

Economics 245B  
Exercise 2

Consider the regression model

$$Y_t = X_t' \beta + U_t, \quad t = 1, \dots, n$$

in which  $X_t$  and  $\beta$  are  $K \times 1$  vectors. While the error is mean zero and i.i.d. unconditionally,  $U_t$  exhibits conditional heteroskedasticity of the ARCH form

$$U_t = \sigma_t V_t,$$

with  $V_t \sim \text{iid } N(0, 1)$  and

$$\sigma_t^2 = \alpha_0 + \alpha_1 U_{t-1}^2$$

To ensure the conditional variance is positive, we impose the sufficient condition  $\alpha_0 > 0$  and  $\alpha_1 \geq 0$ .

a) Construct the likelihood function  $L(\beta, \alpha)$ . How do we impose the constraints?

b) To avoid constrained optimization, one could replace  $(\alpha_0, \alpha_1)$  with  $(\gamma_0^2, \gamma_1^2)$ . (As  $\gamma_i$  need not be positive, unconstrained optimization can be used. The invariance property of ML estimators allows one to then construct the estimate of  $\alpha_i$  by squaring the estimate of  $\gamma_i$ .) Construct  $L^*(\beta, \gamma)$ . What is the relation between the MLE of  $\gamma$  from  $L^*$  and the MLE of  $\alpha$  from  $L$ ?

c) Show that  $\gamma_i = 0$  solves the first-order condition for  $L^*$  for each  $i$  and any data.

d) What do you conclude about the usefulness of reparameterizations of the likelihood as in b)?