6 Dominance and Best Response

This chapter develops and compares the concepts of dominance and best response. The chapter begins with examples in which a strategy is dominated by another pure strategy, followed by an example of mixed strategy dominance. After the formal definition of dominance, the chapter describes how to check for dominated strategies in any given game. The first strategic tension (the clash between individual and joint interests) is illustrated with reference to the prisoners' dilemma, and then the notion of efficiency is defined. Next comes the definition of best response and examples. The last section of the chapter contains analysis of the relation between the set of undominated strategies and the set of strategies that are best responses to some beliefs. An algorithm for calculating these sets is presented.

Lecture Notes

The following may serve as an outline for a lecture.

- Optional introduction: analysis of a game played in class. If a 3×3 dominance-solvable game (such as the one suggested in the notes for Chapter 4) was played in class earlier, the game can be quickly analyzed to show the students what is to come.

- A simple example of a strategy dominated by another pure strategy. (Use a 2×2 game.)

- An example of a pure strategy dominated by a mixed strategy. (Use a 3×2 game.)

- Formal definition of strategy s_i being dominated. Set of undominated strategies for player i, UD_i.

- Discuss how to search for dominated strategies.

- The first strategic tension and the prisoners' dilemma.

- Definition of *efficiency* and an example.

- Best response examples. (Use simple games such as the prisoners' dilemma, the battle of the sexes, Cournot duopoly.)

- Formal definition of s_i being a best response to belief μ_{-i}. Set of best responses for player i, $BR_i(\mu_{-i})$. Set of player i's strategies that can be justified as best responses to some beliefs, B_i.

- Note that forming beliefs is the most important exercise in rational decision making.

- Example to show that $B_i = UD_i$. State formal results.
- Algorithm for calculating $B_i = UD_i$ in two-player games: (1) Strategies that are best responses to simple (point mass) beliefs are in B_i. (2) Strategies that are dominated by other pure strategies are not in B_i. (3) Other strategies can be tested for mixed strategy dominance to see whether they are in B_i. Step 3 amounts to checking whether a system of inequalities can hold.
- Note: Remember that payoff numbers represent preferences over random outcomes.
- Note that Appendix B contains more technical material on the relation between dominance and best response.

The book does not discuss weak dominance until the analysis of the second-price auction in Chapter 27. This helps avoid confusion (students sometimes interchange the weak and strong versions) and, besides, there is little need for the weak dominance concept.

Examples and Experiments

1. *Example of dominance and best response.* To demonstrate the relation between dominance and best response, the following game can be used.

1 \ 2	L	R
T	1, 2	1, 3
M	3, 5	0, 4
B	0, 1	3, 2

   First show that M is the best response to L, whereas B is the best response to R. Next show that T is dominated by player 1's strategy $(0, 1/2, 1/2)$, which puts equal probability on M and B but zero probability on T. Then prove that there is *no* belief for which T is a best response. A simple graph will demonstrate this. On the graph, the x-axis is the probability p that player 1 believes player 2 will select L. The y-axis is player 1's expected payoff of the various strategies. The line corresponding to the expected payoff playing T is below at least one of the lines giving the payoffs of M and B, for every p.

2. *The 70 percent game.* This game can be played by everyone in the class, either by e-mail or during a class session. Each of the n students selects an integer between 1 and 100 and writes this number, along with his or her name, on a slip

of paper. The students make their selections simultaneously and independently. The average of the students' numbers is then computed and the student whose number is closest to 70 percent of this average wins 20 dollars. If two or more students tie, then they share the prize in equal proportions. Ideally, this game should be played between the lecture on Best Response and the lecture on Rationalizability/Iterated Dominance. The few students whose numbers fall within a preset interval of 70 percent of the average can be given bonus points.

7 Rationalizability and Iterated Dominance

This chapter follows naturally from Chapter 6. It discusses the implications of combining the assumption that players best respond to beliefs with the assumption that this rationality is common knowledge between the players. At the beginning of the chapter, the logic of rationalizability and iterated dominance is demonstrated with an example. Then iterated dominance and rationalizability are defined more formally. The second strategic tension—strategic uncertainty—is explained.

Lecture Notes

The following may serve as an outline for a lecture.

- Example of iterated dominance, highlighting hierarchies of beliefs ("player 1 knows that player 2 knows that player 1 will not select…").
- *Common knowledge*: information that each player knows, each player knows the others know, each player knows the others know that they all know…. It is as though the information is publicly announced while the players are together.
- Combining rationality (best response behavior, never playing dominated strategies) with common knowledge implies, and only implies, that players will play strategies that survive iterated dominance. We call these the *rationalizable strategies*.
- Formally, let R^k be the set of strategy profiles that survives k rounds of iterated dominance. Then the rationalizable set R is the limit of R^k as k gets large. For finite games, after some value of k, no more strategies will be deleted.
- Notes on how to compute R: algorithm, order of deletion does not matter.
- The second strategic tension: strategic uncertainty (lack of coordination between beliefs and behavior).

Examples and Experiments

1. *The 70 percent game again.* Analyze the game and show that the only rationalizable strategy is to select 1. In my experience, this always stimulates a lively discussion of rationality and common knowledge. The students will readily agree that selecting 100 is a bad idea. However, showing that 100 is dominated can be quite difficult. It is perhaps easier to demonstrate that 100 is never a best response.

   Note that one player's beliefs about the strategies chosen by the other players is, in general, a very complicated probability distribution, but it can be summarized by the "highest number that the player believes the others will play with positive probability." Call this number x. If $x > 1$, then you can show that the player's best response must be strictly less than x (considering that the player believes at least one other player will select x with positive probability). It is a good example of a game that has a rationalizable solution, yet the rationalizable set is quite difficult to compute. Discuss why it may be rational to select a different number if common knowledge of rationality does not hold.

2. *Generalized stag hunt.* This game can be played in class by groups of different sizes, or it can be played over the Internet for bonus points. In the game, n players simultaneously and independently write "A" or "B" on slips of paper. If any of the players selected B, then those who chose A get nothing and those who chose B get a small prize (say, \$2.00 or 10 points). If all of the players selected A, then they each obtain a larger prize (\$5.00 or 25 points). The game can be used to demonstrate strategic uncertainty, because there is a sense in which strategic uncertainty is likely to increase (and players are more likely to choose B) with n.

   A good way to demonstrate strategic uncertainty is to play two versions of the game in class. In the first version, $n = 2$. In this case, tell the students that, after the students select their strategies, you will randomly choose two of them, whose payoffs are determined by only each other's strategies. In the second version of the game, n equals the number of students. In this case, tell the students that you will pay only a few of them (randomly chosen) but that their payoffs are determined by the strategies of everyone in the class. That is, a randomly drawn student who selected A gets the larger prize if and only if everyone else in the class also picked A. You will most likely see a much higher percentage of students selecting A in the first version than in the second.

8 Location and Partnership

This chapter presents two important applied models. The applications illustrate the power of proper game theoretic reasoning, they demonstrate the art of constructing game theory models, and they guide the reader on how to calculate the set of rationalizable strategies. The location game is a finite (nine location) version of Hotelling's well-known model. This game has a unique rationalizable strategy profile. The partnership game has infinite strategy spaces, but it too has a unique rationalizable strategy profile. Analysis of the partnership game coaches the reader on how to compute best responses for games with differentiable payoff functions and continuous strategy spaces. The rationalizable set is determined as the limit of an infinite sequence. The notion of strategic complementarity is briefly discussed in the context of the partnership game.

Lecture Notes

Students should see the complete analysis of a few games that can be solved by iterated dominance. The location and partnership examples in this chapter are excellent choices for presentation. Both of these require nontrivial analysis and lead to definitive conclusions. It may be useful to substitute for the partnership game in a lecture (one can, for example, present the analysis of the Cournot duopoly game in class and let the students read the parallel analysis of the partnership game from the book). This gives the students exposure to two games that have continuous action spaces.

The following may serve as an outline for a lecture.

- Describe the location game and draw a picture of the nine regions. $S_i = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
- Show that the end regions are dominated by the adjacent ones. Write the dominance condition $u_i(1, s_j) < u_i(2, s_j)$. Thus, $R_i^1 = \{2, 3, 4, 5, 6, 7, 8\}$.
- Repeat. $R_i^2 = \{3, 4, 5, 6, 7\}$, $R_i^3 = \{4, 5, 6\}$, $R_i^4 = \{5\} = R$.
- Applications of the location model.
- Describe the partnership game (or Cournot game, or other). It is useful to draw the extensive form.
- Player i's belief about player j's strategy can be complicated, but, for expected payoff calculations, only the average (mean) matters. Thus, write $BR_1(\overline{y})$ or, for the Cournot duopoly game, $BR_i(\overline{q}_j)$, etc.
- Differentiate (or argue by way of differences) to get the best response functions.
- Sets of possible best responses (B_i).

- Restrictions: implications of common knowledge of rationality. Construct R_i^1, R_i^2, R_i^3. Indicate the limit R.
- Concept of strategic complementarity.

Examples and Experiments

1. *Location games.* You can play different versions of the location game in class (see, for instance, the variations in the Exercises section of the chapter).

2. *Repeated play and convergence.* It may be useful, although it takes time and prizes, to engage your class in repeated play of a simple matrix game. The point is not to discuss reputation, but rather to see whether experimental play stabilizes on one particular strategy profile or subset of the strategy space. This gets the students thinking about an institution (historical precedent, in this case) that helps align beliefs and behavior, which is a nice transition to the material in Chapter 9.

   Probably the easiest way of running the experiment is to have just two students play a game like the following:

1 \ 2	L	C	R
U	5,5	3,2	4,6
M	2,3	4,4	5,2
D	4,9	2,5	8,8

   A game like the one from Exercise 6 of Chapter 9 may also be worth trying. To avoid repeated-game issues, you can have different pairs of students play in different rounds. The history of play can be recorded on the chalkboard. You can motivate the game with a story.

3. *Contract or mediation.* An interesting variant on the convergence experiment can be used to demonstrate that pre-play communication and/or mediation can align beliefs and behavior. Rather than have the students play repeatedly, simply invite two students to play a one-shot game. In one version, they can be allowed to communicate (agree to a self-enforced contract) before playing. In a second version, you or a student can recommend a strategy profile to the players (but in this version, keep the players from communicating between themselves and separate them when they are to choose strategies).

9 Nash Equilibrium

This chapter provides a solid conceptual foundation for Nash equilibrium, based on (1) rationalizability and (2) strategic certainty, where players' beliefs and behavior are coordinated so there is some resolution of the second strategic tension. Strategic certainty is discussed as the product of various social institutions. The chapter begins with the concept of congruity, the mathematical representation of some coordination between players' beliefs and behavior. Nash equilibrium is defined as a weakly congruous strategy profile, which captures the absence of strategic uncertainty (as a single strategy profile). Various examples are furnished. Then the chapter addresses the issue of coordination and welfare, leading to a description of the third strategic tension—the specter of inefficient coordination. Finally, there is an aside on behavioral game theory (experimental work).

Lecture Notes

The following may serve as an outline for a lecture.

- Discuss strategic uncertainty (the second strategic tension). Illustrate with a game (such as the battle of the sexes) where the players' beliefs and behavior are not coordinated, so they get the worst payoff profile.

- Institutions that alleviate strategic uncertainty: norms, rules, communication, etc.

- Stories: (a) repeated social play with a norm, (b) pre-play communication (contracting), and (c) an outside mediator suggests strategies.

- Represent as congruity. Define *weakly congruous*, *best response complete*, and *congruous* strategy sets.

- Example of an abstract game with various sets that satisfy these definitions.

- *Nash equilibrium* (where there is no strategic uncertainty). A weakly congruous strategy profile. *Strict Nash equilibrium* (a congruous strategy profile).

- Examples of Nash equilibrium: classic normal forms, partnership, location, etc.

- An algorithm for finding Nash equilibria in matrix games.

- Pareto coordination game shows the possibility of inefficient coordination. Discuss real examples of inefficient coordination in the world. This is the third strategic tension.

- Note that a institution may thus alleviate the second tension, but we should better understand how. Also, the first and third strategic tensions still remain.

Examples and Experiments

1. *Coordination experiment.* To illustrate the third strategic tension, you can have students play a coordination game in the manner suggested in the previous chapter (see the repeated play, contract, and mediation experiments). For example, have two students play a Pareto coordination game with the recommendation that they select the inferior equilibrium. Or invite two students to play a complicated coordination game (with, say, ten strategies) in which the strategy names make an inferior equilibrium a focal point.

2. *The first strategic tension and externality.* Students may benefit from a discussion of how the first strategic tension (the clash between individual and joint interests) relates the classic economic notion of externality. This can be illustrated in equilibrium, by using any game whose equilibria are inefficient. An n-player prisoners' dilemma or commons game can be played in class. You can discuss (and perhaps sketch a model of) common economic settings where a negative externality causes people to be more active than would be jointly optimal (pollution, fishing in common waters, housing development, arms races).

3. *War of attrition.* A simple war of attrition game (for example, one in discrete time) can be played in class for bonus points or money. A two-player game would be the easiest to run as an experiment. For example, you could try a game like that in Exercise 9 of Chapter 22 with $x = 0$. Students will hopefully think about mixed strategies (or at least, nondegenerate beliefs). You can present the "static" analysis of this game. To compute the mixed strategy equilibrium, explain that there is a stationary "continuation value," which, in the game with $x = 0$, equals zero. If you predict that the analysis will confused the students, this example might be better placed later in the course (once students are thinking about sequential rationality).

10 Oligopoly, Tariffs, Crime, and Voting

This chapter presents six standard applied models: Cournot duopoly, Bertrand duopoly, tariff competition, a model of crime and police, candidate location (the median voter theorem), and strategic voting. Each model is motivated by an interesting, real strategic setting. Very simple versions of the models are described and the equilibria of four of the examples are calculated. Calculations for the other two models are left as exercises.

Lecture Notes

Any or all of the models can be discussed in class, depending on time constraints and the students' background and interest. Other equilibrium models can also be presented, either in addition to or substituting for the ones in the textbook. In each case, it may be helpful to organize the lecture as follows.

- Motivating story and real-world setting.
- Explanation of how some key strategic elements can be distilled in a game theory model.
- Description of the game.
- Overview of rational behavior (computation of best response functions, if applicable).
- Equilibrium calculation.
- Discussion of intuition.

Examples and Experiments

Students would benefit from a discussion of real strategic situations, especially with an eye toward understanding the extent of the first strategic tension (equilibrium inefficiency). Also, any of the applications can be used for classroom experiments.

Here is a game that can be played by e-mail, which may be useful in introducing mixed strategy Nash equilibrium in the next lecture. (The game is easy to describe, but difficult to analyze.) Students are asked to each submit a number from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The students make their selections simultaneously and independently. At a prespecified date, you determine how many students picked each of the numbers and you calculate the mode (the number that was most selected). For example, if ten students picked 3, eight students picked 6, eleven students picked 7, and six students picked 8, then the mode is 7. If there are two or more modes, the highest is chosen. Let x denote the mode. If $x = 9$, then everyone who selected the number 1 gets one bonus point (and the others get zero). If x is not equal to 9, then everyone who selected $x + 1$ gets $x + 1$ bonus points (and the others get zero).

11 Mixed-Strategy Nash Equilibrium

This chapter begins with the observation that, intuitively, a randomized strategy seems appropriate for the matching pennies game. The definition of a mixed-strategy Nash equilibrium is given, followed by instructions on how to compute mixed-strategy equilibria in finite games. The Nash equilibrium existence result is presented.

Lecture Notes

Few applications and concepts rely on the analysis of mixed strategies, so the book does not dwell on the concept. However, it is still an important topic and one can present several interesting examples. Here is a lecture outline.

- Matching pennies—note that there is no Nash equilibrium (in pure strategies). Ask for suggestions on how to play. Ask "Is there any meaningful notion of equilibrium in mixed strategies?"
- Note the $(1/2, 1/2), (1/2, 1/2)$ mixed strategy profile. Confirm understanding of "mixing."
- The definition of a *mixed-strategy Nash equilibrium*—the straightforward extension of the basic definition.
- Two important aspects of the definition: (a) what it means for strategies that are in the support (they must all yield the same expected payoff) and (b) what it means for pure strategies that are *not* in the support of the mixed strategy.
- An algorithm for calculating mixed-strategy Nash equilibria: Find rationalizable strategies, look for a mixed strategy of one player that will make the other player indifferent, and then repeat for the other player.
- Note the mixed-strategy equilibria of the classic normal form games.
- Mixed-strategy Nash equilibrium existence result.

Examples and Experiments

1. *Attack and defend.* Discuss how some tactical choices in war can be analyzed using matching pennies-type games. Use a recent example or a historical example, such as the choice between Normandy and the Pas de Calais for the D-Day invasion of June 6, 1944. In the D-Day example, the Allies had to decide at which location to invade, while the Germans had to choose where to bolster their defenses. Discuss how the model can be modified to incorporate more realistic features.

2. *A socially repeated strictly competitive game.* This classroom experiment demonstrates how mixed strategies may be interpreted as frequencies in a population of players. The experiment can be done over the Internet or in class. The classroom version may be unwieldy if there are many students. The game can be played for money or for points in the class competition.

   For the classroom version, draw on the board a symmetric 2×2 strictly competitive game, with the strategies Y and N for each of the two players. Use a game that has a unique, mixed-strategy Nash equilibrium. Tell the students that some of them will be randomly selected to play this game against one another. Ask all of the students to select strategies (by writing them on slips of paper or using cards as described below). Randomly select several pairs of students and pay them according to their strategy profile. Compute the distribution of strategies for the entire class and report this to all of the students. If the class frequencies match the Nash equilibrium, then discuss this. Otherwise, repeat the gaming procedure several times and discuss whether play converges to the Nash equilibrium.

   Here is an idea for how to play the game quickly. With everyone's eyes closed, each student selects a strategy by either putting his hands on his head (the Y strategy) or folding his arms (the N strategy). At your signal, the students open their eyes. You can quickly calculate the strategy distribution and randomly select students (from a class list) to pay.

3. *Another version of the socially repeated game.* Instead of having the entire class play the game in each round, have only two randomly selected students play. Everyone will see the sequence of strategy profiles and you can discuss how the play in any round is influenced by the outcome in preceding rounds.

4. *Randomization in sports.* Discuss randomization in sport (soccer penalty shots, tennis service location, baseball pitch selection, American football run/pass mix).

In addition to demonstrating how random play can be interpreted and form a Nash equilibrium, the social repetition experiments also make the students familiar with strictly competitive games, which provides a good transition to the material in Chapter 12.