Range-Based Estimation of Stock Price Volatility

Study of range-based estimators begins with specification of a continuous-time process for the share price

$$\ln P(t) = \mu t + \sigma B\,(t)\,,$$

where $B\,(t)$ is standard Brownian motion. The motivation for such a specification is the finding that observed asset prices follow a random walk. A continuous-time process for $\ln P(t)$ underpins the Black-Scholes option pricing formula, in which σ appears but μ does not. As the motivation of the literature is to estimate option prices, attention is focused almost exclusively on estimation of σ, so μ is assumed to equal 0.

A sample path of $\ln P(t)$ for a day, where a day is assumed to correspond to $0 \leq t \leq 1$, is plotted in Figure 1.[1] Traditional measures of σ use only the information contained in closing prices, which are denoted in the figure as C_0 and C_1. (In what follows, C_t is the logarithm of the recorded closing price, H_t is the logarithm of the recorded high, L_t is the logarithm of the recorded low and O_t is the logarithm of the recorded opening price.) In 1976 (published in 1980) Parkinson had the idea that other sample points from the time path might provide more information about σ. Specifically, Parkinson suggested using the range, defined as the high minus the low price over a set calendar interval $(H_1 - L_1)$ in Figure 1. The intuition behind Parkinson is straightforward: The closing prices are measured at the same point on the curve each day, but the high and the low could be at any point on the curve. In essence, the high and low are formed from the entire curve, and so reveal more information, than do points sampled at fixed intervals.

Because the underlying process is constructed from standard Brownian motion, many mathematical results can be brought to bear. Parkinson exploits the work done by Feller in 1951 that determines the distribution of the range. In particular, Parkinson notes

$$E\,(H_1 - L_1) = \left(\frac{8}{\pi}\right)^{\frac{1}{2}} \sigma \text{ and } E\,(H_1 - L_1)^2 = 4 \ln 2 \cdot \sigma^2.$$

[1] Figure 1 is taken from Garman and Klass (1980) who use $z\,(t)$ to denote standard Brownian motion and set $\ln P(t) = B\,(t) = \sigma z\,(t)$, thus $B\,(t)$ rather than $\sigma B\,(t)$ appears at the top of the figure.

Parkinson advocates use of the second moment condition to estimate σ^2 and proposes

$$S_P^2 = \frac{.361}{n}\sum_{t=1}^{n}(H_t - L_t)^2 .$$

He compares his estimator with a traditional estimator of σ^2 that uses closing prices, which is

$$S^2 = \frac{1}{n-1}\sum_{t=1}^{n}(C_t - C_{t-1} - M)^2 \text{ where } M = \frac{1}{n}\sum_{t=1}^{n}(C_t - C_{t-1}),$$

and finds that $\frac{Var(S^2)}{Var(S_P^2)} = 5.2$. Of course, such a comparison of variances only makes sense if both estimators are unbiased. Yet Parkinson's estimator is unbiased only if $\mu = 0$; if there is a non-zero drift (as there almost certainly is) then S_P^2 is biased. (Note S^2 remains unbiased because of the term M that captures a non-zero drift. Such a simple correction will not work for the range estimator of Parkinson.)

Garman and Klass followed up on Parkinson's original idea. In Figure 1, the sample path of the price is divided into a portion in which the market is open and portion in which the market is closed (f). The use of the range incorporates information only over the period during which the market is open (implicitly setting $f = 0$). Because prices are not observed when the market is closed, the only measure of uncertainty during that period is the opening price on day 1, O_1, less the closing price on the previous day, C_0. The first suggestion of Garman and Klass is thus to estimate σ^2 with an estimator of the form

$$\alpha\frac{1}{n}\sum_{t=1}^{n}(O_t - C_{t-1})^2 + (1-\alpha)\frac{.361}{n}\sum_{t=1}^{n}(H_t - L_t)^2 ,$$

where the weight α captures two effects. The first effect is the relative length of market closures, the second effect is the relative accuracy of the open-close estimator to the range estimator. As the market was traditionally closed for more of the day than it was open, the effect of closure would lead to $\alpha > \frac{1}{2}$.[2] Of course, markets are now open for more of the day and the information revelation of firm values may not be constant across a day, so the correct weights to assign on the basis of market closures are ambiguous. The second effect is much more clearly

[2] To capture the length of market closure, Garman and Klass weight the open-close estimator by f^{-1} and the range estimator by $(1-f)^{-1}$. Such a weighting scheme is not intuitive as it gives more weight to the open-close estimator as the length of market closure goes down.

determined. As the range estimator is a more accurate estimator of σ^2 than is a point-in-time estimator such as the open-close estimator, α should be less than $\frac{1}{2}$. In fact, Garman and Klass find that if $f = \frac{1}{2}$ then $\alpha = .17$.

The second suggestion of Garman and Klass is to account for the joint dependence of (O_t, L_t, H_t, C_t). To do so they consider a joint density for these quantities, which depends on σ^2. They assume, sensibly, that the distribution is scale invariant. They further assume, less sensibly, that the density is analytic (that is, can be exactly represented by a Taylor-series expansion with no remainder) in a neighborhood of the origin. Together, these two assumptions imply that the density is a quadratic function of (O_t, L_t, H_t, C_t). Because the underlying process is Brownian motion, one can reduce the quadratic function to

$$.511\,(H_1 - L_1) - .019\left[(C_1 - O_1)(H_1 - O_1 + L_1 - O_1) - 2(H_1 - O_1)(L_1 - O_1)\right] - .383\,(C_1 - O_1)^2 .$$

If we let $f(O_t, L_t, H_t, C_t)$ denote the preceding displayed equation, then Garman and Klass recommend estimating σ^2 with

$$S_{GK}^2 = \alpha \frac{1}{n}\sum_{t=1}^{n}(O_t - C_{t-1})^2 + (1-\alpha)\frac{1}{n}\sum_{t=1}^{n} f(O_t, L_t, H_t, C_t),$$

with $\alpha = .12$. The reduction in α from .17 to .12 makes sense; allowing for joint dependence of the range and close provides a more accurate estimator from the observed market data.

In 1984, Ball and Torous extended the Garman and Klass estimator. Ball and Torous relax the assumption that the joint density of (O_t, L_t, H_t, C_t) is analytic in a neighborhood of the origin, and instead derive the joint density directly from the underlying Brownian motion process. By doing so, Ball and Torous are able to more accurately model the joint density. The key to their construct is the *transition density* that describes the distribution of possible values that a Brownian motion will next take on, or transit to. The joint density of (O_t, L_t, H_t, C_t) is defined to be

$$g\left(O_t, L_t, H_t, C_t; \sigma^2\right) = \frac{\partial^3}{\partial l \partial h \partial c} P\left[-(L_t - O_t) \le l, (H_t - O_t) \le h, (C_t - O_t) \le c\right].$$

At first glance, this is a messy problem. Life is made simpler by noting that for a Brownian motion process the partial derivative with respect to c, which is the transition density, is known. Specifically,

$$\begin{aligned} & \frac{\partial}{\partial c} P\left[-(L_t - O_t) \le l, (H_t - O_t) \le h, (C_t - O_t) \le c\right] \\ = & \frac{\partial}{\partial c} P\left[(C_t - O_t) \le c, -l \le \sigma B(s) \le h \text{ for } 0 \le s \le t\right]. \end{aligned}$$

The right side of the above equality is defined to be the transition density, often denoted $p(c,t)$. The expression for $p(c,t)$ is obtained as the solution to

$$\frac{\partial p(c,t)}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 p(c,t)}{\partial c^2}$$

subject to the constraints $p(-l,t) = p(h,t) = 0$. The solution is the absolutely-convergent infinite series

$$p(c,t) = \sum_{j=1}^{\infty} \left[\frac{2}{h+l}\sin\left(\frac{j\pi l}{h+l}\right)\right] \sin\left[\frac{j\pi(c+l)}{h+l}\right] \exp\left[\frac{-j^2\pi^2\sigma^2 t}{2(h+l)^2}\right].$$

To obtain the joint density, we differentiate each term of $p(c,t)$ with respect to both h and l to deliver

$$g\left(O_t, L_t, H_t, C_t; \sigma^2\right) = \sum_{j=1}^{\infty} \exp\left[-\beta_0(j)\sigma^2\right]\left[\beta_1(j)\sigma^4 + \beta_2(j)\sigma^2 + \beta_3(j)\right],$$

where

$$\begin{aligned}
\beta_0(j) &= \frac{j^2\pi^2}{2(h+l)^2}, \\
\beta_1(j) &= \frac{2j^4\pi^4}{2(h+l)^7}\sin\left[\frac{j\pi(l+c)}{h+l}\right]\sin\left[\frac{j\pi b}{h+l}\right], \\
\beta_2(j) &= \frac{2j^3\pi^3(h-l)}{(h+l)^6}\sin\left[\frac{j\pi(2l+c)}{h+l}\right] - \frac{4j^3\pi^3}{(h+l)^6}\sin\left(\frac{j\pi l}{h+l}\right)\cos\left[\frac{j\pi(l+c)}{h+l}\right] \\
&\quad -\frac{10j^2\pi^2}{(h+l)^5}\sin\left(\frac{j\pi l}{h+l}\right)\sin\left[\frac{j\pi(l+c)}{h+l}\right], \\
\beta_3(j) &= \frac{2j^2\pi^2}{(h+l)^5}(lc-2hl-hc)\cos\left[\frac{j\pi(2l+c)}{h+l}\right] + \frac{4j\pi(l-h)}{(h+l)^4}\sin\left[\frac{j\pi(2l+c)}{h+l}\right] \\
&\quad +\frac{8j\pi c}{(h+l)^4}\sin\left(\frac{j\pi l}{h+l}\right)\cos\left[\frac{j\pi(l+c)}{h+l}\right] \\
&\quad +\left[\frac{4}{(h+l)^3} - \frac{2j^2\pi^2c^2}{(h+l)^5}\right]\sin\left(\frac{j\pi l}{h+l}\right)\sin\left[\frac{j\pi(l+c)}{h+l}\right].
\end{aligned}$$

To form the log-likelihood, Ball and Torous assume that each day is identical and is independent of all other days. The assumption, which is clearly false, implies that intraday dependence in prices plays the crucial role. For a sample of n days the log-likelihood is

$$L\left[\sigma^2 |\left(O_1, L_1, H_1, C_1\right), \ldots, \left(O_n, L_n, H_n, C_n\right)\right] = \sum_{t=1}^{n} \ln g\left(O_t, L_t, H_t, C_t; \sigma^2\right).$$

To maximize the log-likelihood, two issues must be addressed. The first is that the expression for $g\left(O_t, L_t, H_t, C_t; \sigma^2\right)$ is an infinite series. Ball and Torous bound the truncation error and find that an accurate approximation is delivered by the first seven terms

$$g_T\left(O_t, L_t, H_t, C_t; \sigma^2\right) = \sum_{j=1}^{7} \exp\left[-\beta_0(j)\sigma^2\right]\left[\beta_1(j)\sigma^4 + \beta_2(j)\sigma^2 + \beta_3(j)\right].$$

They maximize $\sum_{t=1}^{n} \ln g_T\left(O_t, L_t, H_t, C_t; \sigma^2\right)$. To do so, they use the Newton-Raphson optimization method in which the ith iterated estimator is

$$S_i^2 = S_{i-1}^2 - \left(\frac{\partial^2 L}{\partial\left(\sigma^2\right)^2}\right)^{-1} \frac{\partial L}{\partial \sigma^2},$$

with $S_0^2 = S_{GK}^2$. Let $S_{converge}^2$ denote the final estimator from the optimization method. The resultant estimator of Ball and Torous is

$$S_{BT}^2 = \alpha \frac{1}{n} \sum_{t=1}^{n} \left(O_t - C_{t-1}\right)^2 + (1-\alpha) S_{converge}^2,$$

with $\alpha = .11$.[3]

In 1991, Rogers and Satchell addressed two criticisms of the above estimators. The first criticism is that μ is assumed equal to zero, which implies that the expected return for an investor is zero. The second criticism is that discrete sampling from a continuous process yields an observed range that is smaller than the actual range. They make their modifications to the Garman-Klass estimator, rather than the ML estimator, perhaps because of the ease of construction of S_{GK}^2.

To relax the assumption that $\mu = 0$, Rogers and Satchell make clever use of the exponential distribution to show that for all $0 \leq s \leq 1$ both $EH_s\left(H_s - P_s\right)$ and $EL_s\left(L_s - P_s\right)$ equal $\frac{\sigma^2 s}{2}$. Hence

$$E\left[H_1\left(H_1 - C_1\right) + L_1\left(L_1 - C_1\right)\right] = \sigma^2$$

[3] The value of α corresponds to a relative efficiency for the ML estimator in the middle of the reported efficiencies from Table 1.

for all μ. The downward bias of the observed range is accounted for in the following manner. Because prices are only observed at the discrete intervals at which trade occurs, one can think of constructing a random walk as an approximation to the continuous time process. Let h^{-1} be the number of steps the random walk takes in $[0,1]$. If $H_1 = \sup\{\ln P(s) : 0 \leq s \leq 1\}$ and H is the maximum of the random walk, then

$$H_1 = H + \Delta \text{ and } L_1 = L - \tilde{\Delta},$$

where Δ and $\tilde{\Delta}$ have the same distribution. Thus

$$H_1 (H_1 - C_1) = \Delta^2 + (2H - C_1)\Delta + H(H - C_1),$$

and

$$L_1 (L_1 - C_1) = \tilde{\Delta}^2 - (2L - C_1)\tilde{\Delta} + L(L - C_1).$$

Rogers and Satchell assume that if H occurs at time s^*, a multiple of h, then H_1 occurs in the interval $[s^* - h, s^* + h]$. From this assumption, they show that

$$E\Delta = a\sigma h^{\frac{1}{2}} \text{ and } E\Delta^2 = b\sigma^2 h,$$

with $a = (2\pi)^{\frac{1}{2}} \left[\frac{1}{4} - \frac{2^{\frac{1}{2}} - 1}{6}\right]$ and $b = \frac{1}{12} + \frac{\pi}{16}$. The unbiased estimator of σ^2, which corrects for the bias in measuring the range, is the positive root S_{RS1} of the equation

$$S_{RS1}^2 = 2bS_{RS1}^2 h + 2(H - L)\, aS_{RS1} h^{\frac{1}{2}} + H(H - C_1) + L(L - C_1).$$

The resulting estimator from Rogers and Satchell can be used to estimate σ^2 with

$$S_{RS}^2 = \alpha \frac{1}{n} \sum_{t=1}^{n} (O_t - C_{t-1})^2 + (1 - \alpha) S_{RS1n}^2,$$

where again $\alpha = .11$ and S_{RS1n} is the positive root of

$$S_{RS1n}^2 = 2bS_{RS1n}^2 \frac{1}{n} \sum_{t=1}^{n} h_t + 2aS_{RS1n} \frac{1}{n} \sum_{t=1}^{n} (H_t - L_t)\, h_t^{\frac{1}{2}} + \frac{1}{n} \sum_{t=1}^{n} \left[H_t (H_t - C_t) + L_t (L_t - C_t)\right].$$

As Rogers and Satchell assume that the steps are evenly spaced, h_t^{-1} could be approximated by the number of trades on day t.

Discussion of how to implement the Rogers and Satchell estimator raises several interesting questions. Rogers and Satchell assume both that the number of steps, h^{-1}, is fixed and that the measurements occur at equally spaced intervals. As trading frequency varies across stocks and over time, and as trade occurs randomly throughout a day, both assumptions are violated in recorded trade prices. A more accurate measurement of h_t, and hence of the bias of the estimated range, may be delivered by adjusting h_t to account for random trading. (In 1983 Beckers noted that the relative accuracy of the Garman-Klass and Parkinson estimators varied over days and firms. Other than this, the paper adds little.) To account for random trading, one can disentangle the accuracy of measurement of the high and the low. After all, trading may be much more frequent at one point of the range, say the high, than at the other. To do so, we measure the accuracy of trading at the high separately from the accuracy of trading at the low. The accuracy of trading at the high, h_{Ht}, is simply the largest elapsed time from the trade at the high to either the preceding or following trade. Correspondingly, the accuracy of trading at the low, h_{Lt}, is the largest elapsed time from the trade at the low to either the preceding or the following trade. With these measures of bias, S_{RS2n} is the positive root of

$$\begin{aligned} S_{RS2n}^2 &= bS_{RS2n}^2 \frac{1}{n}\sum_{t=1}^{n}(h_{Ht}+h_{Lt}) + aS_{RS1n}\frac{1}{n}\sum_{t=1}^{n}\left[(2H_t - C_t)\, h_{Ht}^{\frac{1}{2}} + (2L_t - C_t)\, h_{Lt}^{\frac{1}{2}}\right] \\ &\quad + \frac{1}{n}\sum_{t=1}^{n}\left[H_t\,(H_t - C_t) + L_t\,(L_t - C_t)\right]. \end{aligned}$$

To estimate σ^2, one would replace S_{RS1n}^2 with S_{RS2n}^2 in the formula for S_{RS}^2.

An open question, can one make the Rogers and Satchell modifications to the ML estimator of Ball and Torous?

In 1992, Kunitomo proposed an alternative correction to the Parkinson estimator to account for $\mu \neq 0$. The attached figure labeled Kunitomo is similar to the earlier figure from Garman and Klass. The innovation is to connect the open and closing prices with the line labeled AF, which in effect forms a Brownian bridge. Rather than use the range, labeled l in the graph, Kunitomo argues for use of the adjusted range R. Kunitomo derives the distribution of R and shows that if the interval over which the Brownian motion process runs is of length 1, then

$$ER = \sigma\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \text{ and } ER^2 = \sigma^2\frac{\pi^2}{6}.$$

The latter of these is used to construct the estimator

$$\frac{6}{\pi^2}\frac{1}{n}\sum_{t=1}^{n} R_t^2,$$

which, in combination with the close-open component yields

$$S_K^2 = \alpha \frac{1}{n} \sum_{t=1}^{n} (O_t - C_{t-1})^2 + (1-\alpha) S_{K1}^2,$$

with $\alpha = .11$.

Forecast Evaluation

How should one choose among the various estimators of the volatility (diffusion) parameter? The first difficulty is that the object under forecast, σ, is not observable (and is termed a latent variable). The most natural way around the problem is to compare the estimator of the daily volatility with the observed squared daily return, $(C_t - C_{t-1})^2$. (To be formal, $(C_t - C_{t-1})^2$ is the discretely observed time series process of continuously compounded returns with a return horizon of 1 day.) If we let n^* represent the forecast evaluation period and let S^2 represent a candidate estimator, the mean square forecast error is

$$MSE\left(S^2\right) = \frac{1}{n^*} \sum_{t=1}^{n^*} \left[(C_t - C_{t-1})^2 - S_t^2\right].$$

Of course, the same measure can be achieved by selecting the estimator that maximizes the R-squared in the regression

$$(C_t - C_{t-1})^2 = \gamma_0 + \gamma_1 S_t^2 + V_t^2.$$

In 1998, Andersen and Bollerslev noted that the squared daily returns based on closing prices might be a poor measure of σ and so not a valuable forecast criterion. They argued for accumulating squared returns over the course of the day, in order to approximate the quadratic variation of the process, and in so doing develop a better forecast criterion. They argue that variable $\sum_{j=1}^{J} (C_{j,t} - C_{j-1,t})^2$ is a more accurate measure of the diffusion. For example, if there were 8 hours in a trading day and we summed hourly returns, $J = 8$ and $C_{j,t}$ is the jth hourly return on day t. The corresponding MSE and regression evaluations would be achieved by replacing $(C_t - C_{t-1})^2$ with $\sum_{j=1}^{J} (C_{j,t} - C_{j-1,t})^2$, which is a much smoother measure.

Option Pricing

An estimator of σ^2 is essential in pricing an option. Consider the price of a call option with strike price κ and time to maturity τ. If r is the continuously compounded risk-free interest rate and P_t is the current price of the stock, then the Black-Scholes formula for the current price of the call option is

$$P_{ct}\left(\sigma^2\right) = P_t \Phi\left(d_1\right) - \kappa e^{-r\tau} \Phi\left(d_2\right),$$

where Φ is the cumulative distribution function of a standard Gaussian random variable and

$$d_1 = \frac{\ln\left(\frac{P_t}{\kappa}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\left(\sigma^2 \tau\right)^{\frac{1}{2}}} \text{ and } d_2 = d_1 - \left(\sigma^2 \tau\right)^{\frac{1}{2}}.$$

To derive confidence bounds on the option price obtained from S^2, suppose $n^{\frac{1}{2}}\left(S^2 - \sigma^2\right) \sim N\left(0, V\right)$. The estimator of P_{ct} is

$$n^{\frac{1}{2}}\left[P_{ct}\left(S^2\right) - P_{ct}\left(\sigma^2\right)\right] \sim \left.\frac{\partial P_{ct}\left(\sigma^2\right)}{\partial \sigma^2}\right|_{\sigma^2 = S^2} N\left(0, V\right).$$

Simulation of Brownian Motion

Ball and Torous discuss simulation of a Brownian motion process on $[0,1]$. The approximation is obtained for all $t \in [0,1]$ as

$$B_{app}\left(t\right) = \sum_{j=1}^{[kt]} V_j,$$

where $\left\{V_j\right\}_{j=1}^{k}$ is a sequence of i.i.d. uniform random variables with mean 0 and variance $\frac{\sigma^2}{k}$. (A sequence of i.i.d. Gaussian random variables with mean 0 and variance $\frac{\sigma^2}{k}$ could be used as well.) They select $k = 100,000$. To avoid serial correlation from the random number generator, divide the sequence of 100,000 uniform random numbers into 100 sequences of 1000 each and randomly shuffle the sequences. Because the increments to a Brownian motion process are independent, the randomness of the shuffle creates no problems.

References

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