

Chapter 7

Common Value Auctions

So far we have studied auctions for which bidders have private values. In private value auctions each bidder knows how much she values the item, and this value is her private information. In this chapter we will discuss common value auctions. In common value auctions, the actual value of the item for sale is the same for everyone, but bidders have different private information about what that value is.

Many important auctions are common value auctions. Examples include Treasury bill auctions, auctions of timber, spectrum auctions, and auctions of oil and gas leases. In each case, the value of the item is the same to all the bidders, but different bidders have different information about what that value actually is.

In common value auctions the bidders are often subject to the “winner’s curse.” The winner’s curse has been described in many ways. The simplest definition is the following:

Winner’s curse: In a common value auction the bidder with the “best” (most optimistic) information wins. A bidder who fails to take this into account pays, on average, more than the item is worth.

Example: Penny Jar Experiment

However, the winner’s curse also results in common value auctions when bidders fail to account for the way private information influences the bidding behavior of their opponents. This idea is best illustrated by the following example.

Example: Auctioning an Oil Lease

Player A and Player B are each bidding to purchase the rights to develop an oil field. The field had two parts, Part A and Part B, each of which contains either \$0 or \$3 million worth of oil. Each possibility is equally likely and independently determined. Player A is privately informed about the amount of oil in Part A. Player B is privately informed about the amount of oil in Part B. The two players participate in a first-price auction to purchase the rights to both parts of the field.

We will discuss the results of this experiment in class. Here we derive the theoretical predictions. It is useful to go over the informational aspects of this game before we begin. Player A knows the amount of oil in part A, but does not know the amount of oil in part B. She only knows that with probability .5 it is \$0 and with probability .5 it is \$3 million. Likewise for player B. How should the players bid in a first-price auction to buy both parts?

No doubt the answer should depend on a bidder's private information. If a bidder sees that her part of the field is worth \$0, she knows the value of both parts is either \$0 or \$3 million. If she sees \$3 million, she knows the value of both parts is either \$3 or \$6 million. Suppose she bids the expected value of the field conditional on her private information. That is, suppose she bids \$1.5 million if she sees \$0, and \$4.5 million if she sees \$3 million. Sound good? Well, it turns out that would be a bad idea. In fact, even if she bids amounts less than these, say to build in a small profit margin, she will most likely be subject to, what economists call, the "winner's curse."

The problem with the bidding the expected value of the field conditional on your private information (or even some positive amount less than that) is most evident in the case where the bidder sees \$0. Suppose a bidder sees \$0 and bids \$1 million. This is well below the expected value of the field, which is \$1.5 million. Now ask yourself, how should this bidder feel if she finds out she has won the auction? She would probably be happy at first (we all like to win). But then, she might think, if my bid of \$1 million won, how much oil is likely to be in the field. Note that this is a different question than, how much oil is in the field conditional on my private information. Now we are asking how much is the expected value of the field conditional on the value of my winning bid. This of course depends on the strategy of the opponent and so we can't answer this formally until we introduce the idea of equilibrium strategies. However, it is safe to assume at this point that if the opponent saw \$3 million (and knew there was a 50% chance that the field was worth another \$3 million on top of that) she probably would have bid more than \$1. In other words, if a bidder wins with a bid of \$1 million the field is probably worth \$0! Any bidder who bids a positive amount when she sees \$0 will probably lose money if she wins the auction.

It turns out that a bidder should also bid less than \$4.5 million when she sees \$3 million, but this is hard to explain without conducting the full equilibrium analysis. So, let's do just that. We now verify that the unique symmetric equilibrium of this game is for each bidder to bid according to the bid function:

$$b(0) = 0, \quad b(3) = \tilde{x} \in [0, 3]$$

where \tilde{x} is distributed according to $B(x) = \frac{x}{6-x}$.

Suppose bidder B bids according to the proposed equilibrium strategy. Consider the case where bidder A sees 0, and consider an arbitrary bid $b_A > 0$ by

bidder A. The expected payoff for bidder A is

$$\begin{aligned} & \frac{1}{2}(0 - b_A) + \frac{1}{2}B(b_A)(3 - b_A) \\ &= -\frac{1}{2}b_A + \frac{1}{2}\frac{b_A}{6 - b_A}(3 - b_A) < 0, \end{aligned}$$

since $\frac{3-b_A}{6-b_A} < 1$. Hence, $b(0) = 0$ is a best response to player B's strategy.

Next, consider the case where bidder A sees \$3 million. Given the strategy of bidder B, and bid for bidder A between 0 and 3 have an expected payoff of $\frac{3}{2}$:

$$\begin{aligned} & \frac{1}{2}(3 - b_A) + \frac{1}{2}B(b_A)(6 - b_A) \\ &= \frac{1}{2}(3 - b_A) + \frac{1}{2}\frac{b_A}{6 - b_A}(6 - b_A) \\ &= \frac{1}{2}(3 - b_A) + \frac{1}{2}b_A = \frac{3}{2}. \end{aligned}$$

Suppose she bids $b_A > 3$. Then, her expected payoff is

$$\begin{aligned} & \frac{1}{2}(3 - b_A) + \frac{1}{2}(6 - b_A) \\ &= 4\frac{1}{2} - b_A < \frac{3}{2}. \end{aligned}$$

Hence, $b(3) = \tilde{x} \in [0, 3]$, where \tilde{x} is distributed according to $B(x) = \frac{x}{6-x}$ is a best response to player B's strategy.

We have shown that if player B follows the equilibrium strategy it is a best response for player A to do so as well. If we switch the roles of player A and B, the same argument shows that if player A follows the equilibrium strategy it is a best response for player B to do so as well. That's it. We have verified the Nash equilibrium.

Report on Experimental 10: Oil lease experiment.

Exercise 7.1 *Assume that bidders in the oil field experiment can only place whole number bids. Compute the pure strategy Nash equilibrium.*

Answer: The symmetric Nash equilibrium bid function is $b(0) = 0$, $b(3) = 2$.

We will now discuss two, related models of common value auctions. The penny jar example fits the first model, the oil lease example is best captured by the second model.

7.1 Model I

The true value of the item being auctioned is v , but v is unknown to all bidders. Each bidder i receives a signal, s_i , about the true value, which is given by the sum of the true value v and a random variable \tilde{e}_i , which you should think of as a private noise term:

$$s_i = v + \tilde{e}_i,$$

We assume that \tilde{e}_i satisfies $E[\tilde{e}_i] = 0$ and hence each bidder's signal has the property that $E[s_i] = v$. That is, the expected value of each bidder's signal is equal to the true value.

Reality Check: Where do these signals come from? The answer depends on the auction environment. For example, in the case of an auction to buy a gold mine, each bidder has to place an estimate on the amount of gold. This estimate can be derived from private tests of rock samples and these tests have errors associated with the predictions. All we are saying in our requirement that $E[\tilde{e}_i] = 0$, or equivalently that $E[s_i] = v$, is that on average people's estimates are correct. In other words, people do not systematically over estimate or underestimate the true value. If they always over estimated it, for example, it would not be surprising that the winner ends up paying too much. Our goal in illustrating the winner's curse is to show that it arises even if people's estimates of the true value are correct on average.

We will use this model to illustrate the winner's curse. In particular, we will look at what happens to the winning bidder (in terms of her payoff) if she bids too large a fraction of her signal. We will discuss how this depends on the number of bidders and on how big the noise parameter can be relative to the true value.

For concreteness, we will consider a specific random variable \tilde{e}_i , that meets our requirement that $E[\tilde{e}_i] = 0$. Namely, we assume that the each bidder's realized signal e_i is determined by a draw from the uniform distribution on $[-1, 1]$. Hence, the probability that bidder i 's signal is less than or equal to some value $e \in [-1, 1]$, is

$$H(e) = \frac{1 + e}{2}.$$

$H(\cdot)$ is the distribution function for the random variable \tilde{e}_i . It has density function $h(e) = \frac{1}{2}$. Note that $H(-1) = 0$ and $H(1) = 1$, as required. Moreover,

$$E[\tilde{e}_i] = \int_{-1}^1 x \frac{1}{2} dx = \left[\frac{x^2}{4} \right]_{-1}^1 = \frac{1}{4} - \frac{1}{4} = 0,$$

as required.

Now suppose all bidders bid a fraction, m , of their observed signal (We will estimate this fraction using data from a class experiment). We want to show that if m is too close to 1 the winner loses money in expectation. I say "in expectation" because it is not the case that the winner will always lose

money; sometimes improbable things happen and bad decisions can turn out well. However, if a person played the game many times and always bid too close to their value they would lose money.

Since everyone is assumed to bid the same fraction of their signal, the winner will be the bidder with the highest private signal, which is the bidder with the highest realization of the random variable \tilde{e}_i . We denote the random variable for the highest noise term by $\tilde{e}_{(1)}$. In order to compute the winner's expected payoff, we need to compute the expected value of $\tilde{e}_{(1)}$. To do this, we first need to know its density function.

Since $H(e)$ is the distribution of any one noise term (i.e., the realization of any one random variable \tilde{e}_i), the probability that all n realizations of the noise term are less than a value $e \in [-1, 1]$ is

$$H_{(1)} = \left(\frac{1+e}{2} \right)^n,$$

with density

$$h_{(1)}(e) = n \left(\frac{1+e}{2} \right)^{n-1} \frac{1}{2}.$$

Hence

$$\begin{aligned} E[\tilde{e}_{(1)}] &= \int_{-1}^1 e \frac{n}{2} \left(\frac{1+e}{2} \right)^{n-1} de \\ &= \frac{1}{2^n} \int_{-1}^1 ne(1+e)^{n-1} de \end{aligned}$$

Suppose $n = 2$. Then

$$\begin{aligned} E[\tilde{e}_{(1)}] &= \frac{1}{2} \int_{-1}^1 e(1+e)de \\ &= \frac{1}{2} \int_{-1}^1 (e+e^2)de \\ &= \frac{1}{2} \left[\frac{e^2}{2} + \frac{e^3}{3} \right]_{-1}^1 \\ &= \frac{1}{2} \left[\frac{3e^2 + 2e^3}{6} \right]_{-1}^1 \\ &= \frac{1}{2} \left[\frac{5}{6} - \frac{1}{6} \right] = \frac{1}{3}. \end{aligned}$$

Note that even though $E[\tilde{e}_i] = 0$, the expectation of $E[\tilde{e}_{(1)}] > 0$. Consequently, $E[\tilde{v}_{(1)}] = v + E[\tilde{e}_{(1)}] > v$.

Recall that we assume bidders bid some fraction m of their signal. The implication is that the winner will lose money (in expectation) if

$$m(v + E[\tilde{e}_{(1)}]) - v = m\left(v + \frac{1}{3}\right) - v > 0$$

This occurs when

$$m > \frac{v}{v + \frac{1}{3}}.$$

For example, suppose that $v = 3$, then the winning bidder loses money in expectation only if $m > \frac{9}{10}$.

Now let's see what happens if there are more bidders. Does the winner's curse problem get better or worse? Consider $n = 3$. With three bidders, the expected value of the winner's signal is greater because the expected value of $\tilde{e}_{(1)}$ is greater. Namely,

$$\begin{aligned} E[\tilde{e}_{(1)}] &= \frac{n}{2^n} \int_{-1}^1 e(1+e)^{n-1} de \\ &= \frac{3}{8} \int_{-1}^1 e(1+e)^2 de \\ &= \frac{1}{2}. \end{aligned}$$

Hence, $E[\tilde{v}_{(1)}] = v + E[\tilde{e}_{(1)}] = v + \frac{1}{2}$.

Now, assuming all bidders bid a fraction m of their signal, the winner will lose money (in expectation) if

$$m(v + E[\tilde{e}_{(1)}]) - v = m(v + \frac{1}{2}) - v > 0,$$

which occurs when

$$m > \frac{v}{v + \frac{1}{2}}.$$

If $v = 3$ the winning bidder loses money in expectation if $m > \frac{6}{7}$.

Now let's look at what happens when n gets large. Does the winner's curse get better or worse? Put simply, it gets worse. Intuitively, it should be clear that as n gets large the expected value of $\tilde{e}_{(1)}$ will be very close to 1. Think of taking 1000 random draws of numbers between -1 and 1 and ask yourself, "What is the value of the highest draw likely to be?" Hopefully, you said "1." This can also be shown formally. If we further evaluate the integral in the

expression for $E[\tilde{e}_{(1)}]$ we get

$$\begin{aligned}
 E[\tilde{e}_{(1)}] &= \int_{-1}^1 e \frac{n}{2} \left(\frac{1+e}{2} \right)^{n-1} de \\
 &= \frac{1}{2^n} \int_{-1}^1 ne(1+e)^{n-1} de \\
 &= \frac{1}{2^n} \left[e(1+e)^n \Big|_{-1}^1 - \int_{-1}^1 (1+e)^n de \right] \\
 &= \frac{1}{2^n} \left[2^n - 0 - \frac{1}{n+1} (1+e)^{n+1} \Big|_{-1}^1 \right] \\
 &= \frac{1}{2^n} \left[2^n - \frac{1}{n+1} 2^{n+1} \right] \\
 &= 1 - \frac{2}{n+1} \\
 &= \frac{n-1}{n+1}
 \end{aligned}$$

Clearly, as $n \rightarrow \infty$, we have $\frac{n-1}{n+1} \rightarrow 1$.

The point is that for large n , $E[\tilde{v}_{(1)}] = v + E[\tilde{e}_{(1)}] = v + 1$. If $v = 3$ the winning bidder loses money in expectation if $m > \frac{3}{4}$. So even a very cautious bidder, perhaps even one that has been warned about the winner's curse, might lose money!

General Facts:

1. As the number of bidders increases, a bidder must bid a smaller fraction of her signal to avoid the winner's curse.
2. For any given number of bidders, if the range of the noise parameter is smaller, relative to the true value, the winner's curse results less often. (Not shown)

Report on experiment 11: penny jar experiment.

7.2 Model II

Common values can also be modelled as a special case of interdependent values. In the interdependent values model

$$\begin{aligned}
 v_1 &= \alpha s_1 + \gamma s_2 \\
 v_2 &= \alpha s_2 + \gamma s_1
 \end{aligned}$$

where s_1 and s_2 are private signals of bidders 1 and 2, $\alpha \geq 0$ is the weight a bidder puts on her own signal and $\gamma \geq 0$ is the weight she puts on her opponent's signal. We consider the case where $\alpha = \gamma = 1$. In this case, $v_1 = v_2$,

which is a case of common values. Note that the oil lease example fits this model when each signal s_i is determined independently and $s_i = 0$ or 3 with equal probability. In what follows we will consider a more general treatment of the private signals.

Suppose that the signals s_i are drawn independently from the uniform distribution on $[0, 1]$.

Claim 1 *The first-price auction has a symmetric Nash equilibrium in which each bidder bids s_i .*

Proof. Suppose bidder 2 bids s_2 and consider an arbitrary bid b_1 for bidder 1. We need to write down bidder 1's expected payoff as a function of her bid b_1 and show that this is maximized at $b_1 = s_1$. We derive bidder 1's expected payoff as a function of her bid in three steps.

Step 1. Compute the probability that bidder 1 wins with bid b_1 .

Bidder 1 wins only if her bid is higher than bidder 2's bid, i.e., $b_1 > s_2$. Since we assume that signals are uniform on $[0, 1]$, this happens with probability b_1 . Thus, bidder 1 wins the auction with bid b_1 with probability b_1 .

Step 2. Compute bidder 1's expected value of the item if she wins at bid b_1 .

Remember that each bidder's common value for the good is equal to the sum of the private signals. Bidder 1 knows s_1 , but she does not know s_2 . Before the auction begins, the expected value of s_2 is simply $.5$. But this assumes the random variable s_2 can take on any value between 0 and 1. Bidder 1 is interested in the expected value of s_2 only in the event that she wins the auction with a bid b_1 . As we mentioned in step 1, given the proposed bidding strategy of bidder 2, this only happens if $s_2 < b_1$. Hence, we want the expected value of the random variable \tilde{s}_2 conditional on $s_2 < b_1$. This is computed as follows:

$$\begin{aligned} E[\tilde{s}_2 | s_2 < b_1] &= \int_0^{b_1} s_2 \frac{1}{b_1} ds_2 \\ &= \frac{1}{b_1} \left[\frac{(s_2)^2}{2} \right]_0^{b_1} \\ &= \frac{1}{b_1} \frac{(b_1)^2}{2} \\ &= \frac{b_1}{2} \end{aligned}$$

Hence, bidder 1's expected value of the item if she wins with bid b_1 is

$$s_1 + \frac{b_1}{2}. \quad (7.1)$$

Let's spend more time making sure you understand what we just did. Instead of using integration to derive (7.1) let's reason by example. Suppose bidder 1 wins with a bid of $.1$ (or 10 cents). That means the signal of bidder

2 must have been below 10 cents (because we assume he is bidding his signal and his bid was less than 10 cents). Since, we have determined that bidder 2's signal is between 0 and .1, and since all these possibilities are equally likely, the expected value of bidder 2's signal is half its maximum value, or .05, and hence the expected value of the item for bidder 1 is $s_1 + .05$. If, on the other hand bidder 1 were to win with a bid of 90 cents, then bidder 2's signal could be as high as 90 cents. However, the expected value of bidder 2's signal would be half that, or 45 cents, and the expected value of the item for bidder 1 is $s_1 + .45$. In both cases, the expected value of the item to bidder 1 is $s_1 + \frac{b_1}{2}$, just like in the general formula.

Step 3. Compute the expected price bidder 1 pays if she wins with bid b_1 .

Since this is a first-price auction, the price she pays if she wins is her own bid, b_1 .

That's it! We are ready to write down the expected payoff. It is

$$\begin{aligned} & \underbrace{b_1}_{\text{Step 1}} \underbrace{\left(s_1 + \frac{b_1}{2} - b_1\right)}_{\text{Step 2}} \\ & = b_1 \left(s_1 - \frac{b_1}{2}\right). \end{aligned}$$

To maximize this with respect to the choice of b_1 we take the derivative and set it equal to 0:

$$s_1 - b_1 = 0.$$

The solution is $b_1 = s_1$, as required.

The same argument can be used to show that $b_2 = s_2$ is a best response to $b_1 = s_1$. That completes the proof.

Claim 2 *The second-price auction has a symmetric Nash equilibrium in which each bidder bids $2s_i$.*

Proof. Suppose bidder 2 bids $2s_2$ and consider an arbitrary bid b_1 for bidder 1. We need to write down bidder 1's expected payoff as a function of her bid b_1 and show that this is maximized at $b_1 = 2s_1$. We derive bidder 1's expected payoff as a function of her bid in three steps.

Step 1. Compute the probability that bidder 1 wins with bid b_1 .

Bidder 1 wins only if her bid is higher than bidder 2's bid, i.e., $b_1 > 2s_2$. Or, equivalently, bidder 1 wins with bid b_1 if $s_2 < \frac{b_1}{2}$. Since we assume that signals are uniform on $[0, 1]$, this happens with probability $\frac{b_1}{2}$. Thus, bidder 1 wins the auction with bid b_1 with probability $\frac{b_1}{2}$.

Step 2. Compute bidder 1's expected value of the item if she wins at bid b_1 .

Remember that each bidder's common value for the good is equal to the sum of the private signals. Bidder 1 knows s_1 , but she does not know s_2 . She needs to compute the expected value of s_2 conditional on the event that she wins the auction with a bid b_1 , that is, conditional on $s_2 < \frac{b_1}{2}$. This is computed as follows:

$$\begin{aligned} E[\tilde{s}_2 | s_2 < \frac{b_1}{2}] &= \int_0^{\frac{b_1}{2}} s_2 \frac{2}{b_1} ds_2 \\ &= \frac{2}{b_1} \left[\frac{(s_2)^2}{2} \right]_0^{\frac{b_1}{2}} \\ &= \frac{2}{b_1} \frac{(\frac{b_1}{2})^2}{2} \\ &= \frac{b_1}{4} \end{aligned}$$

Hence, bidder 1's expected value of the item if she wins with bid b_1 is

$$s_1 + \frac{b_1}{4}. \tag{7.2}$$

Once again, let's spend a little more time making sure you understand the role bidder 1's winning bid plays in determining her expected value of the item. Suppose bidder 1 wins with a bid of .1 (or 10 cents). That means the signal of bidder 2 must have been below 5 cents (because we assume he is bidding twice his signal and his bid was less than 10 cents). Hence, the value of the item, which is the sum of the signals, is at most $s_1 + .05$. But, we don't really care about the most it can be, we want to base our decision on its expected value. Since, we have determined that bidder 2's signal is between 0 and .05, and since all these possibilities are equally likely, the expected value of bidder 2's signal is half its maximum value, or .025, and hence the expected value of the item for bidder 1 is $s_1 + .025$. If, on the other hand bidder 1 were to win with a bid of 90 cents, then bidder 2's signal could be at most 45 cents. Hence the expected value of bidder 2's signal would be half that, or 22.5 cents and the expected value of the item for bidder 1 is $s_1 + .225$. In both cases the expected value of the item to bidder 1 is $s_1 + \frac{b_1}{4}$, just like in (7.2).

Step 3. Compute the expected price bidder 1 pays if she wins with bid b_1 .

Since this is a second price auction, the price she pays if she wins is bidder 2's bid, which is $2s_2$. The expected value of bidder 2's bid depends on the bid b_1 because bidder 1 does not win the auction unless $b_1 > 2s_2$. We need to compute the expected value of $2s_2$ conditional on $b_1 > 2s_2$. This is done most easily by using a little trick; we use the value $x = 2s_2$ as the running variable in the

integration:

$$\begin{aligned} E[2\tilde{s}_2|2s_2 < b_1] &= \int_0^{b_1} x \frac{1}{b_1} dx \\ &= \frac{1}{b_1} \left[\frac{x^2}{2} \right]_0^{b_1} \\ &= \frac{b_1}{2}. \end{aligned}$$

That's it! We are ready to write down the expected payoff. It is

$$\begin{aligned} &\underbrace{\frac{b_1}{2}}_{\text{Step 1}} \underbrace{\left(s_1 + \frac{b_1}{4} - \frac{b_1}{2}\right)}_{\text{Step 2}} \\ &= \frac{b_1}{2} \left(s_1 - \frac{b_1}{4}\right). \end{aligned}$$

To maximize this with respect to the choice of b_1 we take the derivative and set it equal to 0:

$$\frac{s_1}{2} - \frac{b_1}{4} = 0.$$

The solution is $b_1 = 2s_1$, as required.

The same argument can be used to show that $b_2 = 2s_2$ is a best response to $b_1 = 2s_1$. That completes the proof.

The equilibrium we just verified is appealing because it is symmetric, but it is not unique. While we will not prove it, there are other asymmetric equilibria to this auction. Namely, for any $\lambda > 0$, $b_1(s_1) = (1 + \lambda)s_1$ and $b_2(s_2) = (1 + \frac{1}{\lambda})s_2$ is a Nash equilibrium. See Osborne's 2004 introduction to game theory text for a proof.

7.2.1 Revenue Equivalence

Given these equilibrium bid functions, the expected payment of a bidder is the same in both auction formats. In the first-price auction, bidder i wins with bid s_i with probability s_i , and pays s_i . Hence, bidder i 's expected payment in the first-price common value auction is s_i^2 . In the second-price auction, bidder i wins with probability

$$\text{Pr ob}[2s_i > 2\tilde{s}_j] = \text{Pr ob}[s_i > \tilde{s}_j] = s_i$$

and she pays

$$\begin{aligned} E[2\tilde{s}_j|s_j < s_i] &= \int_0^{s_i} 2s_j \frac{1}{s_i} ds_j \\ &= \frac{2}{s_i} \left[\frac{x^2}{2} \right]_0^{s_i} \\ &= s_i. \end{aligned}$$

Hence, bidder i 's expected payment in the second-price common value auction is s_i^2 .

So, in both the first and second-price, common-value auction expected revenue is

$$2E[\tilde{s}_i^2] = 2 \int_0^1 s_i^2 ds_i = 2 \left[\frac{s_i^3}{3} \right]_0^1 = \frac{2}{3}.$$

Revenue equivalence holds in this setting!