Expected Revenue

Here we calculate the expected revenue under the efficient equilibrium bidding strategies for the first- and second-price auction formats. In a first-price auction with $F(\cdot)$ uniform on $[0,100]$, the symmetric equilibrium bidding strategy has each bidder bid $(n - 1)/n$ times their value.
Hence, expected revenue is simply \((n - 1)/n\) times the expected value of the highest value. I.e., Expected revenue is

\[
\frac{n-1}{n} E[\tilde{v}(1)] = \frac{n-1}{n} \int_0^{100} vf(1)(v)dv
\]

\[
= \frac{n-1}{n} \int_0^{100} \frac{n v^{n-1}}{100^n} dv
\]

\[
= (n - 1) \int_0^{100} \frac{v^n}{100^n} dv
\]

\[
= \frac{(n - 1)}{100^n} \left[ \frac{v^{n+1}}{n + 1} \right]_0^{100}
\]

\[
= \frac{n - 1}{n + 1} \times 100.
\]
In a second-price auction it is a weakly dominant strategy to bid your value, and hence expected revenue is simply the expected value of the second-highest value.

In case where $F(\cdot)$ is uniform on $[0,100]$, 

$$E[\tilde{v}_{(2)}] = \int_0^{100} vf_{(2)}(v)dv$$

$$= \int_0^{100} vn(n-1)(\frac{v^{n-2}}{100^{n-1}} - \frac{v^{n-1}}{100^n})dv$$

$$= n(n-1) \int_0^{100} (\frac{v^{n-1}}{100^{n-1}} - \frac{v^n}{100^n})dv$$

$$= n(n-1) \left( \left[ \frac{v^n}{n100^{n-1}} \right]_0^{100} - \left[ \frac{v^{n+1}}{(n+1)100^n} \right]_0^{100} \right)$$

$$= n(n-1) \left( \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)} \right) \times 100$$

$$= \frac{n-1}{n+1} \times 100$$
Revenue Equivalence of Standard Auction Formats

In the case where each of the $n$ bidders’ values are generated independently from the uniform distribution, the expected revenue to the seller in either a first- or second-price auction is equal to $\frac{n-1}{n+1} \times 100$.

In fact, the result that the expected revenue is the same in these auction formats is true even if values are generated from a non-uniform distribution.

Moreover, revenue equivalence extends to a much broader class of auction formats, which we will refer to as standard auctions.
Define a *standard auction* to be any auction mechanism in which (i) the object is sold to person with highest value, and (ii) the expected payment of bidder with the lowest possible value is 0 in equilibrium.

Assume that all bidders are risk neutral.

Assume that the bidders’ values are independently and identically distributed according to some distribution $F$.

We do not require that $F$ be the uniform distribution.

**Revenue Equivalence Theorem** Any symmetric equilibrium of a standard auction $A$, involving a strictly increasing bid function yields the same expected revenue to the seller.
In-class discussion of revenue estimates from Experiments 1, 2, 5 and 6: 2 and 5-bidder, first and second-price auctions without a reserve or entry fee.
### Table: Revenue Comparison

<table>
<thead>
<tr>
<th></th>
<th>Theoretical</th>
<th>SPA</th>
<th>FPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 bidders</td>
<td>33.33</td>
<td>30.80</td>
<td>56.46</td>
</tr>
<tr>
<td>5 bidders</td>
<td>66.67</td>
<td>60.80</td>
<td>76.21</td>
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## Revenue With a Reserve Price or Entry Fee

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<th>$r=50, c=0$</th>
<th>$r=0, c=25$</th>
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<tr>
<td><strong>observed</strong></td>
<td>30.80</td>
<td>43.96</td>
<td>44.70</td>
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<td><strong>theory</strong></td>
<td>33.33</td>
<td>41.67</td>
<td>41.67</td>
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</table>

**Table:** SPA Revenue

<table>
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<th>$r=0, c=25$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>observed</strong></td>
<td>56.46</td>
<td>38.84</td>
<td>58.20</td>
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<tr>
<td><strong>theory</strong></td>
<td>33.33</td>
<td>41.67</td>
<td>41.67</td>
</tr>
</tbody>
</table>

**Table:** FPA Revenue