t$, and $w(e_t) = f(e_t) - f'(e_t)e_t$. Note that $\frac{\partial u^\circ}{\partial N_t}$ is continuous at $N_t = \hat{E}/e^+$ because $\lim_{N_t \downarrow \hat{E}/e^+} w(e(N_t)) = f(e^+)$ equals $w(e(N_t)) = f(e^+)$ for $N_t \leq \hat{E}/e^+$.

Taking derivatives of $\frac{\partial u^\circ}{\partial N_{t+1}}$:

$$\begin{aligned} \frac{\partial^2 u^\circ}{\partial N_{t+1} \partial N_t} &= \chi \frac{\theta B}{N_t} \left(\frac{w_t}{c_t} - 1 + \frac{b}{\theta} \right), \qquad \text{(A.10)} \\ \frac{\partial^2 u^\circ}{\partial N_{t+1}^2} &= -\chi^2 \frac{\theta B}{N_t c_t} < 0. \end{aligned}$$

To find $\frac{\partial^2 u^\circ}{\partial N_t^2}$, note that $\frac{d}{dN_t} w(e(N_t)) = 0$ for $N_t < \hat{E}/e^+$ and $\frac{d}{dN_t} w(e(N_t)) = -w'(e(N_t)) \frac{e(N_t)}{N_t}$ for $N_t > \hat{E}/e^+$, where $w'(e) = \frac{\partial [f(e) - f'(e)e]}{\partial e} = f'(e) - f''(e)e - f'(e) = -f''(e)e > 0$ for

$e < e^+$. Because $\lim_{N_t \downarrow \hat{E}/e^+} \frac{d}{dN_t} w(e(N_t)) \neq 0$ unless $\lim_{e \to e^+} f''(e) = 0$ (which we do not impose), $w(e(N_t))$ is not generally differentiable at $N_t = \hat{E}/e^+$. For $N_t \neq \hat{E}/e^+$, differentiating $\frac{\partial u^\circ}{\partial N_t}$ yields

$$\begin{aligned}\frac{\partial^2 u^\circ}{\partial N_t^2} &= -w'(e(N_t))\frac{e(N_t)}{N_t}B - \theta\frac{Bc_t}{N_t}\left[\left(\frac{w_t}{c_t} - 1 + \frac{b}{\theta}\right)^2 + \frac{b}{\theta^2}\varpi\right] \\ &= \frac{B}{N_t}\left[\frac{bc_t}{\theta N_t}\varpi - \frac{d}{dN_t}w(e(N_t))\right] - \theta\frac{Bc_t}{N_t}\left(\frac{w_t}{c_t} - 1 + \frac{b}{\theta}\right)^2,\end{aligned}$$

showing (a) by construction.

(b) This follows from (A.10) because $\chi\frac{\theta B}{N_t} > 0$ and because $\frac{w_t}{c_t} - 1 + \frac{b}{\theta} > 0$ is algebraically equivalent to (A.1).

(c) For $N_t \neq \hat{E}/e^+$:

$$\begin{aligned}Det(u^\circ) &\equiv \frac{\partial^2 u^\circ}{\partial N_t^2}\frac{\partial^2 u^\circ}{\partial N_{t+1}^2} - \left(\frac{\partial^2 u^\circ}{\partial N_{t+1}\partial N_t}\right)^2 \\ &= \left[\frac{bc_t}{\theta N_t}\varpi - \frac{d}{dN_t}w(e(N_t))\right]\left[\chi^2\frac{\theta B^2}{N_t c_t}\right].\end{aligned}$$

Note that $\frac{bc_t}{\theta N_t}\varpi > -\frac{d}{dN_t}w(e(N_t))$ implies $Det(u^\circ) > 0$ and $\frac{\partial^2 u^\circ}{\partial N_t^2} < 0$. Hence u° is strictly concave locally if $\frac{bc_t}{\theta N_t}\varpi > -\frac{d}{dN_t}w(e(N_t))$.

For global concavity of u°, a complication is that $\frac{d}{dN_t}w(e(N_t))$ may not exist at $N_t = \hat{E}/e^+$. To show concavity on a set, one must show that

$$\begin{aligned}&\lambda u^\circ(N_t^1, N_{t+1}^1) + (1-\lambda)u^\circ(N_t^2, N_{t+1}^2) \\ \leq\ & u^\circ(\lambda N_t^1 + (1-\lambda)N_t^2, N_{t+1}^1 + (1-\lambda)N_{t+1}^2)\end{aligned}$$

for any $(N_t^1, N_{t+1}^1) \neq (N_t^2, N_{t+1}^2)$ in this set and for any $\lambda \in (0,1)$, with strict inequality required for strict concavity.

Concavity of F and constant returns to scale imply that $N_t f(e(N_t)) = F(N_t, \min(e^+ N_t, \hat{E})$ is concave in N_t. For given (N_t^1, N_{t+1}^1) and (N_t^2, N_{t+1}^2) with $N_t^1 < N_t^2$, define

$$\bar{F}(N) \equiv N_t^1 f(e(N_t^1)) + \frac{N_t^2 f(e(N_t^2)) - N_t^1 f(e(N_t^1))}{N_t^2 - N_t^1}(N - N_t^1).$$

Note that $\bar{F}(N_t^1) = F(N_t^1)$, $\bar{F}(N_t^2) = F(N_t^2)$, and, by concavity of F, $\bar{F}(N) \leq Nf(e(N))$ for all $N \in [N_t^1, N_t^2]$. Also define

$$\bar{u}(N_t, N_{t+1}) \equiv \beta(N_t)u\left(\frac{1}{N_t}(\bar{F}(N_t) - \chi N_{t+1})\right),$$

and note that $\bar{u}(N_t^1, N_{t+1}^1) = u^\circ(N_t^1, N_{t+1}^1)$, $\bar{u}(N_t^2, N_{t+1}^2) = u^\circ(N_t^2, N_{t+1}^2)$, and

$$\begin{aligned} & \bar{u}(\lambda N_t^1 + (1-\lambda)N_t^2, N_{t+1}^1 + (1-\lambda)N_{t+1}^2) \\ \leq\ & u^\circ(\lambda N_t^1 + (1-\lambda)N_t^2, N_{t+1}^1 + (1-\lambda)N_{t+1}^2). \end{aligned}$$

Because $\bar{F}(N)$ is differentiable for all $N \in [N_t^1, N_t^2]$, $\bar{u}$ is twice continuously differentiable. Taking derivatives, $\frac{\partial^2 \bar{u}}{\partial N_t^2} = -\theta \frac{Bc_t}{N_t}\left[\left(\frac{w_t}{c_t} - 1 + \frac{b}{\theta}\right)^2 + \frac{b}{\theta^2}\varpi\right]$ and $Det(\bar{u}) = \left[\frac{bc_t}{\theta N_t}\varpi\right]\left[\chi^2 \frac{\theta B^2}{N_t c_t}\right]$, so $\bar{u}$ is concave everywhere for $\varpi \geq 0$ and is strictly concave for $\varpi > 0$. Hence for all $N \in [N_t^1, N_t^2]$ and all $\varpi \geq 0$,

$$\begin{aligned} & \lambda u^\circ(N_t^1, N_{t+1}^1) + (1-\lambda)u^\circ(N_t^2, N_{t+1}^2) \\ =\ & \lambda \bar{u}(N_t^1, N_{t+1}^1) + (1-\lambda)\bar{u}(N_t^2, N_{t+1}^2) \\ \leq\ & \bar{u}(\lambda N_t^1 + (1-\lambda)N_t^2, N_{t+1}^1 + (1-\lambda)N_{t+1}^2) \\ \leq\ & u^\circ(\lambda N_t^1 + (1-\lambda)N_t^2, N_{t+1}^1 + (1-\lambda)N_{t+1}^2), \end{aligned}$$

where the first inequality follows from concavity of $\bar{u}$ and the second inequality follows from $\bar{F}(N) \leq Nf(e(N))$. Thus for all $\varpi \geq 0$, u° is globally concave. For $\varpi > 0$, the first inequality above is strict by strict concavity of $\bar{u}$, so u° is strictly concave. This proves (c).

(d) This is trivial for $\varpi > 0$ because $-\varepsilon_{n_t,y_t}\kappa(e(N_t)) \leq 0$. For $\varpi \leq 0$, $\kappa(e(N_t)) = 0$ for $N_t \leq \hat{E}/e^+$ implies $X_{con} \subset \{(N_t, N_{t+1}) : N_t > \hat{E}/e^+\}$, so $\frac{d}{dN_t}w(e(N_t))$ exists. Note that $0 < \frac{b}{\theta}\frac{c_t}{f(e(N_t))} < \varepsilon_{n_t,y_t}^{-1}$, so $\varpi > -\varepsilon_{n_t,y_t}\kappa(e(N_t))$ implies $\varpi > -\kappa(e(N_t))/\left(\frac{b}{\theta}\frac{c_t}{f(e(N_t))}\right) = -\frac{d}{dN_t}w(e(N_t))/\left(\frac{bc_t}{\theta N_t}\right)$ and $\frac{bc_t}{\theta N_t}\varpi > -\frac{d}{dN_t}w(e(N_t))$, which implies strict concavity of u°, proving (d). QED.

Proposition 4 (Value function V°): In the optimal population problem of section IV:

(a) For any admissible (θ, b), there is a unique continuous function V° that solves (A.9) for all $N_t \in (0, \infty)$. The associated optimal policy correspondence H is compact-valued and upper hemi-continuous (u.h.c.) on any compact subset of $(0, \infty)$. Moreover, H also solves the Bellman equation defining V^*.

(b) For $\omega \leq 1$, V° is strictly concave and differentiable, and H is single-valued and continuous.

Remark: Because $N_0 = 0$ results trivially in a zero population sequence, we consider only cases with $N_t > 0$. Note that H is generally a correspondence; in the text we only consider cases where H reduces to a function.

Proof: (a) By construction, $\Gamma(N_t)$ is non-empty, compact, and increasing in N_t for all N_t. Note that $N^{\max}(N_t) \leq \hat{E}/f^{-1}(\chi)$ for $N_t \leq \hat{E}/f^{-1}(\chi)$, and $N^{\max}(N_t) < N_t$ for $N_t > \hat{E}/f^{-1}(\chi)$, so population is bounded in $X = [0, \max(N_0, \hat{E}/f^{-1}(\chi))]$ for all t, and $\Gamma(X) \subset X$ is compact. We consider the cases with $\theta < 1$ and $\theta > 1$ separately because when $\theta > 1$, u° is unbounded as $c_t \to 0$.

When $\theta < 1$, consider domain $X = [0, N^{\max}]$ for any $N^{\max} \geq \max(N_0, \hat{E}/f^{-1}(\chi))]$. Because $b_0 < 1$ and $0 \leq u^{\circ}(N_t, N_{t+1}) \leq u^{\circ}(N^{\max}, 0)$ is bounded for all $(N_t, N_{t+1}) \in X \times X$, the Contraction Mapping Theorem implies that a continuous V° exists and that H is compact-valued and u.h.c.

When $\theta > 1$, consider domain $X = [\epsilon, N^{\max}]$ with arbitrary $0 \leq \epsilon < \min(N_0, \frac{\hat{E}/e^+}{f(e^+)/\chi}) < \hat{E}/e^+$. Define $\Gamma_{\epsilon\alpha}(N_t) \equiv \{N_{t+1} \in [\epsilon, N^{\max}(N_t) - \alpha N_t]\}$ with $0 < \alpha < N^{\max}(\epsilon) - \epsilon$. Then: N_{t+1} is bounded away from zero and from $N^{\max}(N_t)$, so u° is bounded; $\Gamma_{\epsilon\alpha}(N_t)$ is non-empty and compact; and $\Gamma_{\epsilon\alpha}(X) \subset X$, which implies the existence of V° and H. For sufficiently small α and ϵ, the choice of $N_{t+1} \in \Gamma(N_t)$ yields optimal values in the interior of $\Gamma_{\epsilon\alpha}(N_t)$ so V° and H do not depend on α and ϵ. The Theorem of the Maximum implies that H is compact-valued and u.h.c.

For any N_t, the Bellman equation for $V^*(N_t)$ is equivalent to the Bellman equation for $V^\circ(N_t)$ multiplied by the exogenous factor $1/\beta(N_t) > 0$. Hence N_{t+1} solves the former if and only if $N_{t+1} \in H(N_t)$, which means H is the optimal policy associated with V^*.

(b) For $\omega < 1$, Lemma 3(c) proves strict concavity of u°, so the claims in (b) follow by standard arguments (e.g., see Stokey, Lucas, and Prescott, 1989). The case $\omega = 1$ requires a more detailed argument. First, Lemma 3(d) with $\varpi = 0 > -\varepsilon_{n_t,y_t}\kappa(e(N_t))$ for $N_t > \hat{E}/e^+$ implies strict concavity of u° on $\bar{X}_{con} = \{(N_t, N_{t+1}) : N_t \geq \hat{E}/e^+\}$, which is a convex set. Hence there is a strictly concave value function V° that solves (A.9) on $\bar{X}_{con}$ with the restriction $N_{t+1} \in \Gamma(N_t) \cap \bar{X}_{con}$. Second, for $N_t = \hat{E}/e^+$, an argument by contradiction shows that $H(\hat{E}/e^+) > \hat{E}/e^+$: otherwise $H(\hat{E}/e^+) = \hat{E}/e^+$ so $\hat{E}$ would not constrain emissions, which would make $n^+ > 1$ optimal. Third, $\frac{\partial^2 u^\circ}{\partial N_{t+1}\partial N_t} > 0$ holds for all N_t, because (A.1) is satisfied for $\theta = b$, so Lemma 3(b) applies. Then the first-order condition $\frac{\partial u^\circ(N_t,N_{t+1})}{\partial N_{t+1}} + b_0 V^{\circ\prime}(N_{t+1}) = 0$, which holds because V° is strictly concave and hence differentiable, implies that N_{t+1} is increasing in N_t. Thus $H(N_t) > \hat{E}/e^+$ for $N_t > \hat{E}/e^+$, so V° solves (A.9) on $\bar{X}_{con}$ without the restriction $N_{t+1} \in \Gamma(N_t) \cap \bar{X}_{con}$. Fourth, because u° is concave by Proposition 3(c), V° is concave for all N_t, which implies that $H(N_t)$ is convex. If $N_t \in H(N_t)$ for any $N_t \leq \hat{E}/e^+$, one would again obtain a contradiction because $\hat{E}$ would not constrain emissions, which would make $n^+ > 1$ optimal. If $N_{t+1} < N_t$ for any $N_{t+1} \in H(N_t)$ and $N_t \leq \hat{E}/e^+$, convexity of $H(N)$ implies that $N \in H(N)$ must apply for some $N \leq N_t$, again leading to a contradiction. Thus $N_{t+1} > N_t$ for all $N_{t+1} \in H(N_t)$ and all $N_t \leq \hat{E}/e^+$, which means population converges to a value in X_{con} from any starting value N_0. Concavity of u° implies strict concavity of $V^\circ(N_t)$ on any set such that $V^\circ(N_{t+1})$ is strictly concave, so by backward recursion, V° is strictly concave for all N_t. QED.

The following lemma is used to prove uniqueness of a steady state.

Lemma 5 (Condition for increasing H): If (A.1) holds on any interval $[N_t^1, N_t^2]$ with $0 < N_t^1 < N_t^2$, then the policy correspondence H is increasing in the sense that $\max H(N_t^1) \leq \min H(N_t^2)$.

Proof: Consider $N_{t+1}^1 = \max H(N_t^1)$ and $N_{t+1}^2 = \min H(N_t^2)$. Note that

$$u^\circ(N_t^1, N_{t+1}^1) + b_0 V^\circ(N_{t+1}^1) \geq u^\circ(N_t^1, N_{t+1}^2) + b_0 V^\circ(N_{t+1}^2)$$

because $N_{t+1}^1 \in H(N_t^1)$, and

$$u^\circ(N_t^2, N_{t+1}^1) + b_0 V^\circ(N_{t+1}^1) \leq u^\circ(N_t^2, N_{t+1}^2) + b_0 V^\circ(N_{t+1}^2)$$

because $N_{t+1}^2 \in H(N_t^2)$. Hence

$$\begin{aligned} u^\circ(N_t^1, N_{t+1}^1) - u^\circ(N_t^1, N_{t+1}^2) &\geq b_0 V^\circ(N_{t+1}^2) - b_0 V^\circ(N_{t+1}^1) \\ &\geq u^\circ(N_t^2, N_{t+1}^1) - u^\circ(N_t^2, N_{t+1}^2), \end{aligned}$$

which implies $u^\circ(N_t^2, N_{t+1}^2) - u^\circ(N_t^1, N_{t+1}^2) \geq u^\circ(N_t^2, N_{t+1}^1) - u^\circ(N_t^1, N_{t+1}^1)$ and

$$\Delta_{21} \equiv \int_{N_t^1}^{N_t^2} \left[u_1^\circ(N_t, N_{t+1}^2) - u_1^\circ(N_t, N_{t+1}^1)\right] dN_t \geq 0.$$

To establish a contradiction, suppose that $N_{t+1}^1 > N_{t+1}^2$. Because $\frac{\partial^2 u^\circ}{\partial N_{t+1} \partial N_t} > 0$ is continuous on a compact set, $M = \min_{[N_t^1, N_t^2] \times [N_{t+1}^2, N_{t+1}^1]} \frac{\partial^2 u^\circ}{\partial N_{t+1} \partial N_t}$ exists and $M > 0$. For any N_t, the mean-value theorem implies

$$\begin{aligned} u_1^\circ(N_t, N_{t+1}^1) - u_1^\circ(N_t, N_{t+1}^2) &= \frac{\partial^2 u^\circ}{\partial N_t \partial N_{t+1}}(N_t, N)(N_{t+1}^1 - N_{t+1}^2) \\ &\geq M(N_{t+1}^1 - N_{t+1}^2) \end{aligned}$$

for some $N \in [N_{t+1}^2, N_{t+1}^1]$. Therefore

$$\Delta_{12} \equiv \int_{N_t^1}^{N_t^2} \left[u_1^\circ(N, N_{t+1}^1) - u_1^\circ(N, N_{t+1}^2)\right] dN \geq M(N_{t+1}^1 - N_{t+1}^2)(N_t^2 - N_t^1) > 0,$$

which implies $\Delta_{21} = -\Delta_{12} < 0$, contradicting $\Delta_{21} \geq 0$. Hence $N^1_{t+1} \leq N^2_{t+1}$, which implies $\max H(N^1_t) \leq \min H(N^2_t)$. QED.

Proposition 6 (Steady states):

(a) For any admissible (θ, b), there is a steady-state population $N^*_{ss} \in H(N^*_{ss})$. Any steady state must satisfy $S^*(e(N^*_{ss})) = 0$ with

$$S^*(e) = (1 - b_0)S(1, e) - b_0 u'(f(e) - \chi)f'(e)e.$$

Moreover, $e^*_{ss} = e(N^*_{ss}) \in (e_{ss}, e^+)$, so $N^*_{ss} \in (\hat{E}/e^+, N_{ss})$.

(b) Sufficient conditions for a unique N^*_{ss} and a unique e^*_{ss} are that $\varpi \geq 0$ (equivalent to $\omega \leq 1$), or that $\varpi > \Phi(e(N_t))$ for all $N_t \in (\hat{E}/e^+, N_{ss})$, or that $\Phi(e)$ is strictly monotone (increasing or decreasing), where $\Phi(e) \equiv -\frac{f''(e)e}{f'(e)} > 0$.

(c) If (A.1) and $\varpi > -\varepsilon_{n_t,y_t}\kappa(e(N_t))$ apply for all $N_t \in [\min\{N : N \in H(N)\}, N_{ss}]$, then $N^*_{ss} = \min\{N : N \in H(N)\}$ is the unique steady state, and $e^*_{ss} = e(N^*_{ss})$.

Proof: (a) Define

$$S^*_u(e) \equiv \frac{S^*(e)}{u'(f(e) - \chi)} = b_0 \left(\frac{1 - b}{1 - \theta}(f(e) - \chi) + \chi - f'(e)e \right) - \chi.$$

Because $u' > 0$, $S^*(e(N^*_{ss})) = 0$ is equivalent to $S^*_u(e(N^*_{ss})) = 0$. Note that $V^\circ(N) \geq \frac{1}{1-b_0}u^\circ(N, N)$ for all N because $N \in \Gamma(N)$. Hence

$$\begin{aligned} V^\circ(N) &= \max_{N' \in \Gamma(N)} \{u^\circ(N, N') + b_0 V^\circ(N')\} \\ &\geq \max_{N_{ss} \in \Gamma(N)} \{u^\circ(N, N_{ss}) + \frac{b_0}{1 - b_0} u^\circ(N_{ss}, N_{ss})\} \equiv V^\circ_{ss}(N) \end{aligned}$$

for all N, where V°_{ss} can be interpreted as utility given a constant population starting one period ahead. Since u° is differentiable, the optimal N^*_{ss} must satisfy the first-order condition

$$u^\circ_2(N, N^*_{ss}) + \frac{b_0}{1 - b_0}\left[u^\circ_1(N^*_{ss}, N^*_{ss}) + u^\circ_2(N^*_{ss}, N^*_{ss})\right] = 0. \tag{A.11}$$

If $N^*_{ss} \in H(N^*_{ss})$ then $V^\circ(N^*_{ss}) = \frac{1}{1-b_0}u^\circ(N^*_{ss}, N^*_{ss})$, so $N_{ss} = N^*_{ss}$ must maximize $V^\circ_{ss}(N^*_{ss})$ and hence satisfies the first-order condition (A.11), which when $N = N_{ss} =$

N^*_{ss} reduces to $b_0 u_1^\circ(N,N) + u_2^\circ(N,N) = 0$, which in turn reduces to $b_0 \left[\varpi(f(e) - \chi) + (f(e) - f'(e)e)\right] - \chi = S_u^*(e) = 0$. Steady states $N \leq \hat{E}/e^+$ are ruled out by $S_u^*(1) > 0$; steady states with $N > \hat{E}/f^{-1}(\chi)$ are ruled out by $N \notin \Gamma(N)$ for $N > \hat{E}/f^{-1}(\chi)$; and $N = \hat{E}/f^{-1}(\chi)$ is ruled out because $u_1^\circ \to \infty$ as $N^*_{ss} \to N^{\max}(N^*_{ss})$ contradicts (A.11). Therefore $N^*_{ss} \in (\hat{E}/e^+, \hat{E}/f^{-1}(\chi))$.

Existence of N^*_{ss}: Because H is u.h.c., $h(N) \equiv \min_{N'}\{N' \in H(N)\}$ exists for all N. The set $X^{\leq}_{ss} \equiv \{N \geq \hat{E}/e^+ : h(N) \leq N\}$ is non-empty because $N^{\max}(\hat{E}/f^{-1}(\chi)) \leq \hat{E}/f^{-1}(\chi)$. Because H is u.h.c., $N^{\leq} \equiv \inf X^{\leq}_{ss} \in X^{\leq}_{ss}$. To show that $h(N^{\leq}) = N^{\leq}$, note that $h(N^{\leq}) > N^{\leq}$ is ruled out because $N^{\leq} \in X^{\leq}_{ss}$. If $h(N^{\leq}) < N^{\leq}$, then by Lemma 5, $\max H(N) \leq h(N^{\leq})$ for $N \in (h(N^{\leq}), N^{\leq})$, so $N \in X^{\leq}_{ss}$, contradicting $N^{\leq} = \inf X^{\leq}_{ss}$. Thus $h(N^{\leq}) = N^{\leq}$, so $N^{\leq}$ is a steady state.

(b) Uniqueness of N^*_{ss}: Note that $S_u^*(e^+) = \frac{S(1,e^+)(1-b_0)}{u'(f(e^+)-\chi)} > 0$ and $S_u^*(e_{ss}) = b_0\left(-f'(e)e\right) < 0$. The continuity of S_u^* implies that $S_u^*(e) = 0$ has at least one root e^*_{ss} with $e_{ss} < e^*_{ss} < e^+$. Moreover,

$$\frac{\partial S_u^*(e)}{\partial e} = b_0\left(\varpi f'(e) - f''(e)e\right) = b_0 f'(e)\left(\varpi + \Phi(e)\right).$$

If Φ is either strictly increasing or decreasing, $\varpi + \Phi(e) = 0$ has at most one root, denoted e_0, so S_u^* has at most one increasing and one decreasing segment on $[0, e^+]$. Because $S_u^*(e_{ss}) = b_0\left(-f'(e)e\right) < 0$ and $S_u^*(e^+) = \frac{S(1,e^+)(1-b_0)}{u'(f(e^+)-\chi)} > 0$, S_u^* must increase on a subset of $[e_{ss}, e^+]$. There are three possibilities: (i) if $\varpi + \Phi(e) \neq 0$ for $e \in (0,1)$, then S_u^* must be strictly increasing everywhere, implying $\varpi + \Phi(e) > 0$, so $S_u^*(e) = 0$ has a single root $e^*_{ss} = e(N^*_{ss}) \in (e_{ss}, e^+) \subset [0, e^+]$; (ii) if $e_0 \in (0,1)$ exists and Φ is strictly decreasing, then $\frac{\partial S_u^*(e)}{\partial e} > 0$ for $e \in [0, e_0)$ and $\frac{\partial S_u^*(e)}{\partial e} < 0$ for $e \in (e_0, e^+]$, so $S_u^*(e) \geq S_u^*(e^+) > 0$, which means $S_u^*(e) = 0$ has a single root on $(e_{ss}, e_0) \subset [0, e^+]$; (iii) if $e_0 \in (0,1)$ exists and Φ is strictly increasing, then $\frac{\partial S_u^*(e)}{\partial e} < 0$ for $e \in [0, e_0)$ and $\frac{\partial S_u^*(e)}{\partial e} > 0$ for $e \in (e_0, e^+]$, so $S_u^*(e) = 0$ has a single root on $[e_{ss}, e^+]$. Uniqueness of $N^*_{ss} = e^{-1}(e^*_{ss})$ follows.

(c) Combining $\varpi > -\varepsilon_{n_t,y_t}\kappa(e)$ and (A.1), which implies $\varepsilon_{n_t,y_t} < \frac{f'(e)e}{f(e)}$, one obtains $\varpi > -\varepsilon_{n_t,y_t}\kappa(e) > -\kappa(e)\frac{f'(e)e}{f(e)} = \frac{f\prime(e)e^2}{f(e)} = -\Phi(e)$, so uniqueness of N^*_{ss} and e^*_{ss} follow from part (b). QED.

Remark: Production functions considered in the calibration section have strictly increasing $\Phi(e)$, so there is a unique steady state for any (θ, b).

Proposition 7 (Convergence and conditions for a unique optimal path):

(a) Suppose $N_0 < N^{*1}_{ss}$, where $N^{*1}_{ss} \equiv \min\{N : N \in H(N)\}$, and suppose (A.1) holds on $[N_0, N^*_{ss}]$ in that (A.1) holds for all $N_t \in [N_0, N^*_{ss}]$ and all $N_{t+1} \in H(N_t)$. Then population converges monotonely from N_0 to N^*_{ss}.

(b) If there is a unique steady state N^*_{ss} and (A.1) holds on a interval X that includes N^*_{ss}, then population converges monotonely to N^*_{ss} from an initial value $N_0 \in X$.

(c) For any admissible (θ, b), if an interval X satisfies (b) and $X \subset \{N_t : 1/\omega > 1 - \varepsilon_{n_t,y_t}\kappa(e(N_t))$ for $N_{t+1} \in H(N_t)\}$, then V° is strictly concave and differentiable on X, and H is single-valued and continuous.

Proof: (a) For any $N_t < N^{*1}_{ss}$, (A.1) implies that H is increasing as defined in Lemma 5, so $N_{t+1} \leq N^{*1}_{ss}$ for all $N_{t+1} \in H(N_t)$. By construction $N^{*1}_{ss} = \min X^{\leq}_{ss}$, with $X^{\leq}_{ss}$ as defined in the proof of Lemma 5. Hence $N_t \notin X^{\leq}_{ss}$ for all $N_t < N^{*1}_{ss}$, which implies $N_{t+1} > N_t$. Thus $N_{t+1} \in (N_t, N^{*1}_{ss}]$ for all $N_t < N^{*1}_{ss}$, which proves monotone convergence.

(b) If there is only one steady state, $N_{t+1} < N_t$ for $N_{t+1} \in H(N_t)$ must hold for all $N_t > N^*_{ss}$ (otherwise $N^{\max}(\hat{E}/f^{-1}(\chi)) \leq \hat{E}/f^{-1}(\chi)$ would imply a second steady state on $(N^*_{ss}, \hat{E}/f^{-1}(\chi)]$, a contradiction), so $N_{t+1} \in [N^*_{ss}, N_t)$. Thus population converges monotonely from above; for $N_t < N^*_{ss}$, part (a) applies.

(c) From proposition 3, $\varpi > -\varepsilon_{n_t,y_t}\kappa(e(N_t))$ implies strict concavity of u°. From part (b) $H(N_t) \subset X$ for all $N_t \in X$, so (A.9) on X has a solution in the space of strictly concave functions, which implies differentiability and that H is single-valued and continuous. QED.

Remark: Because Proposition 6(b) shows that N^*_{ss} is unique for $\omega \leq 1$, Proposition 7(b) implies convergence for $\omega \leq 1$, which is asserted in the text.

Remark: The condition on ϖ in (c) is equivalent to $1/\omega > 1 - \varepsilon_{n_t,y_t}\kappa(e(N_t))$, which is asserted in the text.

Remark: If ω is in a neighborhood of one, then N^*_{ss} is unique from 6(c) and 7(b-c) apply. If $\omega \approx 1$, then $\varepsilon_{n_t,y_t} \approx 1$ so condition (A.1) holds for all e. If κ and Φ are bounded away from zero (which is true for the Cobb-Douglas and abatement-cost production functions used in calibrations), then the conditions $\varpi > -\Phi(e(N_t))$ and $\varpi > -\varepsilon_{n_t,y_t}\kappa(e(N_t))$ apply on $X = [\hat{E}/e^+, N_{ss}]$, so N^*_{ss} is unique and V° strictly concave. Thus $\omega = 1$ is not a borderline case.

Steady States With Time Costs and Backstop Technology (Section V)

In section V-A, we claim that e_{ss} and e^*_{ss} in the model with time costs are unique under assumptions similar to those made in Section IV. In section V-B, we claim that $e_{ss} > 0$ and $e^*_{ss} > 0$ exist whenever $f(0) < f^B$. Because the calibration analysis allows for both time costs and backstops, we prove existence and uniqueness of $e_{ss} > 0$ and $e^*_{ss} > 0$ for any $f(0) < f^B$ in a setting with both time costs and backstops. Recall that $f^B = \frac{\chi}{\phi - \psi}$ and, by assumption, $\chi + \psi f(e^+) < \phi f(e^+)$, where $\phi = 1/\left(1 + \frac{1-\theta}{1-b}\frac{1-b_0}{b_0}\right)$.

Proposition 8: In an economy with time costs and backstops, if $f(0) < f^B$, then: (a) a natural steady state $e_{ss} > 0$ exists; and (b) an optimal steady state $e^*_{ss} > 0$ exists.

Proof: (a) Define $S_u(n,e) \equiv S(n,e)/u'((1-\psi n)f(e) - \chi n)$. Then the steady-state condition $S(n,e) = 0$ holds iff $S_u(n,e) = 0$, so e_{ss} must satisfy

$$\begin{aligned} S_u(1,e) &= -[\chi + w(e)\psi] + \frac{\beta(1)}{1-\beta(1)}\frac{1-b}{1-\theta}[(1-\psi)f(e) - \chi] \\ &= \frac{1}{1-\phi}[(\phi-\psi)f(e) + (1-\phi)\psi f'(e)e - \chi] \\ &= f'(e)e\psi + \frac{\phi-\psi}{1-\phi}[f(e) - f^B] = 0. \end{aligned} \tag{A.12}$$

Assumption $\chi + \psi f(e^+) < \phi f(e^+)$ implies $(\phi - \psi)f(e^+) - \chi > 0$ so $S_u(1, e^+) > 0$. Because $f'(e)e \to 0$ as $e \to 0$ by concavity of production, $S_u(1, 0) < 0$ if and only if $f(0) < f^B$. Thus given $f(0) < f^B$, continuity of S_u implies a root $e_{ss} \in (0, e^+)$ with $S_u(1, e_{ss}) = 0$.

(b) Define

$$\begin{aligned} S_u^*(e) &\equiv \frac{S^*(e)}{u'((1-\psi)f(e)-\chi)} = (1-b_0)S_u(1,e) - b_0(1-\psi)f'(e)e \\ &= f'(e)e(\psi - b_0) + (1-b_0)\frac{\phi-\psi}{1-\phi}[f(e) - f^B], \end{aligned} \tag{A.13}$$

Then e_{ss}^* must satisfy $S_u^*(e_{ss}^*) = 0$. Note that $S_u^*(e^+) = (1 - b_0)S_u(1, e^+) > 0$ and $S_u^*(0) = \frac{\phi-\psi}{1-\phi}[f(0) - f^B] < 0$. Hence by continuity, $S_u^*(e_{ss}^*) = 0$ for some $e_{ss}^* \in (0, e^+)$. QED.

Proposition 9: In an economy with time costs and backstops, suppose $f(0) < f^B$. Then:

(a) e_{ss} is unique if

$$\Phi(e) \equiv \frac{(-f''(e))e}{f'(e)}$$

is strictly monotone in e (increasing or decreasing).

(b) e_{ss}^* is unique if $\varpi > -\frac{b_0-\psi}{b_0(1-\psi)}\Phi(e)$ for all $e \in (0, e^+)$, or if $\Phi(e)$ is strictly monotone in e (increasing or decreasing).

Proof: (a) From (A.12) and (A.13), $f(0) < f^B$ implies $S_u(1, 0) < 0$ and $S_u^*(0) < 0$. Recall that $S_u(1, e^+) = S_u^*(e^+) > 0$. From (A.12):

$$\begin{aligned}\frac{\partial}{\partial e}S_u(1,e) &= f''(e)e\psi + f'(e)\psi + \frac{\phi-\psi}{1-\phi}f'(e) \\ &= f'(e)\left[\psi + \frac{\phi-\psi}{1-\phi} - \psi\Phi(e)\right].\end{aligned}$$

Assumptions $\chi + \psi f(e^+) < \phi f(e^+)$ and $\psi \geq 0$ imply $\psi + \frac{\phi-\psi}{1-\phi} > 0$. There are three cases:

(a-i) If $\psi = 0$, then $\frac{\partial}{\partial e}S_u(1,e) > 0$ for $e < e^+$ follows from $\frac{\phi-\psi}{1-\phi} > 0$, so $S_u(1,e) = 0$ has at most one root (which exists by Proposition 8).

(a-ii) If $\psi > 0$ and Φ increases strictly, then $\psi + \frac{\phi-\psi}{1-\phi} - \psi\Phi(e)$ decreases strictly so $\frac{\partial}{\partial e}S_u(1,e)$ decreases strictly. Because $S_u(1,0) < S_u(1,e^+)$, $\frac{\partial}{\partial e}S_u(1,e) > 0$ for some $e \in (0,e^+)$. Thus either $\frac{\partial}{\partial e}S_u(1,e) > 0$ for all $e \in (0,e^+)$ so again $S_u(1,e) = 0$ has at most one root, or $\frac{d}{de}S_u(1,e) = 0$ has a unique root $e^0 \in (0,e^+)$. Then $\frac{\partial}{\partial e}S_u(1,e) > 0$ for $e < e^0$ and $\frac{\partial}{\partial e}S_u(1,e) < 0$ for $e > e^0$, so $S_u(1,e) \geq S_u(1,e^+) > 0$ for $e \geq e^0$. Hence $S_u(1,e) = 0$ has at most one root, which must lie in the interval $(0,e^0)$.

(a-iii) If $\psi > 0$ and Φ decreases strictly, then $\psi + \frac{\phi-\psi}{1-\phi} - \psi\Phi(e)$ increases strictly so $\frac{\partial}{\partial e}S_u(1,e)$ increases strictly. By reasoning similar to case (ii), $S_u(1,e) = 0$ has at most one root. (The argument differs only in that if $\frac{d}{de}S_u(1,e) = 0$ has a root $e^0 \in (0,e^+)$, then $\frac{\partial}{\partial e}S_u(1,e) < 0$ for $e < e^0$, and $\frac{\partial}{\partial e}S_u(1,e) > 0$ for $e > e^0$, so the root of $S_u(1,e) = 0$ lies in the interval (e^0, e_{ss}).

Thus e_{ss} is unique in all cases.

(b) From (A.13),

$$\begin{aligned}\frac{\partial}{\partial e}S_u^*(e) &= f'(e)\left[(\psi - b_0)(1-\Phi(e)) + (1-b_0)\frac{\phi-\psi}{1-\phi}\right] \\ &= f'(e)\left[b_0(1-\psi)\varpi + (b_0-\psi)\Phi(e)\right], \qquad \text{(A.14)}\end{aligned}$$

using

$$\psi - b_0 + (1-b_0)\frac{\phi-\psi}{1-\phi} = (1-\psi)\frac{\phi-b_0}{1-\phi} = (1-\psi)b_0\varpi.$$

If $\Phi(e)$ is strictly monotone, the argument is similar to the proof of (a). Namely, If $b_0 = \psi$, then $\psi < \phi$ implies $b_0 < \phi$ and $\varpi > 0$, so $\frac{\partial}{\partial e} S_u^*(e) > 0$ for all e as in (a-i). If $b_0 \neq \psi$, then $(b_0 - \psi)\Phi(e)$ either increases strictly or decreases strictly, so the argument is analogous to either case (a-ii) or (a-iii).

Even if $\Phi(e)$ is not monotone, $\varpi > -\frac{b_0-\psi}{b_0(1-\psi)}\Phi(e)$ is sufficient for a unique e_{ss}^* because it implies $b_0(1-\psi)\varpi + (b_0 - \psi)\Phi(e) > 0$ in (A.14) and hence $\frac{\partial}{\partial e} S_u^*(e) > 0$. QED.

Remark: Recall from Proposition 6(b) that without time costs, $\varpi > -\Phi(e)$ implies uniqueness of e_{ss}^*. Condition $\varpi > -\frac{b_0-\psi}{b_0(1-\psi)}\Phi(e)$ in Proposition 9(b) is a generalization. Note that $\Phi(e) \geq 0$, so Proposition 9(b) always holds for $\varpi \geq 0$ and $\psi < b_0$.

Proposition 10: For the Cobb-Douglas technology defined in Section VI, e_{ss} and e_{ss}^* are unique.

Proof: By assumption, $f(e) = f_0 e^{f_1}(f_2 - e)^{1-f_1}$, where $f_1 \in (0,1)$ and $f_2 = 1/f_1 > 1$. The derivatives are $f'(e) = f(e)\left[\frac{f_1}{e} - \frac{1-f_1}{f_2-e}\right] = \frac{f(e)}{e}\frac{1-e}{f_2-e}$ and $f''(e) = -\frac{f(e)}{e^2}\frac{(1-f_1)f_2}{(f_2-e)^2} < 0$. Hence

$$\Phi(e) = \frac{(-f''(e))e}{f'(e)} = \frac{(1-f_1)f_2}{(1-e)(f_2-e)},$$

which is increasing in e. Thus e_{ss} and e_{ss}^* are unique by Proposition 9. QED.

Proposition 11: For the abatement-cost technology defined in Section VI, e_{ss} and e_{ss}^* are unique provided $g_0 = f(0) < f^B$.

Proof: By assumption, $f(e) = 1-(1-g_0)(1-e)^{g_1}$ with $0 \leq g_0 < 1$ and $g_1 > 1$. The derivatives are $f'(e) = g_1(1-g_0)(1-e)^{g_1-1}$ and $f''(e) = -g_1(1-g_0)(g_1-1)(1-e)^{g_1-2}$. Hence

$$\Phi(e) = \frac{(-f''(e))e}{f'(e)} = (g_1 - 1)\frac{e}{(1-e)},$$

which is increasing in e. Thus e_{ss} and e_{ss}^* are unique by Proposition 9. QED.

Proposition 12: In an economy with time costs and backstops, if $f(0) > f^B$, then: (i) a steady state $e_{ss} > 0$ does not exist; (ii) fertility converges to the unique root n_{ss} of $S(n_{ss}, 0) = 0$, which satisfies $n_{ss} > 1$; and (iii) optimal fertility converges to the same value $n_{ss}^* = n_{ss}$. Moreover, if $f(0) = f^B$, then: (iv) the steady-state conditions reduce to $S(1, 0) = 0$ in the natural economy and $S^*(0) = 0$ in the optimal economy.

Proof: (i) In (A.12), $f(0) > f^B$ and $f'(e)e \geq 0$ imply $S_u(1, e) > 0$ for all e, so $S(1, e) > 0$ and there is no e_{ss} that satisfies $S(1, e_{ss}) = 0$.

(ii) Recall that $S(n, e) = -u'(c)(\chi + \psi w(e)) + \frac{\beta'(n)}{1-\beta(n)} u(c)$ with $c = (1 - \psi n) f(e) - \chi n$. For $e = 0$, $S(n, 0) \to -\infty$ as $n \to f(0)/(\psi f(0) + \chi)$ from below, because the latter implies $c \to 0$. When $\theta < 1$, $\beta'(n) \to \infty$ as $n \to 0$, which implies $S(n, 0) \to \infty$. When $\theta > 1$, $\frac{\beta'(n)u}{1-\beta(n)} \to \infty$ as $n \to \beta^{-1}(1)$ from above, which implies $S(n, 0) \to \infty$. Hence the continuity of S implies that n_{ss} satisfying $S(n_{ss}, 0) = 0$ exists. Moreover, $\frac{d}{dn} S(n, 0) = u''(c)(\psi f(e) + \chi)^2 + \frac{\beta''(n)}{1-\beta(n)} u(c) < 0$ at n such that $S(n, 0) = 0$, so $S(n, 0)$ can cross zero only once. Hence n_{ss} is unique with $S(n, 0) > 0$ for $n < n_{ss}$ and $S(n, 0) < 0$ for $n > n_{ss}$. To show $n_{ss} > 1$, note that $n_{ss} \leq 1$ would imply $S(1, 0) \leq 0$, which contradicts the result in (i) that $S(1, e) > 0$ for all e. Therefore $S(1, 0) > 0$. Given $S(n_{ss}, 0) = 0$ with $n_{ss} > 1$, derivations analogous to those in the proof of Proposition 2 imply that utility converges to $U_{ss} = \frac{1}{1-\beta(n_{ss})} u[(1 - \psi n_{ss}) f(0) - \chi n_{ss}]$ and that population follows a path $N_{t+1} = N(N_t)$ that approaches a ray with slope $n_{ss} > 1$, which means fertility converges to n_{ss}.

(iii) For $f(0) > f^B$, a necessary condition for optimal steady-state fertility is that $S_{nu}^*(n, e) = 0$, where

$$S_{nu}^*(n, e) \equiv (1 - \beta(n)) S_u(n, e) - \frac{\beta(n)}{n} [(1 - \psi n) f'(e) e] \tag{A.15}$$

replaces $S_u^*(e)$ when $n \neq 1$. (Note that $S_{nu}^*(1, e) = S_u^*(e)$.) Because $f'(e)e \to 0$ as $e \to 0$, $S_{nu}^*(n, 0) = S_u(n, 0)$ for all n. Hence for $f(0) > f^B$, $n_{ss}^* = n_{ss} > 1$.

(iv) For $f(e) = f^B$, (A.12) and (A.13) imply $S_u(1, 0) = S_{nu}^*(1, 0) = S_u^*(0) = 0$, so $S(1, 0) = S^*(0) = 0$. QED.

Numerical Procedures (Section VI)

To study the natural population after a cap is imposed, we first solve the condition for the natural steady state $S(1, \tilde{e}_{ss}) = 0$ for $\tilde{e}_{ss}$. Because $\hat{E}$ is fixed, the value of $\tilde{e}_{ss}$ implies values of growth-adjusted population $\tilde{N}_{ss}$, income $\tilde{y}_t = (1 - \tilde{\psi}\tilde{n}_t)f(\tilde{e}_{ss})$, and wages $w(\tilde{e}_{ss}) = f(\tilde{e}_{ss}) - f'(\tilde{e}_{ss})\tilde{e}_{ss}$.

To find the dynamic path to the natural steady state, we use value-function iteration to solve for utility along the perfect-foresight path as a function of population, denoted $U(\tilde{N})$. Specifically, we define a grid of values of $\tilde{N}$ (equally spaced on a log-scale) and compute initial utility values at each gridpoint of $U_0(\tilde{N}) = u(f(e((1-\tilde{\psi})\tilde{N}))-\tilde{\chi})/(1-\beta(1))$, which is feasible because $\tilde{n} = 1$ is feasible for all $\tilde{N} \leq \tilde{N}_{ss}$. At each iteration $i = 1, 2, ...$ we calculate equilibrium fertility $\tilde{n} = \eta_i(\tilde{N})$ at each gridpoint by Gauss-Newton iteration on the household first-order condition

$$V_n(\tilde{n}, w, TR, U) = 0,$$

evaluated at the equilibrium values $w = w(e((1-\tilde{\psi}\tilde{n})\tilde{N}))$, $TR = \hat{E}f'(e((1-\tilde{\psi}\tilde{n})\tilde{N}))/\tilde{N}$, and $U = U_{i-1}(\tilde{n}\tilde{N})$, where the latter is the value function from the previous iteration approximated using cubic-spline interpolation between gridpoints. Given values $\eta_i(\tilde{N})$ at iteration i, we then compute

$$\begin{aligned} U_i(\tilde{N}) &= u((1 - \tilde{\psi}\eta_i(\tilde{N}))f(e((1 - \tilde{\psi}\eta_i(\tilde{N}))\tilde{N})) - \tilde{\chi}\eta_i(\tilde{N})) \\ &\quad +\tilde{\beta}(\eta_i(\tilde{N}))U_{i-1}(\eta_i(\tilde{N})\tilde{N}), \end{aligned}$$

and proceed to the next iteration. When an i is reached at which $U_i(\tilde{N}) - U_{i-1}(\tilde{N})$ is sufficiently small at all gridpoints, we take $U(\tilde{N}) = U_i(\tilde{N})$ and $\eta(\tilde{N}|\hat{E}) = \eta_i(\tilde{N})$ at gridpoints, and use cubic-spline interpolation to find values of $U(\tilde{N})$ and $\eta(\tilde{N}|\hat{E})$ between gridpoints.

To compute specific population paths, we iterate on $\eta(\tilde{N}|\hat{E})$ from a starting value $\tilde{N}_0$. Finally we convert the growth-adjusted natural population sequence into the actual natural population sequence by reversing the transformations in Section V. Note that U is well-defined because we showed in earlier sections that a unique natural population sequence exists starting from any initial population.

We study the optimal population using similar steps. We start from the condition for the optimal steady state $S^*(1, \tilde{e}^*_{ss}) = 0$ to find $\tilde{e}^*_{ss}$. Optimal dynamics are simpler than natural dynamics because there is no externality. Value-function iteration is based on

$$V_i^*(\tilde{N}) \equiv \max_{n \in \{n | (1-\tilde{\psi}n) f(e((1-\tilde{\psi}n)\tilde{N})) \geq \chi n\}} \left\{ \begin{array}{c} u((1-\tilde{\psi}n) f(e((1-\tilde{\psi}n)\tilde{N})) - \chi n) \\ +\tilde{\beta}(n) V_{i-1}^*(n\tilde{N}) \end{array} \right\},$$

starting from $V_0(\tilde{N}) = u(f(e((1-\tilde{\psi})\tilde{N})) - \tilde{\chi})/(1-\beta(1))$, which is feasible as in the natural problem. Again we use cubic-spline interpolation to evaluate V^*_{i-1} between grid points. When an i is reached at which $V_i^*(\tilde{N}) - V_{i-1}^*(\tilde{N})$ is sufficiently small at all gridpoints, we take $V^*(\tilde{N}) = V_i^*(\tilde{N})$ and $\eta^*(\tilde{N}|\hat{E}) = \arg\max u((1-\tilde{\psi}n) f(e((1-\tilde{\psi}n)\tilde{N})) - \chi n) + \tilde{\beta}(n) V^*(n\tilde{N})$ at gridpoints, and use cubic-spline interpolation to find values of $V^*(\tilde{N})$ and $\eta(\tilde{N}|\hat{E})$ between gridpoints. Note that, for easier comparison with the natural path, we use value-function iteration for V^* instead of Alvarez' V°; V^* is well-defined because it is a transformation of V°.

Details on the Calibration of Child Costs (Section VI)

USDA (2007) reports expenditures of \$204,000 per child by husband-wife families with two children and before-tax annual income between \$45,800 and \$77,100. Department of Education (2007) reports total K-12 spending of \$599 billion (table 26), enrollment in 2-year and 4-years colleges of 17,922,000 (table 177), and average undergraduate tuition, fees, and room and board of \$15,434 (table 320) for 2007. From Census (2008a,b), the population aged 0-17 was 73,902,000, and actual and replacement U.S. fertility were 1.9 and 2.1, implying $n = 0.9$ for the U.S. From table 26 of the Economic Report of the President, NNP was $12,381 billion in 2007. Thus aggregate expenditures on children as a fraction of NNP in 2007 were

$$\left[(73,902,000(\frac{204,000}{18}) + 599 * 10^9 + (15,434)(17,922,000)\right] /(12,381 * 10^9)$$
$$= 0.138.$$

Participation rates are from the Bureau of Labor Statistics (2008). Specifically, we take the ratio of the average participation rate (over males and females) to the male participation rate to equal the relative decrease in labor due to time costs, $.685/.76 = 1 - \psi n$ and use $n = 0.9$ to infer $\psi = 0.11$ and therefore $\tilde{\psi} = \psi\lambda/\alpha = 0.098$. We then recover $\chi = 0.138$ from[44]

$$n\chi = (.9)\chi = (0.138)(1 - \psi n)(1) = (0.138)(\frac{0.685}{0.76}).$$

Note that costs of K-12 education, which are typically provided free by government, amounted to 4.8 percent of NNP in the U.S. in 2007. Neglecting other subsidies and taxes on children, this suggests a preexisting tax on children of $-.048$. It would be possible to adjust the calibration to take account of a preexisting tax with the value $\tau = -.048$; such a treatment would be correct if the subsidy is a historical accident. If the existing subsidy instead corrects some other market failure, on the other hand, assuming a preexisting value $\tau = -.048$ without modeling the market failure would be inappropriate. To avoid this, we assume there is no preexisting tax.

[44]The calculated value of χ is close to but not the same as the calculated fraction of NNP going to child expenditures; to three decimals, each is 0.138.

Expanded Model with Explicit Environmental Dynamics and Time-Varying Optimal Caps (Footnote in Section I)

The assumptions of a constant emissions cap and zero damages are major simplifications that allow us to focus on the population externality. To evaluate the simplifications, we consider an expanded setting with explicit environmental dynamics and a known damage process, in which emissions caps and child taxes are the optimal policy.

Suppose damage depends on a vector of environmental state variables X_t, so $Y_t = F(L_t, E_t)(1-\delta_t)$ with $\delta_t = \delta(X_t)$ where δ is a known function. Assume environmental dynamics follow a known Markov process that depends on emissions, $X_{t+1} = \Theta(E_t, X_t)$. The state of the system is described by (N_t, X_t). Policy is a combined choice of optimal (next-period) population and optimal emissions, (N_{t+1}, E_t). Optimal policy must solve the dynamic programming problem

$$V^\circ(N_t, X_t) = \max_{(N_{t+1},E_t)\in\Gamma(N_t,X_t),} \{u^\circ(N_t, N_{t+1}, E_t, X_t) + b_0 V^\circ(N_{t+1}, \Theta(E_t, X_t))\}, \tag{A.16}$$

where

$$u^\circ(N_t, N_{t+1}, E_t, X_t) \equiv \beta(N_t) u\left[(1-\delta(X_t)) f(E_t/N_t) - \chi N_{t+1}/N_t\right]$$

and $\Gamma(N_t, X_t) = [0, \frac{1}{\chi} Y^{\max}(N_t, X_t)] \times [0, e^+ N_t]$ with $Y^{\max}(N_t, X_t) \equiv \frac{1}{\chi} N_t (1-\delta(X_t)) f(e^+))$. (The problem is stated in terms of the basic model with $f(0) \geq 0$; time costs and growth adjustment could be included but would needlessly add clutter.)

In the expanded setting, existence of a steady state and conditions for convergence are nontrivial and depend on environmental-science information on X_t and Θ. If one assumes there is a set of emissions policies that yields stable steady states, however, then it is straightforward to characterize the steady states. Necessary conditions for optimal (N_{t+1}, E_t) are

$$\frac{\partial u^\circ}{\partial N_{t+1}}(N_t, N_{t+1}, E_t, X_t) + b_0 \frac{\partial V^\circ(N_{t+1}, \Theta(E_t, X_t))}{\partial N_{t+1}} = 0 \tag{A.17}$$

$$\frac{\partial u^\circ}{\partial E_t}(N_t, N_{t+1}, E_t, X_t) + b_0 \Theta_E(E_t, X_t) \bullet \left(\frac{\partial V^\circ(N_{t+1}, \Theta(E_t, X_t))}{\partial X_{t+1}} \right) = 0, \tag{A.18}$$

where $\Theta_E \equiv \frac{\partial X_{t+1}}{\partial E_t}$ and $\frac{\partial V^0}{\partial X_{t+1}}$ is a vector of partial derivatives. ($\bullet$ denotes inner product.) Let $(N_{SS}, E_{SS}, X_{SS}, \delta_{SS})$ denote steady-state values. Because X_t plausibly has higher dimension than E_t, feasible X_{SS} must be a fixed point of Θ for given E_{SS} in the sense that setting $E_t = E_{SS}$ for all $t \geq t_0$ (starting from some $t_0 \geq 0$) implies convergence to an environmental steady state $X_t \to X_{SS}$. Let $X_{SS} \equiv \mathcal{X}(E_{SS})$ capture the dependence of X_{SS} on emissions, and define $\delta_{SS} \equiv \delta(\mathcal{X}(E_{SS}))$.

From the envelope condition $\frac{\partial V^\circ(N_t, X_t)}{\partial N_t} = \frac{\partial u^\circ}{\partial N_t}(N_t, N_{t+1}, E_t, X_t)$, a necessary condition for optimal population at given $(E_{SS}, X_{SS}, \delta_{SS})$ is

$$\frac{\partial u^\circ}{\partial N_{t+1}}(N_t, N_{t+1}, E_{SS}, \mathcal{X}(E_{SS})) + b_0 \frac{\partial u^\circ}{\partial N_t}(N_t, N_{t+1}, E_{SS}, \mathcal{X}(E_{SS})) = 0. \tag{A.19}$$

This is the same optimality condition as in the main model, except that the income term in u° is $(1-\delta_{SS})f(e)$ instead of $f(e)$ in the main model. Because $(1-\delta_{SS})$ is fixed, all results about existence and uniqueness of a steady state population still apply. Notably, there is a finite steady-state population N_{SS} provided $(1-\delta_{SS})f(0) < f_B$, and the ratio E_{SS}/N_{SS} satisfies the limit condition $S^*(E_{SS}/N_{SS}) = 0$ with $f(e)$ in the definition of S^* multiplied by the factor $(1-\delta_{SS})$.

The dynamics of jointly optimal population and emissions are characterized by the optimality conditions (A.17) and (A.18). If one assumes policy functions $N_{t+1} = N^+(N_t, X_t)$ and $E_t = E^+(N_t, X_t)$ exist ($^+$ denotes jointly optimal), one can obtain an optimal path of population and environmental variables by iterating on $N_{t+1} = N^+(N_t, X_t)$ and $X_{t+1} = \Theta(E^+(N_t, X_t), X_t)$ starting at (N_0, X_0). This yields time series $(N_t^+, X_t^+, E_t^+, \delta_t^+)$ for all $t \geq 0$ and optimal limit values $(N_{SS}^+, X_{SS}^+, E_{SS}^+, \delta_{SS}^+)$. The optimal path can be implemented with time-varying emissions caps, $\hat{E}_t = E_t^+$, and optimal child taxes. Given the emission caps, the child taxes are (by logic parallel to that in Section IV):

$$\tau_t^+ = \frac{\beta(n_t^+)}{n_t^+} \sum_{i=1}^{\infty} \left[\prod_{j=1}^{i-1} \beta(n_{t+j}^+) \right] \frac{u'(c_{t+i}^+)}{u'(c_t^+)} (1 - \delta(X_{t+i}^+)) f'(e_{t+i}^+) e_{t+i}^+, \tag{A.20}$$

where $e_{t+i}^+ = \frac{E_{t+i}^+}{N_{t+i}^+}$ and $n_{t+j}^+ = \frac{N_{t+i+1}^+}{N_{t+i}^+}$.

Except for the $(1-\delta(X_{t+i}^+))$ terms, equation (A.20) for τ_t^+ is identical to the equation for τ_t^* in Section IV, if $\hat{E}$ is set equal to E_{SS}^+. In steady state, τ_{SS}^+ is reduced by the factor $(1 - \delta_{SS}^+)$, and income $y_{SS}^+ = (1 - \delta_{SS}^+))f(e_{SS}^+)$ is reduced by the same factor. The calibrated tax rates $\{\tau/y\}_{SS}^*$ in the text are therefore identical to the tax rates τ_{SS}^+/y_{SS}^+ one would obtain from the expanded model.

The time path of tax rates in the expanded model depends on the time paths of emissions and damages as specified in (A.20) and generally differs from the path in the model with constant $\hat{E}$. However, the latter approximates the former if: $\hat{E} = E_{SS}^+$ so steady-state emissions are optimal; differences between $\hat{E}_t$ and E_{SS}, and between $\delta(X_t)$ and δ_{SS}, are disregarded so $\hat{E}_t = E_{SS}$ and $\delta(X_t) = \delta_{SS}$ for $t \geq 0$, and the (nuisance) factor $1 - \delta_{SS}$ is subsumed into $f(e)$.

The following logic suggests that transitional tax rates in the model with constant $\hat{E}$ are conservative. From (A.20):

$$\frac{\tau_t^+}{y_t^+} = \frac{\beta(n_t^+)}{n_t^+} \sum_{i=1}^{\infty} \left[\prod_{j=1}^{i-1} \beta(n_{t+j}^+) \right] \frac{u'(c_{t+i}^+)}{u'(c_t^+)} \frac{y_{t+i}^+}{y_t^+} \frac{f'(e_{t+i}^+) e_{t+i}^+}{f(e_{t+i}^+)}.$$

Note that the emissions shares $\frac{f'(e_t)e_t}{f(e_t)}$ are the central determinants in our calibrations. Suppose optimal emissions approach E_{SS}^+ from below, so $E_t^+ \leq E_{SS}^+$, and that N_t^+ is sufficiently close to N_t^* that $E_t^+ < E_{SS}^+ = \hat{E}$ implies $\hat{E}/N_t^* = e_t^* < e_t^+$. Then $\frac{f'(e_t^*)e_t^*}{f(e_t^*)} < \frac{f'(e_t^+)e_t^+}{f(e_t^+)}$, which suggests $\frac{\tau_t^*}{y_t^*} < \frac{\tau_t^+}{y_t^+}$. In this case, our calibrated tax rates would underestimate the optimal tax rates along the path with optimal emissions caps. That is, our calibrations would understate the population externality.

Permanent Emission Rights (Footnote in Section I)

The population externality can be thought of as resulting from imperfect property rights: there would be no population externality if government were to issue permanent, bequeathable emissions rights instead of permits valid for only a single period. With single-period permits, a marginal birth means more people share a given total amount of emissions rights in the next period so others receive fewer emission rights. With permanent rights, on the other hand, total rights to emit in the next period are given so the new person gets no rights from others. Instead, when a household has an additional child, it is the emission rights of the household's earlier children that are reduced. This provides a disincentive to have children equal to that induced by optimal Pigovian taxes.

To formalize this, suppose the government at $t = 0$ simply gives the household rights to emit $\pi_0 = \hat{E}/N_0$ units of greenhouse gases each period in perpetuity. A household in period t with permanent rights to emit π_t units rents these to the firm at the competitive price $p_t = f'$ in the period, then leaves an equal share to each child so emission rights evolve as $\pi_{t+1} = \pi_t/n_t$. The latter captures the reduction in an earlier child's inheritance from a marginal child , and implies $\pi_t = e(N_t)$ for all t. The household earns wage $w_t = f - e_t f'$ and consumes $c_t = w_t + \pi_t p_t - \chi n_t = f(e_t) - \chi n_t$. Then:

Proposition 13: With permanent property rights, the natural population sequence with permanent, bequeathable emissions rights is the optimal sequence.

Proof: In maximizing utility, the household at t chooses n_t taking π_t as given and also taking the future path of aggregate population and hence of per-capita emissions as given. Denote the utility value to parents of a child who inherits π_t by $v(\pi_t|N)$. The fertility that individually maximizes utility must satisfy the Bellman equation

$$v(\pi_t|N_t) = \max_{n_t}\{u(c_t) + \beta(n_t)v(\frac{\pi_t}{n_t}|\bar{n}_t N_t)\}, \tag{A.21}$$

with $\bar{n}_t = N_{t+1}/N_t$ given and $c_t = f(e(N_t)) + [\pi_t - e(N_t)]f'(e(N_t)) - \chi n_t$. The Euler equation is

$$V_n^{\pi}(n_t|N_t, \bar{n}_t, \pi_t) \equiv -u'(c_t)\chi + \beta'(n_t)v(\frac{\pi_t}{n_t}|\bar{n}_t N_t) - \frac{\beta(n_t)}{n_t}\frac{\pi_t}{n_t}v_{\pi}(\frac{\pi_t}{n_t}|\bar{n}_t N_t) = 0. \quad \text{(A.22)}$$

Optimality follows if the household's optimal fertility choice n_t satisfies the optimality condition in the optimal-population problem,

$$V_n^* = -u'(f(e(N_t)) - \chi n_t)\chi + \beta'(n_t)V^*(N_{t+1}) + \frac{\beta(n_t)}{n_t}N_{t+1}\frac{dV^*}{dN}(N_{t+1}) = 0, \quad \text{(A.23)}$$

where V_n^* and

$$\frac{dV^*}{dN}(N_{t+1}) = -\frac{1}{N_{t+1}}\sum_{i=1}^{\infty}\left[\prod_{j=1}^{i-1}\beta(n_{t+j})\right]u'(c_{t+i})f'(e(N_{t+i}))e(N_{t+i}) \quad \text{(A.24)}$$

were derived in Section IV-A.

The first two terms in (A.22) and in (A.23) are the marginal child cost and marginal private benefit of children. These correspond to the first two (private) terms in the household's first-order condition in the natural economy ($V_n = 0$). In equilibrium, $\pi_t = \hat{E}/N_t = e_t$, $\bar{n}_t = n_t$, and $c_t = f(e_t) - \chi n_t$ for all $t \geq 0$. Hence c_t in (A.22) equals $f(e(N_t)) - \chi n_t$ in (A.23), so the first terms are equal. The second terms are equal if $v(\frac{\pi_t}{n_t}|\bar{n}_t N_t) = v(e(N_{t+1})|N_{t+1}) = V^*(N_{t+1})$, which is verified below. The third term in (A.22) is the dilution of children's inheritances of emissions rights from a marginal child. To re-express this term, the envelope theorem implies

$$v_{\pi}(\pi_t|N_t) = u'(c_t)f'(e(N_t)) + \frac{\beta(n_t)}{n_t}v_{\pi}(\frac{\pi_t}{n_t}|N_{t+1}). \quad \text{(A.25)}$$

Multiplying (A.25) at $t+1$ by $\pi_{t+1} = \pi_t/n_t = e_{t+1}$ and expanding as a sum:

$$\begin{aligned}\frac{\pi_t}{n_t}v_{\pi}(\frac{\pi_t}{n_t}|N_{t+1}) &= u'(c_{t+1})f'(e(N_{t+1}))e_{t+1} + \frac{\beta(n_{t+1})}{n_{t+1}}e_{t+1}v_{\pi}(\frac{\pi_{t+1}}{n_{t+1}}|N_{t+2}) \\ &= \sum_{i=1}^{\infty}\left[\prod_{j=1}^{i-1}\beta(n_{t+j})\right]u'(c_{t+i})f'(e_{t+i})e_{t+i},\end{aligned}$$

so, using (A.24), $\frac{\pi_t}{n_t} v_\pi(\frac{\pi_t}{n_t}|N_{t+1}) = -N_{t+1}\frac{dV^*}{dN}(N_{t+1})$. Thus the third terms in (A.22) and (A.23) are equal. Hence $V_n^\pi = 0$ is satisfied at the same fertility choice $n_t = n_t^*$ that satisfies $V_n^* = 0$, provided $v(e(N_{t+1})|N_{t+1}) = V^*(N_{t+1})$. To verify this, suppose $v(e(N)|N) = V^*(N)$ at some time t_1. Then if $\pi_t = e(N_t)$ at $t = t_1 - 1$, by the argument above, $n_t = n_t^*$ solves the Bellman equation (A.21). Hence the optimal value is $v(e(N_t)|N_t) = u'(c_t^*) + \beta(n_t^*)v(e(n_t^*N_t)|n_t^*N_t) = u'(c_t^*) + \beta(n_t^*)V^*(n_t^*N_t) = V^*(N_t)$, which confirms that $v(e(N)|N)$ satisfies the same Bellman equation as $V^*(N)$. QED.

Remark: The analysis ignores tradeability of permanent emissions rights, but in the representative-agent economy, net trades among households would be zero.

Remark: Permanent emission rights would be created by government and be rights to the revenue from one unit of emissions per-period, forever, which would effectively privatize the public revenue stream generated by a cap. Permanent emission rights would mean the government at $t = 0$ binds all future governments. Permanence could fail if future governments were to change the level of the cap or were to tax or reallocate emission rights, which may benefit a majority of voters. With heterogeneous agents, for instance, dynasties with a heritable preference to have more children would over time form a relatively impoverished majority that would gain from redistribution.

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