Recursive Decentralization of Dynamic Mirrleesean Economies

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Abstract

I study the competitive equilibria in dynamic Mirrleesean economies where the agents can trade income contingent assets on a period-by-period basis. I show that if the private productivity shocks are I.I.D., markets can achieve efficiency without any government intervention. If the productivity shocks are Markov then taxation is still not needed, although the government needs to monitor the asset trades to prevent the agents from making certain trades.

1 Introduction

In this paper I study competitive equilibria in dynamic Mirrleesean economies with variable labor supply and publicly unobserved productivity shocks. I show that when people can trade income dependent assets with financial intermediaries, no taxes are needed to implement the efficient allocations. However, when the productivity shocks are Markov, there is a need for an external monitoring agency that will prevent agents from making certain trades. The role of the agency is nevertheless fundamentally different from the role of a tax authority, because the monitoring agency does not collect any resources from the agent. I also show that the external monitoring agency is not needed if the shocks are I.I.D.

One of the main implications of the decentralization I present is that, since the income dependent assets are traded on the spot markets, long term contracts between an individual and the financial intermediary are not required. This result is

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important since long-term contracts are usually perceived to be necessary in order to obtain efficiency in dynamic private information economies. The idea behind the decentralization is that, before the current shock is realized, the agents's objectives are aligned with government's objectives - both are interested in maximizing the expected utility. If one assumes that the agents can only trade risk free bond, as in Albanesi and Sleet [1] or Kocherlakota [11], then the agents do not have enough instruments to achieve as good insurance as the government can provide, and taxes are needed. But with income dependent assets, they can insure themselves by choosing a asset portfolio with income dependent assets in the previous period. Income dependent assets act thus in the same way as a tax function.

There are two implications of this result. First, conclusions about (nonzero) optimal taxation in such framework critically depend on the assumption that some assets are not allowed to be traded. In other words, if one studies taxation in a framework where the set of assets to be traded is exogenously restricted, he is in fact not studying an economy with private information frictions but an economy with completely different type of friction - exogenously missing markets for income contingent assets. Second, the paper hopefully increases the appeal of private information as a source of frictions for positive studies of dynamic economies. If markets can achieve efficient allocations without any need to resort to long term binding contracts or taxation, then it seems more plausible to hypothesize that efficient allocations are actually achieved in some way, as in Kocherlakota [12] or in Maziero and Ales [13].

The argument of this paper is very closely related to the one of Prescott and Townsend [15],[16]. They show that competitive equilibria in many private information economies, including a static version of the one considered here, are efficient if markets for type dependent contracts are considered. One contribution of this paper is to make an explicit connection with the optimal taxation literature. Unlike in Prescott and Townsend, I consider an indirect decentralization where the agents are not required to report their type to the financial intermediary and mere observation of their incomes by financial intermediaries is enough to execute the trades.

The paper is also closely related to the decentralization in Golosov and Tsyvinski [6]. There are three main differences. First, agents in Golosov and Tsyvinski sign long-term contracts with financial intermediaries, who provide them with consumption and direct their production. The agents do not trade any asset themselves. In contrast, this paper relies on a sequence of short term asset trades executed directly by the agents. Second, Golosov and Tsyvinski do not require the external monitoring agency to be present, while monitoring is needed for the decentralization of this paper (unless the shocks are I.I.D.). Thus, external monitoring can be seen as a substitute for long term contracting. The third difference is that Golosov and Tsyvinski study direct decentralization where financial intermediaries condition their payments on agents' reports about their productivity, while this paper studies both direct decentralization and indirect decentralization.

There are two related decentralizations for dynamic moral hazard economies. A decentralization with income dependent assets is provided by Kocherlakota [10]. Similarly to this model in case of Markov shocks, a monitoring agency is required to prevent the agents from making certain trades. Another type of decentralization is provided by Grochulski [7]. Grochulski studies a dynamic moral hazard economy and shows that the efficient allocations can be implemented in a market economy where the agents trade unsecured debt with financial intermediaries and are subject to discharge under certain conditions of bankruptcy law. The bankruptcy law is exogenous and serves a role similar to the external monitoring agency in this paper.

The paper is divided in three parts. In the first part, I provide a recursive representation of the constrained efficient allocations. The recursive representation is novel. I recursively characterize the social planner's primal problem and use a vector of social planner's expected costs *conditional on the current shock* as a state variable. I call the costs conditional on the current shock as ex-post costs, to distinguish them from the expected costs at the beginning of the period, called ex-ante costs. The reason why the ex-post costs are used as a state variable is that they turn out to be closely linked to the income dependent assets the agents will hold in a competitive equilibrium. In contrast, ex-ante costs do not, in general, have any physical counterpart in a competitive equilibrium.

In the second part I study a direct decentralization where agents trade one period contracts with financial intermediaries. The contracts consists of two elements: for each possible report of the agent they specify the asset payment and the income the agent must produce. While the direct decentralization is not the main objective of the paper and is in principle not needed, it provides a useful intermediate step between the social planner's problem and the indirect decentralization. I show why the external monitoring agency is needed in order to achieve efficiency. If the external monitoring agency is not present, the agent might gain from reporting an incorrect type, while at the same time choosing consumption and a next period contract that is inconsistent with the behavior of the agent whose report has been mimicked. The external monitoring agency is needed to prevent such joint deviations by restricting the agent in her ability to choose next period contracts.

The indirect decentralization is analyzed in the third part. Compared to the direct decentralization, the indirect one has two advantages. First, I show that the contracts traded are simpler and consist only of asset portfolios. Second, type reporting is not needed. It is implicit in a sense that the financial intermediaries infer the agent's type indirectly from the agent's behavior. This makes asset trading more realistic than contract trading in the direct decentralization. The cost of indirect decentralization is that the efficient incomes must always be invertible, so that the financial intermediaries could infer the agents' current type from their income. This additional restriction is again not needed if the shocks are I.I.D.

The paper is organized as follows. Next section provides a simple example to show the main intuition behind the results. Section (3) sets up the environment, defines the constrained efficient allocations and characterizes them recursively. Section (4) studies direct decentralization with report dependent assets while section (5) studies indirect decentralization with income dependent assets. Section (6) discusses additional roles left for taxation and concludes. Proofs can be found in the Appendix.

2 An Example

Consider the following static economy where production and consumption takes place only in one period. The agents receive publicly unobserved productivity shocks $\theta \in \Theta$ where Θ is a finite set. The probability of getting a shock $\theta \in \Theta$ is $\pi(\theta)$. The utility function depends on consumption c and labor l and is given by $u(c) - v(l)$. The income y is given by $y = \theta l$. There are no government expenditures in this economy.

Let $c : \Theta \to \mathcal{R}_+$ and $y : \Theta \to \mathcal{R}_+$ be consumption and income assigned to the agents by the social planner. The social planner maximizes the expected utility of the agent by choosing an allocation $\{c, y\}$:

$$\max_{\{c,y\}} \sum_{\theta\in\Theta} [u(c(\theta)) - v(\frac{y(\theta)}{\theta})]\pi(\theta),$$

subject to an incentive compatibility constraint

$$\theta \in \arg\max_{\sigma\in\Theta} u(c(\sigma)) - v(\frac{y(\sigma)}{\theta}) \quad \forall \theta \in \Theta \tag{1}$$

and a resource constraint requiring the cost of an allocation to be equal to some initial wealth level $\bar{B}$:

$$\bar{B} = \sum_{\theta\in\Theta} [c(\theta) - y(\theta)]\pi(\theta). \tag{2}$$

Denote the efficient allocation by a pair of functions $c^*(\bar{B}, \theta)$ and $y^*(\bar{B}, \theta)$. I will now compare two different ways of decentralizing the efficient allocation. The first one involves government and uses income taxes to redistribute the resources. The second one involves no government, but allows the agents to trade income dependent assets before the uncertainty about productivity is revealed.

2.1 Decentralization with Income Taxes

The optimal tax problem is standard and has been studied many times, beginning with Mirrlees [14]. The government imposes an income tax function $T(\bar{B}, y)$ where $\bar{B}$ is individuals' initial wealth level and y is income they realize. After learning about their productivity, the agents maximize utility by choosing income and consumption subject to a budget constraint

$$c = \bar{B} + y - T(\bar{B}, y). \tag{3}$$

Define the optimal tax function $T^*(\bar{B}, y)$ by using the efficient allocation as follows: $T^*[\bar{B}, y^*(\bar{B}, \theta)] = \bar{B} + y^*(\bar{B}, \theta) - c^*(\bar{B}, \theta)$. The optimal tax function is extended appropriately beyond the range of $y^*(\bar{B}, \cdot)$. The taxation principle implies that when individuals with initial wealth $\bar{B}$ and productivity θ face the tax function T^*, they will find it optimal to choose the efficient income $y^*(\bar{B}, \theta)$ and consume the efficient amount $c^*(\bar{B}, \theta)$.

2.2 Decentralization with Income Dependent Assets

Consider now the following market arrangement. Before the productivity shock is revealed, the agents are given a chance to trade with financial intermediaries, using their wealth $\bar{B}$ and buying a portfolio of income contingent assets. The portfolio of income contingent assets a pays $a(y)$ units of good if the agent chooses to produce y units of income. Financial intermediaries act competitively and price portfolios of income contingent assets as a whole, rather than pricing individual assets separately. Let $P(a)$ be the price of an asset portfolio a. The agent can choose any portfolio she can afford. The budget constraint is given by

$$\bar{B} = P(a).$$

It is useful to think through the agent's problem backwards. Suppose that the agent has chosen an asset portfolio a. When the agent produces y units of goods in period 1 she is paid $a(y)$ units of goods from the financial intermediary and hence consumes

$$c = y + a(y). \tag{4}$$

Let $y^i(a,\theta)$ and $c^i(a,\theta)$ be the utility maximizing income and consumption choice of an agent who has previously chosen a portfolio a.

Comparing (3) and (4) one can see that the agent's asset portfolio a acts in essentially identical way as the government's net income $y - T$. Thus, in principle, the agent's ability to self insure is the same as the government's ability to redistribute resources. The question is whether the agent will have the right incentives to insure herself. The answer is yes, and it follows from two arguments. First, ex-ante, both the government's objective and the agent's objective is identical: both want to maximize expected utility. Second, since the financial intermediaries maximize profits, the equilibrium price of any portfolio a must be equal to the expected costs for the financial intermediary. Thus, the price of a portfolio a is given by

$$P(a) = \sum_{\theta \in \Theta} a[y^i(a,\theta)]\pi(\theta)d\theta. \tag{5}$$

In particular, the portfolio $a^i(\bar{B},\cdot) = \bar{B} - T^*(\bar{B},\cdot)$ that "mimics" the optimal tax function is feasible to the agent. To see this, suppose that, given portfolio $a^i(\bar{B},\cdot)$, the agent wants to choose the efficient income level $y[a^i(\bar{B},\cdot),\theta] = y^*(\bar{B},\theta)$:

$$\begin{aligned} P[a^i(\bar{B},\cdot)] &= \sum_{\theta \in \Theta} a^i[\bar{B}, y(a^i(\bar{B},\cdot),\theta)]\pi(\theta). \\ &= \sum_{\theta \in \Theta} [\bar{B} - T^*(\bar{B}, y^*(\bar{B},\theta))]\pi(\theta) = \bar{B}, \end{aligned}$$

where the last equality follows from the fact that the aggregate tax payments under T^* are zero. If the agent chooses the portfolio $a^i(\bar{B},\cdot)$, one can easily show that she will indeed want to choose the efficient income level after the shock is realized. The efficient allocation is therefore implemented with income dependent assets

The example shows the main idea behind decentralization with income dependent assets: before the shock is realized, the agents have the right incentives to insure themselves. This idea will carry over to the more general model. The example also correctly suggests that asset trading before the shock is realized is a key in obtaining the efficient allocations and shows how the financial intermediaries price portfolios by inferring the agent's probabilities of incomes.

On the other hand, the example is too simplistic on several dimensions. The fact that there is no period with both asset trading and production is, as we shall see, restrictive, and the results in a multiperiod model will need to be somewhat modified. There are also important differences between economies with Markov shocks and economies with I.I.D. shocks and these differences have been blurred by the simple example. And, finally, while there is an equivalence between decentralization with taxes and decentralization with income dependent assets in the static example, such equivalence exists for recursive formulations of dynamic economies only if the shocks are I.I.D.

For all these reasons, it is important to consider a more general multiperiod model with Markov shocks. I will also link the competitive equilibrium allocations directly to the efficient allocations, and circumvent the discussion of decentralization with income taxes completely.

3 Constrained Efficient Allocations

There is a finite number of time periods T and the economy starts at time zero. Period utility is given by a function $u(c) - v(l)$ where $u : \mathcal{R}_+ \to \mathcal{R}$ is strictly increasing and strictly concave, and $v : \mathcal{R}_+ \to \mathcal{R}$ is increasing and strictly convex, c is consumption and l is labor supply. The agents discount future utility at rate β, $0 < \beta \leq 1$.

Each period the agent receives a productivity shock $\theta \in \Theta$ where Θ is a finite set with I elements. The productivity shocks follow a Markov process with probability of a shock θ' next period if the current shock is θ given by $\pi(\theta'|\theta)$. The initial shock $\theta_{-1} \in \Theta$ is supposed to be publicly known and is the same for everyone. For any $t \leq T$ one can construct the probability of any sequence of shocks $\theta^t = (\theta_0, \theta_2, ...\theta_t) \in \Theta^{t+1}$ given θ_{-1} and it will be denoted by $\pi^t(\theta^t|\theta_{-1})$. The productivity shocks affect the agent's income: if the agent supplies labor l and her productivity is θ then she produces income $y = \theta l$.

The social planner has access to a credit market. He faces a sequence of intertemporal prices of consumption $q = \{q_t\}_{t=0}^{T}$, where $q_t > 0$ is a relative price of consumption between periods t and $t+1$. Let $Q_t = \prod_{i=0}^{t-1} q_i$. In what follows I will take the sequence of prices as a parameter.[1]

[1] In this paper I will not attempt to endogenize the intertemporal prices of consumption, as in Albanesi and Sleet [1] and Atkeson and Lucas [2]. Endogenizing q is not likely to change any of the main results of this paper.

3.1 A Sequence Formulation

The social planner chooses an *allocation* $Z = \{C_t, Y_t\}_{t=0}^{T}$, given by a collection of report and time contingent consumption and income plans $C_t : \bar{\mathcal{B}} \times \Theta \times \Theta^{t+1} \to \mathcal{R}_+$ and $Y_t : \bar{\mathcal{B}} \times \Theta \times \Theta^{t+1} \to \mathcal{R}_+$. The allocation is indexed by the present value of costs $\bar{B}_0 \in \bar{\mathcal{B}} \subseteq \mathcal{R}$ and the initial shock $\theta_{-1} \in \Theta$.[2] Let $U(Z, \theta_{-1})$ be the lifetime utility an agent gets from the allocation Z:

$$U(Z,\theta_{-1}) = \sum_{t=0}^{T} \sum_{\theta^t \in \Theta^{t+1}} \beta^t \{u[C_t(\theta^t)] - v[\frac{Y_t(\theta^t)}{\theta}]\} \pi^t(\theta^t|\theta_{-1}).$$

The allocation must satisfy two constraints. First, it must be incentive compatible: any possible reporting strategy $\sigma = \{\sigma_t\}_{t=0}^{T}$, where $\sigma_t : \bar{\mathcal{B}} \times \Theta \times \Theta^{t+1} \to \Theta$ must be dominated by truthtelling:

$$U(Z,\theta_{-1}) \geq U(Z \circ \sigma, \theta_{-1}) \tag{6}$$

for all possible reporting strategies σ. Second, the social planner is restricted by a requirement that the present value of the costs must be equal to $\bar{B}_0$:

$$\sum_{t=0}^{T} \sum_{\theta^t \in \Theta^{t+1}} Q_t [C_t(\theta^t) - Y_t(\theta^t)] \pi^t(\theta^t|\theta_{-1}) = \bar{B}_0. \tag{7}$$

The efficient allocation Z^* is an allocation that attains the maximum of the agent's expected utility among all incentive compatible allocations that have present value $\bar{B}_0$:

$$U^*(\bar{B}_0, \theta_{-1}) = \max_Z U(Z, \theta_{-1}) \qquad \text{s.t. (6),(7).} \tag{8}$$

For a given allocation Z, the present value of the social planner costs follows a certain stochastic process. Define $B_t(Z, \theta^t)$ to be the present value of the social planner's costs after a history θ^t has been realized. It is given by

$$B_t(Z,\theta^t) = C_t(\theta^t) - Y_t(\theta^t) + \sum_{j=t+1}^{T} \sum_{\theta^{j-t} \in \Theta^{j-t}} \frac{Q_j}{Q_t} [C_j(\theta^t, \theta^{j-t}) - Y_t(\theta^t, \theta^{j-t})] \pi^{j-t}(\theta^{j-t}|\theta_t).$$

In what follows, I will assume the following:

[2] The dependence on $\bar{B}_0$ and θ_{-1} will be kept implicit whenever convenient to reduce the notational burden

Assumption 1 *The efficient allocation satisfies the following equal treatment property: If, for any* $(\theta^{t-1}, \hat{\theta}^{t-1}) \in \Theta^{2t}$*, the allocation satisfies* $B_t(Z, \theta^{t-1}, \cdot) = B_t(Z, \hat{\theta}^{t-1}, \cdot)$ *then* $C_t(\theta^{t-1}, \theta_t) = C_t(\hat{\theta}^{t-1}, \theta_t)$*,* $Y_t(\theta^{t-1}, \theta_t) = Y_t(\hat{\theta}^{t-1}, \theta_t)$ *and* $B_{t+1}(\theta^{t-1}, \theta_t, \cdot) = B_{t+1}(\hat{\theta}^{t-1}, \theta_t, \cdot)$.

The assumption says that if two histories lead to the same cost vector in period t then the consumption and income in period t will also be identical and, moreover, the ex-post costs tomorrow will be identical as well. The assumption is crucial for the existence of a recursive formulation. If it is not satisfied, the efficient allocation exhibits memory that cannot be captured by the cost vector and hence cannot be replicated by a recursive contract.

Assumption (1) is clearly not an assumption about primitives of the economy. One can however show that it will be satisfied for instance if the period utility is linear in θ (or if it is linear in some nonlinear transformation of θ).[3]

3.2 A Recursive Formulation

I will now show that the utility maximization problem can be written recursively, using the vector of costs as a state variable. The social planner will be constrained by the requirement that, for each possible realization of the shock $\theta \in \Theta$, the costs must not exceed a given value $b(\theta)$. Since the costs are contingent upon the current shock, I will refer to them as *ex-post costs*. Typical ex-post costs will be denoted as $b \in \mathcal{R}^I$.

Each period, the social planner chooses an *allocation rule*, which consists of report contingent consumption and income, as well as a vector of next period ex-post costs. The allocation rule is therefore a collection of functions $z = \{c_t, y_t, b_t\}_{t=0}^T$ where $c_t : \mathcal{R}^I \times \Theta \to \mathcal{R}_+$ is consumption in period $t \geq 0$, $y_t : \mathcal{R}^I \times \Theta \to \mathcal{R}_+$ is income in period $t \geq 0$ and b_t are ex-post costs at the beginning of period t, defined as $b_t : \mathcal{R}^I \times \Theta \to \mathcal{R}^I$ for $t \geq 1$ and as $b_0 : \bar{\mathcal{B}} \times \Theta \times \Theta \to \mathcal{R}^I$ in period zero.

The ex-post resource constraint the allocation rule must satisfy is given, for all $t \leq T$ by[4]

$$b(\theta) = c_t(\theta) - y_t(\theta) + q_t \sum_{\theta' \in \Theta} b_{t+1}(\theta, \theta')\pi(\theta'|\theta) \qquad \forall \theta \in \Theta, \tag{9}$$

[3]The assumption is therefore satisfied for instance if $v(l) = l^{1+\rho}$, $\rho > 0$.

[4]To economize on notation, I will keep the dependence of the allocation on the ex-post costs b implicit whenever convenient. Also, when there is a risk of confusion or whenever it is necessary, I will use the notation $b(\cdot)$ to highlight the fact that b is a vector, and not a scalar.

where $b_{T+1} = 0$. Let $V_t(b, \theta_-)$ be the maximum lifetime utility at the beginning of period $t \leq T$ that the social planner can deliver to the agent given that the ex-post costs are $b \in \mathcal{R}^I$ and the last period productivity shock is $\theta_- \in \Theta$. An allocation is incentive compatible if for all $\theta \in \Theta$ the utility of an agent is maximized by reporting truthfully: for all $t \leq T$,

$$u[c_t(\theta)] - v[\frac{y_t(\theta)}{\theta}] + \beta V_{t+1}[b_{t+1}(\theta, \cdot), \theta] \geq u[c_t(\hat{\theta})] - v[\frac{y_t(\hat{\theta})}{\theta}] + \beta V_{t+1}[b_{t+1}(\hat{\theta}, \cdot), \theta] \quad \forall \theta, \hat{\theta} \in \Theta, \tag{10}$$

where $V_{T+1} = 0$. Let $\mathcal{B}_t \subset \mathcal{R}^I$ be a set of the ex-post costs for which there exists some allocation rule in period t that satisfies both the ex-post resource constraint and the incentive compatibility constraint:

$$\mathcal{B}_t = \{b \in \mathcal{R}^I : \exists (c_t, y_t, b_{t+1}) : \Theta \to \mathcal{R}_+^2 \times \mathcal{R}^I \text{ such that } (c_t, y_t, b_{t+1}) \text{ satisfies (9) and (10)}\}.$$

The ex-post costs chosen by the social planner are required to be taken from the set $\mathcal{B}_{t+1}$ for all $t \leq T-1$:

$$b_{t+1}(\theta, \cdot) \in \mathcal{B}_{t+1} \qquad \forall \theta \in \Theta. \tag{11}$$

The efficient allocation rule maximizes the lifetime utility of the agent, that satisfies the following Bellman equation:

$$V_t(b, \theta_-) = \max_{c_t, y_t, b_{t+1}} \sum_{\theta \in \Theta} \{u[c_t(\theta)] - v[\frac{y_t(\theta)}{\theta}] + \beta V_{t+1}[b_{t+1}(\theta, \cdot), \theta]\} \pi(\theta|\theta_-) \tag{12}$$

subject to the ex-post resource constraint (9), incentive compatibility constraint (10) and the constraint (11). The efficient allocation rule is denoted by z^*.

The social planner's problem in period zero is different, because the social planner is only constrained by the ex-ante costs $\bar{B}_0$ rather than by a vector of ex-post costs. Equivalently, one can think of the social planner as choosing, before the period zero shock is realized, the ex-post costs b_0 subject to an ex-ante resource constraint

$$\bar{B}_0 = \sum_{\theta \in \Theta} b_0(\theta) \pi(\theta|\theta_{-1}). \tag{13}$$

The optimal ex-post costs in period zero thus solve

$$b_0^*(\bar{B}_0, \theta_{-1}, \cdot) = \arg\max_{b(\cdot) \in \mathcal{B}_0} V_0[b(\cdot), \theta_{-1}] \qquad \text{s.t. (13)}. \tag{14}$$

The lifetime utility the agent receives from the allocation rule z^*, given that the ex-post costs in period zero are chosen optimally is given by $V^*(\bar{B}_0, \theta_{-1}) = V_0[\hat{b}_0^*(\bar{B}_0, \theta_{-1}, \cdot), \theta_{-1}]$.

The definition of the allocation rule does not allow for the possibility that the efficient allocation rule might depend on the last period shock θ_-. This may, at first sight, appear to be a significant restriction since the agent's lifetime utility $V_t(b, \theta_-)$ clearly depends on θ_-. Next lemma shows that such restriction is indeed justified. In the optimum, the allocation rule never depends on the last period shock.

Lemma 2 *The efficient allocation rule in period t, $z_t = (c_t, y_t, b_{t+1})$ is independent of the last period shock θ_- for all $t \leq T$.*

This result is a fairly straightforward implication of the fact that the problem can be redefined as a problem where the social planner chooses the allocation rule *after* the current shock is realized. But after the current shock is realized, neither the objective function nor the constraints depend on θ_-. In other words, all the dependence of the efficient allocation rule on θ_- is already captured in the choice of the current ex-post costs b.

The recursive formulation constructed in this section can be seen as an alternative to the recursive formulation of Fernandes and Phelan [4]. There are two main differences between both formulations. First difference is obvious: Fernandes and Phelan provide a recursive formulation of a dual problem, while I provide a recursive formulation of the primal problem. But the most important difference from the point of view of this paper is the second one: while Fernandes and Phelan use a vector of ex-ante variables as a state, my formulation uses a vector of ex-post variables as its state. This is essential because the ex-ante variables are all but one counterfactuals: In Fernandes and Phelan they represent the lifetime utility of an agent who has deviated from truthtelling although noone deviates in equilibrium.[5] The problem with the counterfactual state variables is that they are probably impossible to map to a meaningful state variable in a competitive equilibrium. In contrast, none of the ex-post costs is counterfactual, because the ex-post costs are independent of the true productivity shock the agent has received last period. This, as one will see later, allows the ex-post costs to be linked with state dependent assets in a competitive equilibrium.

[5]Similarly, one could write a recursive representation of the primal problem, where the vector of states would be the present values of ex-ante costs of an agent who has deviated from truthtelling.

It is fair to say that for computational reasons the dual formulation of Fernandes and Phelan is better suited. First, in an infinite horizon economy, the Bellman operator in a dual recursive formulation is a contraction, while the Bellman operator in the primal recursive formulation may not be: the presence of the value function in the incentive compatibility constraint prevents the application of the Blackwell's Theorem. Second, the state space of the dual recursive problem can sometimes be significantly reduced,[6] while there is no hope to reduce the state space of the primal recursive problem similarly. But for the purpose of a theoretical analysis presented in this paper, the primal recursive formulation is a better fit.

3.3 Relationship between The Sequence Problem and the Recursive Problem

This section studies the relationship between the sequence representation and the recursive representation of the social planner's problem. To study the relationship properly, one first needs to find a way of mapping allocations to allocation rules and vice versa.

For a given allocation rule z define an allocation Z as follows. Take any $\bar{B}_0$ and $\theta_{-1} \in \Theta$. Let $\hat{B}_0(\bar{B}_0, \theta_{-1}, \theta_0) = b_0^*(\bar{B}_0, \theta_{-1}, \theta_0)$ For $t \geq 1$, let $\hat{B}_t(\bar{B}_0, \theta_{-1}, \theta^t)$ solve a difference equation $\hat{B}_{t+1}(\bar{B}_0, \theta_{-1}, \theta^t, \theta_{t+1}) = b_{t+1}[\hat{B}_t(\bar{B}_0, \theta_{-1}, \theta^{t-1}, \cdot), \theta_t, \theta_{t+1}]$. Set

$$C_t(\theta^t) = c_t[\hat{B}_t(\bar{B}_0, \theta_{-1}, \theta^{t-1}, \cdot), \theta_t]$$
$$Y_t(\theta^t) = y_t[\hat{B}_t(\bar{B}_0, \theta_{-1}, \theta^{t-1}, \cdot), \theta_t].$$

Call $Z = \{C_t, Y_t\}_{t=0}^T$ an allocation generated by the allocation rule z.

Conversely, for a given allocation Z define an allocation rule z as follows. Set $b_0^*(\bar{B}_0, \theta_{-1}, \cdot) = B_0(Z, \cdot)$. Fix $b \in \mathcal{R}^I$, $\theta_- \in \Theta$ and construct a set $H_t(b) = \{\theta^{t-1} \in \Theta^{t-1} : B_t(Z, \theta^{t-1}, \cdot) = b\}$ for $t \leq T$. The set H_t contains all histories up to period t such that the present value of costs is b. If the set $H_t(b)$ is empty for some b, set $y_t(b, \theta) = 0$, $c_t(b, \theta) = b(\theta)$ and $b_{t+1}(b, \theta, \cdot) = 0$ for all $\theta \in \Theta$. Otherwise, set

$$c_t(b, \theta) = C_t(\theta^{t-1}, \theta) \text{ for all } \theta^{t-1} \in H_t(b)$$
$$y_t(b, \theta) = Y_t(\theta^{t-1}, \theta) \text{ for all } \theta^{t-1} \in H_t(b)$$
$$b_{t+1}(b, \theta, \cdot) = B_{t+1}(\theta^{t-1}, \theta, \cdot) \text{ for all } \theta^{t-1} \in H_t(b).$$

[6]If shocks are I.I.D., then the vector of states reduces to the promised utility. If a first order approach can be applied, it is shown by Kapicka [9] that the state space reduces to the promised utility and a one dimensional costate variable.

Call $z = \{c_t, y_t, b_t\}_{t=0}^T$ to be an allocation rule generated by the allocation Z. Given assumption (1), the allocation rule z is correctly and uniquely defined.

Next result shows the relationship between the sequence problem and the recursive problem. Both problems are shown to be equally capable of solving the social planner's problem. In addition, the allocation generated by the efficient allocation rule is efficient.[7]

Theorem 3 *i) Suppose that z is the efficient allocation rule that solves (12) and (14) for $\bar{B}_0 \in \bar{\mathcal{B}}$ and $\theta_{-1} \in \Theta$ given. Then the allocation Z that is generated by z is the efficient allocation that solves (8) for B_0 and θ_{-1}.*

ii) Suppose Z is the efficient allocation that solves (8) for $\bar{B}_0 \in \bar{\mathcal{B}}$ and $\theta_{-1} \in \Theta$ given. Then the allocation rule z that is generated by Z delivers lifetime utility $V^*(\bar{B}_0, \theta_{-1}) = U^*(\bar{B}_0, \theta_{-1})$.

In the next two sections, I will consider a problem of decentralizing the efficient allocations with state dependent assets in a competitive equilibrium. The decentralizations I consider will critically rely on the recursive formulation of the social planner's problem. The reason is that the social planner's ex-post costs will be directly linked to the agent's holdings of income or report contingent assets in a competitive equilibrium. This result is complementary to the result obtained by Albanesi and Sleet [1], who link ex-ante costs with holdings of a risk free bond. At the same time, this result is a key ingredient that allows for a decentralization of the constrained efficient allocation without any need to use taxes.

4 Direct Decentralization

The market structure I will consider is the following. Each period, the agents trade one period financial contracts with financial intermediaries. The contracts are contingent on the agent's next period report and, for each report, specify two things: asset payoff and income that the agent must produce. In this sense, the contracts "tie" agent's assets with income. A generic financial contract will be denoted by $\phi = (a, y)$, where $a \in \mathcal{R}^I$ is the report contingent asset and $y : \Theta \to \mathcal{R}_+$ is the report

[7] The theorem is asymmetric in a sense that the allocation rule generated by an efficient allocation is not efficient because it does *not* solve the recursive social planner's problem for all possible values of ex post costs. It only provides a correct solution for the ex-post costs that are selected by the efficient allocation.

contingent income. The space of financial contracts is $\Phi = \mathcal{R}^I \times \mathcal{R}^I_+$. The contracts are executed at the beginning of the period. That is, at the beginning of the period the agent provides a report about her shock. That determines the income she must produce and the asset payoff. After that, the agent decides about her consumption and chooses a new financial contract.

The markets for the financial contracts are differentiated according to the current period report of the agent. Prices at each market are linear in quantities and are determined by the financial intermediaries who supply them competitively. Let $p = \{p_t\}_{t=0}^T$, $p_t : \Theta^2 \to \mathcal{R}$ be a vector of asset prices, where $p_t(\theta, \theta')$ denotes the price of an asset that pays one unit of consumption good tomorrow if an agent reports θ' tomorrow and has reported θ today.

I follow Prescott and Townsend [16] and Bisin and Gottardi [3] by assuming that constraints that arise from the agents' private information are imposed directly on the set of trades the agents can take. In addition to the constraints arising from private information, the agent is also limited in her choices by an external monitoring agency. The agency has the power to restrict agent's choice of next period assets, given her current contract and current report. The external monitoring agency will be needed in some, but not all, cases to ensure that the competitive equilibrium allocation is constrained efficient. The precise role of the external monitoring agency is discussed later.

I will now formally introduce all the elements of the market economy and define the competitive equilibrium for the economy, henceforth called a *direct competitive equilibrium*.

External Monitoring Agency. The external monitoring $m = \{m_t\}_{t=0}^T$ is given by a sequence of correspondences $m_t : \Phi \times \Theta \to\to \mathcal{R}^I$, where $m_{t+1}(\phi, \hat{\theta})$ is the set of assets the agent can choose in period t if she enters current period with financial contract $\phi \in \Phi$ and reports $\hat{\theta} \in \Theta$ to the financial intermediary.

Agent's problem. It is convenient to consider the agent's problem in two stages. In the first stage, the agent chooses her report to the financial intermediary. In the second stage, the agent produces the required income and chooses current consumption and next period contract.

Suppose that the agent enters period t with a financial contract ϕ. Her lifetime utility from reporting $\hat{\theta} \in \Theta$ to the financial intermediary when her current shock is $\theta \in \Theta$ is given by a value function $\hat{W}^d_t[\phi(\hat{\theta}), \theta]$.[8] Maximum lifetime utility of a θ

[8]The value function reflects the fact that the report $\hat{\theta}$ enters the value function only through the payment of the financial intermediary $\phi(\hat{\theta})$.

type agent from a contract ϕ, $W_t^d(\phi, \theta)$, is attained by a report that solves:

$$W_t^d(\phi, \theta) = \max_{\hat{\theta} \in \Theta} \hat{W}_t^d[\phi(\hat{\theta}), \theta]. \tag{15}$$

This defines the optimal reporting strategy $\theta^d = \{\theta_t^d\}_{t=0}^T$, where $\theta_t^d : \Phi \times \Theta \rightarrow \Theta$. The value function $\hat{W}_t^d[\phi(\hat{\theta}), \theta]$ is determined in the second stage:

$$\hat{W}_t^d[\phi(\hat{\theta}), \theta] = \max_{c_t, \phi_{t+1}} u(c_t) - v[\frac{y(\hat{\theta})}{\theta}] + \beta \sum_{\theta' \in \Theta} W_{t+1}^d(\phi_{t+1}, \theta')\pi(\theta'|\theta) \tag{16}$$

subject to three constraints. First, the agent's choices must satisfy her budget constraint:

$$a(\hat{\theta}) = c_t - y(\hat{\theta}) + \sum_{\theta' \in \Theta} p_t(\hat{\theta}, \theta')a_{t+1}(\theta'). \tag{17}$$

Second, since the financial intermediary is uninformed about the agent's productivity shock, it will trade only such contracts that are incentive compatible in a sense that the agent will want to report their true shock to the financial intermediary next period:

$$\phi_{t+1} \in X_{t+1}, \tag{18}$$

where X_{t+1} is the consumption possibility set, given by

$$X_{t+1} = \{\phi \in \Phi, \quad \hat{W}_{t+1}^d[\phi(\theta), \theta] \geq \hat{W}_{t+1}^d[\phi(\hat{\theta}), \theta] \quad \forall \theta, \hat{\theta} \in \Theta^2\}.$$

Third, the external monitoring agency restricts the agent's choice of next period assets:

$$a_{t+1}(\cdot) \in m_{t+1}(\phi, \hat{\theta}).$$

The optimal consumption and financial contract that solve the stage 2 problem (16) are denoted by $\hat{c}_t^d(\phi(\hat{\theta}), \theta)$ and $\hat{\phi}_{t+1}^d(\phi(\hat{\theta}), \theta)$. Together with the optimal reporting strategy θ^d one can derive the optimal policy functions $z^d = \{c_t^d, \phi_t^d\}_{t=0}^T$, where $c_t^d : \Phi \times \Theta \rightarrow \mathcal{R}_+$ is given by $c_t^d(\phi, \theta) = \hat{c}_t^d[\phi(\theta_t^d(\phi, \theta)), \theta]$ for $t \geq 0$ and $\phi_t^d : \Phi \times \Theta \rightarrow \Phi$ is given by $\phi_t^d(\phi, \theta) = \hat{\phi}_t^d[\phi(\theta_t^d(\phi, \theta)), \theta]$ for $t \geq 1$.

Period zero is special because the agents do not enter that period with any particular financial contract. Instead, the agents start with some initial wealth $\bar{A}_0$. The market arrangements are such that before the period zero shock is realized, the agents are allowed to buy their period zero financial contract ϕ_0, subject to a budget constraint

$$\bar{A}_0 = \sum_{\theta_0 \in \Theta} p_0(\theta_{-1}, \theta_0)a_0(\theta_0). \tag{19}$$

The agents choose ϕ_0 to maximize $\sum_{\theta_0 \in \Theta} W_0^d(\phi, \theta_0)\pi(\theta_0|\theta_{-1})$ subject to (19) and a requirement that $\phi_0 \in X_0$. The first period financial contract chosen by the agents is denoted by $\phi_0^d(\bar{A}_0, \theta_{-1})$.

Financial Intermediaries. Financial intermediaries sell financial contracts. They can also trade with each other at intertemporal prices $q = \{q_t\}_{t=0.}^T$ The trades of a financial intermediary who operates on a market for agents who have reported $\theta \in \Phi$ are denoted by $x = \{x_t\}_{t=0}^T$ where $x_t : \Theta \to \mathcal{R}$. The financial intermediary maximizes profits: for all $t \leq T$, the supply of financial assets x_t^d solves

$$\max_{x_t} q_t \sum_{\theta' \in \Theta} \sum_{\theta \in \Theta} x_t(\theta')\pi(\theta'|\theta) - \sum_{\theta' \in \Theta} \sum_{\theta \in \Theta} x_t(\theta')p_t(\theta, \theta')$$

where the objective function is discounted by q_t to reflect the fact that assets are paid only in the next period. It follows that the equilibrium prices are determined by the probabilities and the intertemporal price of consumption:

Lemma 4 *The asset prices satisfy, for all $t \leq T$,*

$$p_t(\theta, \theta') = q_t \pi(\theta'|\theta) \qquad \forall \theta', \theta \in \Theta^2. \tag{20}$$

The direct competitive equilibrium is defined next.

Definition 5 *A direct competitive equilibrium with external monitoring m is given by prices p, agent's allocation z^d, trades of financial intermediaries x^d such that the agent's allocation solves (15) and (16), prices are given by (20) and markets clear.*

The competitive equilibrium is said to be incentive compatible if, whenever the agents start with a financial contract from their consumption possibility set, they prefer to report truthfully to the financial intermediary. Next lemma shows that the direct competitive equilibrium is incentive compatible. The proof is straightforward and is omitted.

Lemma 6 *For all $\phi \in X_t$ all $\theta \in \Theta$ and all $t \leq T$, $\theta_t^d(\phi, \theta) = \theta$.*

4.1 Efficiency Properties of the Direct Competitive Equilibrium

In this section I study the relationship between the competitive equilibrium allocations and the constrained efficient allocations. A direct competitive equilibrium is said to be constrained efficient if it satisfies the following definition:

Definition 7 *A direct competitive equilibrium is constrained efficient if* $\phi_0^d(\bar{A}_0, \theta_{-1}) = [b_0^*(\bar{A}_0, \theta_{-1}), y_0^*(b_0^*(\bar{A}_0, \theta_{-1}), \cdot)]$ *and, for any* $t \geq 0$*, the following property holds: whenever* $\phi \in X_t$ *satisfies* $\phi = [b, y_t^*(b, \cdot)]$ *then for all* $\theta \in \Theta$*,*

$$c_t^d(\phi, \theta) = c_t^*(b, \theta),$$
$$\phi_{t+1}^d(\phi, \theta) = [b_{t+1}^*(b, \theta, \cdot), y_{t+1}^*(b_{t+1}^*(b, \theta, \cdot), \cdot)].$$

That is the competitive equilibrium is efficient under two conditions: First, the agents choose the efficient contract[9] at the beginning of period zero. Second, in any period, given that they start with the efficient contract, the agent choose the constrained efficient consumption level and the next period contract will be efficient as well.

The external monitoring is said to be *efficient* if the set m contains only asset portfolios that the social planner would choose in a given situation: for all $\theta \in \Theta$, all $t \leq T-1$, $m_{t+1}[(b, y), \theta] = \{b_{t+1}^*(b, \theta, \cdot)\}$ if $y = y_t^*(b, \theta)$, and $m_{t+1}[(b, y), \theta]$ is empty otherwise. The empty set is interpreted as a large penalty for choosing any portfolio, large enough to deter the agent from choosing such income. If the monitoring is efficient, then the first welfare theorem holds, as shown by the next theorem.

Theorem 8 *The direct competitive equilibrium with efficient external monitoring* m *is constrained efficient.*

The reason why the first welfare theorem holds is that with report dependent assets, the agents have enough instruments to insure themselves through financial markets. The fact that there is private information is in a sense irrelevant because the government is as uninformed as the financial intermediaries are. The reason why the agents *choose* to insure themselves efficiently is that ex ante, their objective is the same as the objective of the government, and the equilibrium prices correctly reflect probabilities of various shocks.

The only restriction needed is the restriction imposed by the external monitoring agency. The efficient external monitoring is in general needed because there is the following asymmetry between the social planner's problem and the direct competitive equilibrium. In the social planner's problem, reporting $\hat{\theta}$ by a θ-type agent in period t implies that the agent must consume $c_t(b, \hat{\theta})$ but also that her next period ex-post costs are $b_{t+1}(b, \hat{\theta}, \cdot)$. That is, if a θ-type agent decides to consume an amount that is efficient for a $\hat{\theta}$-type agent, she is forced to have the same ex-post costs as well.

[9]A contract is efficient if the agents choose an asset portfolio equal to the constrained efficient ex-post costs and the contract income vector equals the constrained efficient income vector.

A symmetric statement does not apply in the direct competitive equilibrium, since if a θ-type agent reports $\hat{\theta}$ to the financial intermediary, she is, in the absence of the external monitoring, free to choose any portfolio and consumption that satisfy the budget constraint (17). The consumption $\hat{c}_t^*(\phi(\hat{\theta}), \hat{\theta})$ and next period assets $\hat{a}_{t+1}^*(\phi(\hat{\theta}), \hat{\theta})$ are still available to her, but they will in general not be chosen for a simple reason that a $\hat{\theta}$-type agent has a probability distribution of future shocks that differs from the one of a θ-type agent.[10] Therefore $\hat{c}_t^*(\phi(\hat{\theta}), \theta) \neq \hat{c}_t^*(\phi(\hat{\theta}), \hat{\theta})$ and $\hat{a}_{t+1}^*(\phi(\hat{\theta}), \theta) \neq \hat{a}_{t+1}^*(\phi(\hat{\theta}), \hat{\theta})$.

The only exception occurs when the probability distribution of future shocks does not depend on the current shocks, i.e. the productivity shocks are I.I.D. In that case $\hat{c}_t^*(\phi(\hat{\theta}), \theta)$ and $\hat{a}_{t+1}^*(\phi(\hat{\theta}), \theta)$ are both independent of θ and one gets the following result:

Corollary 9 *If the productivity shocks are I.I.D. then the direct competitive equilibrium is constrained efficient even without external monitoring.*

If the shocks are Markov, the social planner's problem has features of adverse selection: when the agents make their decision about next period financial contract, they already know their current type. It has been shown by Bisin and Gottardi [3] and Prescott and Townsend [15] that the first welfare theorem sometimes fails in models with adverse selection. Why doesn't this problem occur in the Mirrleesean economy? The reason is that competitive equilibria fail to be efficient only when the solution to the social planner's problem features pooling and, consequently, cross subsidization across types. However, pooling is never a solution in a Mirrleesean economy because it is never incentive compatible. Thus, competitive equilibria with efficient external monitoring are always constrained efficient.[11]

5 Indirect Decentralization

The indirect decentralization differs from the direct one in one important aspect: the asset payoffs are contingent directly on the agent's next period income. Compared

[10] Note also that the restriction that $m_{t+1}[(b, y), \theta]$ is empty if $y \neq y_t^*(b, \theta)$ is in principle necessary. In its absence, the agent might choose a next period contract with an output vector that is only slightly different from the efficient one. That would "free" the agent from the regulations of the monitoring agency and make deviations profitable.

[11] The failure of the first welfare theorem in Bisin and Gottardi [3] and Prescott and Townsend [15] is unrelated to the problems that are remedied by the external monitoring agency. The external monitoring agency is needed because of the dynamic features of the model. It does not collect any resources and is therefore unable to provide any cross-subsidization.

to the financial contracts in the direct decentralization, the contracts are therefore significantly simplified: they are just income contingent asset portfolios. No reporting to the financial intermediary is needed and there is no need to make an explicit agreement about the income to be produced.

Since there is only I possible productivity shock values, the agents will in each period choose at most I different levels of income. One can therefore exogenously restrict the agent's choice of an asset portfolio to be such that only assets for only I income levels $\tilde{y}_1, \tilde{y}_2, ...\tilde{y}_I$, chosen by the agent, are purchased. For all other income levels the asset payment is implicitly specified to be a large negative number, in effect preventing the agent from choosing such income levels. Such an assumption is innocuous, but simplifies the algebra significantly. A typical portfolio of income dependent assets is thus $a \in \mathcal{A} = \mathcal{R}^I$. The timing is as follows. At the beginning of the period the agents learn about their current shock and choose the income they want to produce. The income is observed by the financial intermediary who pays the agent the amount specified by the contract. Then the agents decide how to split their resources between consumption and next period asset portfolio.

The markets for financial contracts are differentiated according to the current portfolio and the current income of the agent. It is assumed that the contracts are traded as a bundle. That is, there is one price for the whole asset portfolio. The contract prices are determined competitively. Let $P = \{P_t\}_{t=0}^T$, $P_t : \mathcal{A}^2 \times \mathcal{R}_+ \rightarrow \mathcal{R}$ be a sequence of asset prices, where $P_t(a, y; a')$ is the price of an asset portfolio a' if the agent holds asset portfolio a today and chooses income y.

Similarly to the direct decentralization, an external monitoring agency will in general be needed to achieve efficiency. The role of the external monitoring agency has been discussed in the previous section. Since the discussion carries over without any change, the external monitoring agency will not be discussed in detail again.

In this section, I will assume that the efficient allocation rule is such that the income can be inverted to infer the underlying productivity shock. This assumption will be essential if the shocks are Markov, since otherwise the financial intermediary will not be able to infer what the agent's current productivity shock is and therefore what the probability distribution over future shocks is.

Assumption 10 *If* $\theta \neq \hat{\theta}$ *then* $y_t^*(b, \theta) \neq y_t^*(b, \hat{\theta})$ *for all* $b \in \mathcal{R}^I$ *,all* $t \leq T - 1$.

The competitive equilibrium is called an indirect competitive equilibrium, to distinguish it from the direct one. The elements of the indirect competitive equilibrium are defined as follows.

External Monitoring Agency. The external monitoring $M = \{M_t\}_{t=0}^T$ is given by a sequence of correspondences $M_t : \mathcal{A} \times \mathcal{R}_+ \rightarrow\rightarrow \mathcal{A}$. where $M_{t+1}(a, y)$ is the set of asset portfolios the agent can choose in period t if she enters current period with assets $a \in \mathcal{A}$ and produces income $y \in \mathcal{R}_+$.

Agent's problem. Suppose that the agent enters period t with an asset portfolio a. Her lifetime utility from such a contract when her true shock is θ is given by a value function $W_t^i(a, \theta)$, which satisfies

$$W_t^i(a, \theta) = \max_{c_t, y_t, a_{t+1}} u(c_t) - v(\frac{y}{\theta}) + \beta \sum_{\theta' \in \Theta} W_{t+1}^i(a_{t+1}, \theta')\pi(\theta'|\theta) \tag{21}$$

subject to her budget constraint

$$a(y_t) = c_t - y_t + P_t(a, y_t; a_{t+1}), \tag{22}$$

and the restrictions imposed by the external monitoring agency:

$$a_{t+1}(\cdot) \in m_{t+1}(a, y).$$

Denote the optimal policy functions by $z^i = \{c_t^i, y_t^i, a_t^i\}_{t=0}^T$, where $c_t^i : \mathcal{A} \times \Theta \to \mathcal{R}_+$ is the optimal consumption in period $t \geq 0$, $y_t^i : \mathcal{A} \times \Theta \to \mathcal{R}_+$ is the optimal income in period $t \geq 0$, and $a_t^i : \mathcal{A} \times \Theta \to \mathcal{A}$ is the optimal asset portfolio at the beginning of period $t \geq 1$.

The agents start period zero with some wealth $\bar{A}_0$ and are allowed to buy an asset portfolio before the period zero shock is realized. They can buy any asset portfolio a_0 that satisfies the budget constraint

$$\bar{A}_0 = \sum_{\theta \in \Theta} P_0(\theta_{-1}, y)a_0(y), \tag{23}$$

where the period zero prices depend directly on θ_{-1} because θ_{-1} is publicly observed. The agents choose a_0 to maximize $\sum_{\theta \in \Theta} W_0^i(a, \theta)\pi(\theta|\theta_{-1})$ subject to (23). The period zero asset portfolio that solves this problem is denoted by $a_0^i(\bar{A}_0, \theta_{-1}, \cdot)$.

Given that the efficient allocation rule is supposed to have invertible income in all situations by assumption (10), one may conjecture that the same property holds for the solution of the agent's problem: if $\theta \neq \hat{\theta}$ then $y_t^i(a, \theta) \neq y_t^i(a, \hat{\theta})$, any $t \leq T$. The conjecture will be proven when the first welfare theorem is proven.

Financial Intermediaries. The trades of a financial intermediary are denoted by $x = \{x_t\}_{t=0}^T$, $x_t : \mathcal{A} \times \mathcal{R}_+ \to \mathcal{A}$ where $x_{t+1}(a, y)$ is the number of portfolios sold to agents who have current portfolio a and produce y. In equilibrium, only the markets for agents with current assets a and current income $y_t^i(a, \theta)$ for some

$\theta \in \Theta$ will operate. The equilibrium price of an asset portfolio is determined by the profit maximization of financial intermediaries: for a and $y_t^i(a,\theta)$ given, the financial intermediary solves

$$\max_{x_{t+1}} x_{t+1} q_t \sum_{\theta' \in \Theta} \sum_{\theta \in \Theta} a_{t+1}[y_{t+1}^i(a_{t+1},\theta')]\pi(\theta'|\theta) - x_{t+1} P_t(a, y; a_{t+1}),$$

where the financial intermediaries implicitly determine the current and future productivity shock from agent's choice of current and future income. This is possible by assumption (10). The equilibrium portfolio prices are given in the next lemma:

Lemma 11 *In equilibrium, the asset prices satisfy, for all $t \leq T-1$, and all $a \in \mathcal{A}$,*

$$P_t[a, y_t^i(a,\theta); a_{t+1}] = q_t \sum_{\theta' \in \Theta} a_{t+1}[y_{t+1}^i(a_{t+1},\theta')]\pi(\theta'|\theta) \qquad \forall \theta, \theta' \in \Theta^2. \qquad (24)$$

The indirect competitive equilibrium is defined as follows:

Definition 12 *The indirect competitive equilibrium with external monitoring M is given by prices P, agent's allocation z^i, trades of financial intermediaries x such that the agent's allocation solves (21), prices are given by (24) and markets clear.*

5.1 Efficiency Properties of the Indirect Competitive Equilibrium

The efficiency of an indirect competitive equilibrium is defined as follows:

Definition 13 *An indirect competitive equilibrium is constrained efficient if $a_0^i(\bar{A}_0, \theta_{-1}) = b_0^*(\bar{A}_0, \theta_{-1})$ and for any $t \geq 0$, whenever a satisfies $a[y_t^*(b,\cdot)] = b(\cdot)$ then for all $\theta \in \Theta$*

$$\begin{aligned} c_t^i(a,\theta) &= c_t^*(b,\theta), \\ y_t^i(a,\theta) &= y_t^*(b,\theta), \\ a_{t+1}^i[a,\theta, y_{t+1}^*(b_{t+1}^*(b,\theta,\cdot),\cdot)] &= b_{t+1}^*(b,\theta,\cdot). \end{aligned}$$

That is, the indirect competitive equilibrium is efficient if the following two conditions hold. First, at the beginning of period zero the agent chooses an asset portfolio that is efficient in a sense that the asset payments the social planner would get coincide with the constrained efficient ex-post costs of the social planner. Second, if the agent starts with an efficient portfolio, she chooses the efficient consumption, income and next period asset portfolio.

The external monitoring is said to be *efficient* if, for all $t \leq T-1$, $M_{t+1}(a, y) = \{b^*_{t+1}(b, \theta, \cdot)\}$ whenever $a[y^*_t(b, \cdot)] = b(\cdot)$ and $y = y^*_t(b, \theta)$ for some $\theta \in \Theta$, and $m_{t+1}(a, y)$ is empty otherwise. That is, the set M_{t+1} contains only the asset portfolio that the social planner would choose, given the agent's type (as inferred from agent's income and current asset portfolio). If the agent chooses an income that is not optimal for any type, then the set M_{t+1} is empty, in effect prohibiting the agent from doing so. The first welfare theorem again links the equilibrium allocations to the constrained efficient ones:

Theorem 14 *The indirect competitive equilibrium with efficient external monitoring M is constrained efficient.*

Conceptually, the theorem is a first welfare theorem, but it also contains elements of a taxation principle.[12] Compared to the first welfare theorem (8), it shows that indirect mechanisms are as capable of achieving efficiency as direct mechanisms (provided that income is invertible). The proof of the theorem is based on several related arguments. First, the external monitoring agency forces a $\theta-$type agent to select the efficient income and asset portfolio of some agent, but not necessarily of the same type. If however the agent chooses the efficient income and asset portfolio of, say, $\hat{\theta}-$type agent, she also faces prices that the $\hat{\theta}-$type agent faces and, as a consequence, must choose the consumption of the $\hat{\theta}-$type agent as well. The fact that the efficient allocation rule is incentive compatible then implies that it is in the agent's interest to choose the efficient allocation of her own type.

When the shocks are I.I.D., one can again considerably simplify the structure of the problem. As one can expect, the external monitoring agency is superfluous, as in the case of direct competitive equilibrium. In addition, the pricing mechanism is simplified as well, because the financial intermediaries do not need to know the agent's current productivity shock to assess the agent's future prospects. It follows that the invertibility assumption is not needed as well.

Corollary 15 *If the productivity shocks are I.I.D. then*

i) the indirect competitive equilibrium is constrained efficient even without external monitoring,

ii) the competitive equilibrium prices $P_t(a, y; a')$ are independent of a and y for all $t \geq 0$, and

iii) Assumption (10) that y^* is invertible is not required.

[12] See Hammond [8] for a statement of the taxation principle.

6 Concluding Remarks

This paper studies if, and to what extent, can the constrained efficient allocations in dynamic Mirrleesean economies be recursively decentralized with assets that depend on agent's current income (or current report). The recursive decentralization is of interest, because it does not require long term contracting between the agents and the financial intermediaries: if the first welfare theorem holds, the efficient allocations can be implemented with a series of spot asset trading.

I present two alternative decentralizations, a direct one where the agent send reports about their productivity shocks to the financial intermediary directly, and an indirect one where the information about the agent's income is all the financial intermediaries need to observe. I show that both environments are capable of decentralizing the efficient allocations. However, if the shocks are Markov, slightly stricter conditions are required for the indirect decentralization: it is required that the agent's income is invertible so that the agent's current productivity can be uncovered from the income. The main benefit of the indirect decentralization is that the contracts traded are significantly simpler than the contracts traded in a direct decentralization, and that the trading arrangements seem more natural because no direct reporting takes place.

The conclusion of the paper is that the constrained efficient allocations can be recursively decentralized without any use of taxes. But, in general, the government is still needed to monitor the asset trades and to prevent the agents from making "joint deviations" from the optimum, in consumption and asset trades. The only exception is when the shocks are I.I.D., in which case no government monitoring is needed and markets can achieve efficiency without any government involvement. The need for the external monitoring agency is however not directly related to the fact that decentralization is recursive. Rather, it is likely to be needed in any decentralization in which the agents can trade assets, and their budget constraint allows them to make the joint deviations.

One may claim that the conclusion that taxation is not needed to achieve efficiency is not a very sensible one, because the asset trades that emerge in equilibrium are probably not observed in actual economies. But such criticism is misguided: if the assets predicted by the competitive equilibrium are not traded (or are traded only to a limited extent), one needs to look for reasons why that happens, rather than restricting the asset trades exogenously. Recently, Ramsey taxation has been exposed to the following criticism: why should one believe the results if they are based on exogenous restrictions on the set of tax instruments? Mirrleesean taxation with exogenous restrictions on asset trades would be prone to a very similar criticism:

why should one believe the results if asset trades are exogenously restricted?

What are then some possible reasons why the asset markets might be distorted or nonexistent and a role for taxes reappears? One argument is related to the fact that period zero is different from all other periods: the agent are given a chance to trade assets before their shock is known. There is no other period with similar trading arrangements. One could argue that such trading is in fact not feasible: the agents are already born with their shock and there is simply no time for such trading. In that case, income taxation would be called for, but only in period zero. Or, one could argue the other way: if the agents have a chance to trade assets at the beginning of period zero, they should also get a chance to retrade their asset portfolios at the beginning of any period. In that case, asset or income taxation would in general be needed. The case for taxation is related to the fact that the recursive social planner's problem is not time consistent. Consequently, the agent's problem is not time consistent as well: the agent would like to retrade the assets, before her shock is revealed. But asset trading at the beginning of a period would undermine the commitment that is implicitly contained in the current trading arrangements and taxation would be needed to correct that. A special situation arises again for I.I.D. shocks in which case the social planner's problem is time consistent and trading at the beginning of a period would do no harm. Taxation would also be needed if the asset trades with financial intermediaries were not exclusive or if consumption were unobservable (this situation has been studied by Golosov and Tsyvinski [6] for different trading arrangements).

Yet another reason for taxation, unrelated to the existence of asset trades, would occur if the government has some expenditures it needs to finance. Then the government will want to impose a lump-sum tax on the initial assets at the amount equal to the present value of expenditures. Beyond that, the problem is unchanged and the first welfare theorems continue to hold. However, if the government has a stream of expenditures to finance but is not allowed to borrow and save, taxation in all periods might be needed. The pattern of optimal taxation in such case is left for future research. For all these cases, the results of this paper are a useful starting point for such considerations.

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Appendix

Proof of Lemma (2). Since none of the constraints (9), (10) and (11) depends directly on θ_-, the optimal allocation rule in period t maximizes for all $\theta \in \Theta$,

$$\max_{c_t(\theta), y_t(\theta), b_{t+1}(\theta)} u[c_t(\theta)] - v[\frac{y_t(\theta)}{\theta}] + \beta V_{t+1}[b_{t+1}(\theta, \cdot), \theta] \tag{25}$$

subject to (9), (10) and (11). The objective function (25) is independent of θ_- and so is the optimal allocation rule. ■

The following lemma will be used in the proof of Theorem (3):

Lemma 16 *Let*

$$U_t(Z, \hat{\theta}^{t-1}, \theta_{t-1}) = \sum_{j=0}^{T} \sum_{\theta_j^t \in \Theta^{1+j}} \beta^j \{u[C_{t+j}(\hat{\theta}^{t-1}, \theta_t^{t+j})] - v[\frac{Y_{t+j}(\hat{\theta}^{t-1}, \theta_t^{t+j})}{\theta_{t+j}}]\} \pi^{j+1}(\theta_t^{t+j} | \theta_{t-1}). \tag{26}$$

An allocation Z is incentive compatible if and only if for all $t \leq T$, for all histories $\hat{\theta}^{t-1} \in \Theta^t$ and all current period shocks $\theta_t \in \Theta$,

$$u[C_t(\hat{\theta}^{t-1}, \theta_t)] - v[\frac{Y_t(\hat{\theta}^{t-1}, \theta_t)}{\theta_t}] + \beta U_{t+1}[Z, (\hat{\theta}^{t-1}, \theta_t), \theta_t] \geq u[C_t(\hat{\theta}^t)] - v[\frac{Y_t(\hat{\theta}^t)}{\theta_t}] + \beta U_{t+1}(Z, \hat{\theta}^t, \theta_t) \tag{27}$$

for all $\hat{\theta}_t \in \Theta$.

Proof. Necessity of (27) is straightforward. I will show sufficiency by induction. Note first that for any history of reports $\hat{\theta}^{t-1} \in \Theta^t$, any reporting strategy σ such that $\sigma^{t-1}(\theta^{t-1}) = \hat{\theta}^{t-1}$ for some history of shocks $\theta^{t-1} \in \Theta^t$, the lifetime utility $U_t(Z \circ \sigma, \hat{\theta}^{t-1}, \theta_{t-1})$ can be written recursively as follows:

$$U_t(Z \circ \sigma, \hat{\theta}^{t-1}, \theta_{t-1}) = \sum_{\theta_t \in \Theta} \{u[C_t(\hat{\theta}^{t-1}, \sigma_t(\theta^t))] - v[\frac{Y_t(\hat{\theta}^{t-1}, \sigma_t(\theta^t))}{\theta_t}] + \beta U_{t+1}[Z \circ \sigma, (\hat{\theta}^{t-1}, \sigma_t(\theta^t)), \theta_t]\} \pi(\theta_t|\theta_{t-1}). \tag{28}$$

Consider now any reporting function in period zero $\tilde{\sigma}_0 : \Theta \to \Theta$. The incentive compatibility constraint (6) and (27) imply that

$$\begin{aligned} U(Z, \theta_{-1}) &= \sum_{\theta_0 \in \Theta} \{u[C_0(\theta_0)] - v[\frac{Y_0(\theta_0)}{\theta_0}] + \beta U_1(Z, \theta_0, \theta_0)\} \pi(\theta_0|\theta_{-1}) \\ &\geq \sum_{\theta_0 \in \Theta} \{u[C_0(\tilde{\sigma}_0(\theta_0))] - v[\frac{Y_1(\tilde{\sigma}_0(\theta_0))}{\theta_0}] + \beta U_2[Z, \tilde{\sigma}_0(\theta_0), \theta_0]\} \pi(\theta_0|\theta_{-1}) \\ &= U(Z \circ \sigma^{(0)}, \theta_{-1}), \end{aligned}$$

where $\sigma^{(0)}$ is a reporting strategy involving only first period deviations, defined by $\sigma_0^{(0)}(\theta_0) = \tilde{\sigma}_0(\theta_0)$ and $\sigma_j^{(0)}(\theta^j) = \theta_j$ for $j \geq 1$ for all $\theta^j \in \Theta^{j+1}$.

Assume now that, for any $t \geq 1$, $U(Z, \theta_{-1}) \geq U(Z \circ \sigma^{(t-1)}, \theta_{-1})$,where $\sigma^{(t-1)}$ is a reporting strategy involving only deviations in periods $0, 1, ..t-1$. Let $\hat{\theta}^{t-1} = \sigma^{(t-1)(t-1)}(\theta^{t-1})$ by the history of reports in the first $t-1$ following a shock history $\theta^{t-1} \in \Theta^t$, as prescribed by the reporting strategy $\sigma^{(t-1)}$. Let $\tilde{\sigma}_t : \Theta \to \Theta$ be an arbitrary reporting function in period t. Applying the inequality (27) in period t, taking expectations and using (28) one gets that

$$U_t(Z, \hat{\theta}^{t-1}, \theta_{t-1}) \geq U_t(Z \circ \sigma^{(t)}, \hat{\theta}^{t-1}, \theta_{t-1}). \tag{29}$$

By expanding the expression for $U(Z \circ \sigma^{(t-1)}, \theta_{-1})$ using (28) repeatedly t times, one gets that

$$U(Z\circ\sigma^{(t-1)},\theta_{-1}) = \sum_{j=0}^{t-1}\sum_{\theta^j\in\Theta^j}\beta^{j-1}\{u[C_j(\sigma^{(t-1)j}(\theta^j))] - v[\frac{Y_j(\sigma^{(t-1)j}(\theta^j))}{\theta_j}]\}\pi(\theta^j|\theta_{-1})$$
$$+\beta^{t-1}\sum_{\theta^{t-1}\in\Theta^{t-1}} U_t[Z,\sigma^{(t-1)(t-1)}(\theta^{t-1}),\theta_t]\pi(\theta^{t-1}|\theta_{-1})$$
$$\geq \sum_{j=0}^{t-1}\sum_{\theta^j\in\Theta^j}\beta^{j-1}\{u[C_j(\sigma^{(t-1)j}(\theta^j))] - v[\frac{Y_j(\sigma^{(t-1)j}(\theta^j))}{\theta_j}]\}\pi(\theta^j|\theta_{-1})$$
$$+\beta^{t-1}\sum_{\theta^{t-1}\in\Theta^{t-1}} U_t[Z\circ\sigma^{(t)},(\sigma^{(t)(t-1)}(\theta^{t-1})),\theta_{t-1}]\pi(\theta^{t-1}|\theta_{-1})$$
$$= U(Z\circ\sigma^{(t)},\theta_{-1}),$$

where the inequality follows from the inequality (29). Hence $U(Z,\theta_{-1}) \geq U(Z\circ\sigma^{(t)},\theta_{-1})$ and the incentive compatibility holds for any reporting strategies involving deviations in the first t periods. ■

Proof of Theorem (3). The proof proceeds in two steps. First, it is shown in step one that if z solves the recursive social planner's problem then Z generated by z satisfies (6) and (7). Hence $U^*(\bar{B}_0,\theta_{-1}) \geq V^*(\bar{B}_0,\theta_{-1})$. Second, a converse result is shown: If Z solves the social planner's problem then z generated by Z satisfies (9), (10) and (11). Therefore $V^*(\bar{B}_0,\theta_{-1}) \geq U^*(\bar{B}_0,\theta_{-1})$. Taken together, $V^*(\bar{B}_0,\theta_{-1}) = U^*(\bar{B}_0,\theta_{-1})$, which proves the second part of the result. That Z generated by z is efficient follows from the fact that $U(Z,\theta_{-1}) = V^*(\bar{B}_0,\theta_{-1}) = U^*(\bar{B}_0,\theta_{-1})$, where the first equality follows from step one.

Step 1: I will show that 1a) Z is incentive compatible and that 1b) it has a present value of costs $\bar{B}_0$.

(1a) Incentive compatibility. Note first that if an allocation rule z solves the recursive social planner's problem and Z is generated by z then

$$U_t(Z,\hat{\theta}^{t-1},\theta_{t-1}) = V_t[\hat{B}_t(\bar{B}_0,\theta_{-1},\hat{\theta}^{t-1},\cdot),\theta_{t-1}], \tag{30}$$

where is defined in (26). By lemma (16), it is enough to show that the allocation Z

satisfies (27). One gets that

$$
\begin{aligned}
& u[C_t(\hat{\theta}^{t-1},\theta_t)] - v(\frac{Y_t(\hat{\theta}^{t-1},\theta_t)}{\theta_t}] + \beta U_{t+1}[Z,(\hat{\theta}^{t-1},\theta_t),\theta_t] \\
&= u[c_t(\hat{B}_t(\bar{B}_0,\theta_{-1},\hat{\theta}^{t-1},\cdot),\theta_t)] - v[\frac{y_t(\hat{B}_t(\bar{B}_0,\theta_{-1},\hat{\theta}^{t-1},\cdot),\theta_t)}{\theta_t}] + \beta V_{t+1}[\hat{B}_{t+1}(\bar{B}_0,\theta_{-1},(\hat{\theta}^{t-1},\theta_t),\cdot),\theta_t] \\
&= u[c_t(\hat{B}_t(\bar{B}_0,\theta_{-1},\hat{\theta}^{t-1},\cdot),\theta_t)] - v[\frac{y_t(\hat{B}_t(\bar{B}_0,\theta_{-1},\hat{\theta}^{t-1},\cdot),\theta_t)}{\theta_t}] + \beta V_{t+1}[b_{t+1}[\hat{B}_t(\bar{B}_0,\theta_{-1},\hat{\theta}^{t-1},\cdot),\theta_t,\cdot],\theta_t] \\
&\geq u[c_t(\hat{B}_t(\bar{B}_0,\theta_{-1},\hat{\theta}^{t-1},\cdot),\hat{\theta}_t)] - v[\frac{y_t(\hat{B}_t(\bar{B}_0,\theta_{-1},\hat{\theta}^{t-1},\cdot),\hat{\theta}_t)}{\theta_t}] + \beta V_{t+1}[b_{t+1}[\hat{B}_t(\bar{B}_0,\theta_{-1},\hat{\theta}^{t-1},\cdot),\hat{\theta}_t,\cdot],\theta_t] \\
&= u[c_t(\hat{B}_t(\bar{B}_0,\theta_{-1},\hat{\theta}^{t-1},\cdot),\hat{\theta}_t)] - v[\frac{y_t(\hat{B}_t(\bar{B}_0,\theta_{-1},\hat{\theta}^{t-1},\cdot),\hat{\theta}_t)}{\theta_t}] + \beta V_{t+1}[\hat{B}_{t+1}(\bar{B}_0,\theta_{-1},(\hat{\theta}^{t-1},\hat{\theta}_t),\cdot),\theta_t] \\
&= u[C_t(\hat{\theta}^{t-1},\hat{\theta}_t)] - v(\frac{Y_t(\hat{\theta}^{t-1},\hat{\theta}_t)}{\theta_t}] + \beta U_{t+1}[Z,(\hat{\theta}^{t-1},\hat{\theta}_t),\theta_t],
\end{aligned}
$$

where the first and last equality follows from the definition of Z and from (30), second and next-to-last equality follows from the definition of $\hat{B}_{t+1}$, and the inequality follows from the incentive compatibility constraint (10). Thus, Z satisfies (27).

(1b) Resource Constraint. It is easy to show that

$$B_t(Z,\theta^t) = \hat{B}_t(\bar{B}_0,\theta^t). \tag{31}$$

The present value of costs of the allocation Z is given by

$$
\begin{aligned}
\sum_{\theta_0\in\Theta} B_0(Z,\theta_0)\pi(\theta_0|\theta_{-1}) &= \sum_{\theta_0\in\Theta} \hat{B}_0(\bar{B}_0,\theta_{-1},\theta_0)\pi(\theta_0|\theta_{-1}) \\
&= \sum_{\theta_0\in\Theta} b_0^*(\bar{B}_0,\theta_{-1},\theta_0)\pi(\theta_0|\theta_{-1}) \\
&= \bar{B}_0,
\end{aligned}
$$

where the first equality follows from (31), second equality follows from the definition of $\hat{B}_0$ and the last equality from the fact that b_0^* satisfies the ex-ante resource constraint (13). Thus, Z generated by z satisfies both the resource constraint. It clearly delivers the same lifetime utility as z, given by $V^*(\bar{B}_0,\theta_{-1})$. Therefore $U^*(\bar{B}_0,\theta_{-1}) \geq V^*(\bar{B}_0,\theta_{-1})$.

Step 2: One needs to show that z generated by Z satisfies 2a) the ex -post resource constraint (9), 2b) the incentive compatibility constraint (10) and 2c) the constraint (11).

(2a) Ex-post resource constraint. One can show that $B_t(Z,\theta^t)$ satisfies

$$B_t(Z,\theta^t) = C_t(\theta^t) - Y_t(\theta^t) + q_t \sum_{\theta' \in \Theta} B_{t+1}(Z,\theta^t,\theta_{t+1})\ \pi(\theta_{t+1}|\theta_t). \tag{32}$$

Choose $b \in \mathcal{R}^I$ and suppose that $\theta^{t-1} \in H_t(b)$. The allocation rule z generated by Z satisfies

$$\begin{aligned}
&c_t(b,\theta_t) - y_t(b,\theta_t) + q_t \sum_{\theta' \in \Theta} b_{t+1}(b,\theta_t,\theta_{t+1})\pi(\theta_{t+1}|\theta_t) \\
&= C_t(\theta^t) - Y_t(\theta^t) + q_t \sum_{\theta' \in \Theta} B_{t+1}(\theta^t,\theta_{t+1})\ \pi(\theta_{t+1}|\theta_t) \\
&= B_t(Z,\theta^t) \\
&= b(\theta),
\end{aligned}$$

where the first equality follows from definition of z, the second one from (32) and the last one from the definition of z again. Hence the allocation rule satisfies the ex-post resource constraint.

(2b) Incentive compatibility. Define recursively the value of the generated allocation rule z by

$$\tilde{V}_t(b,\theta_-) = \sum_{\theta \in \Theta} \{u[c_t(\theta)] - v[\frac{y_t(\theta)}{\theta}] + \beta \tilde{V}_{t+1}[b_{t+1}(\theta,\cdot),\theta]\}\pi(\theta|\theta_-).$$

It is easy to see that if $\hat{\theta}^{t-1} \in H_t(b)$ then $U_t(Z,\hat{\theta}^{t-1},\theta_{t-1}) = \tilde{V}_t(b,\theta_-)$. Since Z is incentive compatible, lemma (16) implies that it satisfies (27). Therefore, for any $b \in \mathcal{R}^I$ and $\hat{\theta}^{t-1} \in H_t(b)$,

$$\begin{aligned}
&u[c_t(b,\theta_t)] - v[\frac{y_t(b,\theta_t)}{\theta_t}] + \beta \tilde{V}_{t+1}[b_{t+1}(b,\theta_t,\cdot),\theta_t] \\
&= u[C_t(\hat{\theta}^{t-1},\theta_t)] - v[\frac{Y_t(\hat{\theta}^{t-1},\theta_t)}{\theta_t}] + \beta \tilde{V}_{t+1}[B_{t+1}(\hat{\theta}^{t-1},\theta_t,\cdot)\ ,\theta] \\
&= u[C_t(\hat{\theta}^{t-1},\theta_t)] - v[\frac{Y_t(\hat{\theta}^{t-1},\theta_t)}{\theta_t}] + \beta U_{t+1}[Z,(\hat{\theta}^{t-1},\theta_t),\theta_t] \\
&\geq u[C_t(\hat{\theta}^t)] - v[\frac{Y_t(\hat{\theta}^t)}{\theta_t}] + \beta U_{t+1}(Z,\hat{\theta}^t,\theta_t) \\
&= u[C_t(\hat{\theta}^t)] - v[\frac{Y_t(\hat{\theta}^t)}{\theta_t}] + \beta \tilde{V}_{t+1}[B_{t+1}(\hat{\theta}^{t-1},\hat{\theta}_t,\cdot),\theta_t] \\
&= u[c_t(b,\hat{\theta}_t)] - v[\frac{y_t(b,\hat{\theta}_t)}{\theta_t}] + \beta \tilde{V}_{t+1}[b_{t+1}(b,\hat{\theta}_t,\cdot),\theta_t],
\end{aligned}$$

where the first and the last equality follows from the definition of z, the second and next-to-last inequality follows from the fact that $(\hat{\theta}^{t-1}, \theta_t) \in H_{t+1}[B_{t+1}(\hat{\theta}^{t-1}, \theta_t, \cdot)]$ by construction and the inequality follows from (27).

(2c) Inclusion in $\mathcal{B}$. Suppose that (11) is violated for some $t < T-1$. Then it follows from (2a) and (2b) that either (32) or (27) is violated in $t+1$, a contradiction. Hence z generated by Z satisfies (9), (10) and (11).

Define $\tilde{V}^*(\bar{B}_0, \theta_{-1}) = \max_{b \in \mathcal{B}_1} \tilde{V}_0[b, \theta_{-1}]$ s.t. (13). It is also easy to see that $\tilde{V}^*(\bar{B}_0, \theta_{-1}) = U^*(\bar{B}_0, \theta_{-1})$. Since z may not necessarily maximize the value of the recursive social planner's problem, $V^*(\bar{B}_0, \theta_{-1}) \geq U^*(\bar{B}_0, \theta_{-1})$. ■

Proof of Theorem (8). The proof is by induction. Suppose the agent enters period T with a contract ϕ_T that satisfies the assumption of definition (13), i.e. $\phi_T = [b, y^*_T(b_T, \cdot)]$. Suppose that the agent reports $\hat{\theta}$ while her true shock is θ. Then, as follows from the ex-post budget constraint (9), she consumes $\hat{c}^d_T[\phi(\hat{\theta}), \theta] = b(\hat{\theta}) - y^*_T(b, \hat{\theta}) = c^*_T(b, \hat{\theta})$. By lemma (6), the agent reports truthfully, $\hat{\theta} = \theta$, and so $c^d_T(\phi, \theta) = c^*_T(b, \theta)$. Thus, the direct competitive equilibrium is constrained efficient in period T.[13] In addition, for $t = T$, the following equality of value functions holds.

$$\sum_{\theta \in \Theta} W^d_t(\phi_t, \theta)\pi(\theta|\theta_-) = V_t(b, \theta_-). \tag{33}$$

Now suppose that the direct competitive equilibrium is constrained efficient for all time periods greater than some $t \leq T$ and that the equality of value functions (33) holds in period $t+1$. Suppose also that the agent enters the period with a contract ϕ_t that satisfies the assumption of definition (13), i.e. $\phi_t = [b, y^*_t(b, \cdot)]$. Efficient external monitoring implies that if a θ-type agent reports $\hat{\theta}$, she must choose next period assets $\hat{a}^d_{t+1}[\phi_t(\hat{\theta}), \theta, \cdot] = b^*_{t+1}(b, \hat{\theta}, \cdot)$. The consumption is then determined as a residual from the budget constraint:

$$\begin{aligned} \hat{c}^d_t[\phi_t(\hat{\theta}), \theta] &= b(\hat{\theta}) + y^*_t(b, \hat{\theta}) - \sum_{\theta' \in \Theta} p(\hat{\theta}, \theta') b^*_{t+1}(b, \hat{\theta}, \theta') \qquad (34a) \\ &= c^*_t(b, \hat{\theta}), \end{aligned}$$

where the first equality follows from the budget constraint and the fact that, by reporting $\hat{\theta}$, the agent faces asset prices $p(\hat{\theta}, \cdot)$, and the second one follows from lemma (4) and the ex-post resource constraint (9). The optimal financial contract

[13] The competitive equilibrium is said to be constrained efficient in period t if the definition (7) applies in period t.

$\hat{\phi}^d_{t+1}$ is given by the asset vector $\hat{a}^d_{t+1}[\phi_t(\hat{\theta}),\theta]$ and the income vector $\hat{y}^d_{t+1}[\phi_{t-1}(\hat{\theta}),\theta,\cdot]$ that is chosen to maximize the expected lifetime utility of the agent among all feasible contracts:

$$\hat{y}^*_{t+1}[\phi_t(\hat{\theta}),\theta,\cdot] = \arg\max_{y(\cdot)} \sum_{\theta'\in\Theta} W_{t+1}\{[\hat{a}^d_{t+1}[\phi_t(\hat{\theta}),\theta,\cdot],y(\cdot)],\theta'\}\pi(\theta'|\theta)$$
$$s.t. \quad [\hat{a}^d_{t+1}[\phi_t(\hat{\theta}),\theta,\cdot],y(\cdot)] \in X_{t+1}.$$

It follows that $\hat{y}^d_{t+1}[\phi_t(\hat{\theta}),\theta,\cdot] = y^*_{t+1}[b_{t+1}(b,\hat{\theta},\cdot),\cdot]$. To see this, suppose by contradiction that $\hat{y}^d_{t+1}[\phi_t(\hat{\theta}),\theta,\cdot] \neq y^*_{t+1}(b_{t+1}(b,\hat{\theta},\cdot),\cdot)$. Then it must be true that

$$\sum_{\theta\in\Theta} W^d_{t+1}\{\hat{\phi}^d_{t+1}[\phi_t(\hat{\theta}),\theta,\cdot],\theta'\}\pi(\theta'|\theta) > \sum_{\theta\in\Theta} W_{t+1}\{[b^*_{t+1}(b,\hat{\theta},\cdot),y^*_{t+1}(b_{t+1}(b,\hat{\theta},\cdot),\cdot)],\theta'\}\pi(\theta'|\theta)$$
$$= V_{t+1}[b_{t+1}(b,\hat{\theta},\cdot),\theta]$$

where the equality follows from (33). But since $\hat{\phi}^d_{t+1}[\phi_t(\hat{\theta}),\theta,\cdot] \in X_{t+1}$, one can construct from the equilibrium allocation a new social planner allocation rule $\{\tilde{z}^*_j\}^T_{j=t+1}$ that satisfies (9), (10) and (11) for all $j = t+1...T$ and yields strictly higher utility. Hence $\{z^*_j\}^T_{j=t+1}$ is not efficient, a contradiction. Finally, by lemma (6) the agent reports truthfully and so.

$$c^d_t(\phi_t,\theta) = c^*_t(b,\theta), \tag{35}$$
$$\phi^d_{t+1}(\phi_t,\theta) = [b^*_{t+1}(b,\theta,\cdot),y^*_{t+1}(b^*_{t+1}(b,\theta,\cdot),\cdot)]. \tag{36}$$

In addition, the equality of the value function is preserved in period t since

$$\sum_{\theta\in\Theta} W_t(\phi_t,\theta)\pi(\theta|\theta_-)$$
$$= \sum_{\theta\in\Theta}\{u[c^d_t(\phi_t,\theta)] - v[\frac{y^*_t(b,\theta)}{\theta}] + \beta\sum_{\theta'\in\Theta} W^d_{t+1}[\phi^d_{t+1}(\phi_t,\theta),\theta']\pi(\theta'|\theta)\}\pi(\theta|\theta_-)$$
$$= \sum_{\theta\in\Theta}\{u[c^*_t(b,\theta)] - v[\frac{y^*_t(b,\theta)}{\theta}] + \beta V_{t+1}[b^*_{t+1}(b,\theta),\theta]\pi(\theta|\theta_-)\}$$
$$= V_t(b,\theta_-). \tag{37}$$

where the first equality follows from the Bellman equation (16), second equality follows from (35), (36) and (33) and the last equality follows from the Bellman equation (12). Hence the competitive equilibrium implements the constrained efficient allocation in period t and the equality of value functions (37) holds in period t.

Thus, the equality of value functions holds in period zero. Similar arguments then show that the agent chooses the initial financial contract such that $\phi_0^d(\bar{A}_0, \theta_{-1}) = [b_0^*(\bar{A}_0, \theta_{-1}), y_0^*(b_0^*(\bar{A}_0, \theta_{-1}), \cdot)]$. ■

Proof of Theorem (14). I will prove the theorem by induction. I first show that the theorem holds in period T. Fix b and suppose the agent enters period T with assets a satisfying $a[y_T^*(b, \cdot)] = b(\cdot)$. It follows from the efficient external monitoring that $y_T^i(a, \theta) = y_T^*(b, \hat{\theta})$ for some $\hat{\theta} \neq \theta$. The consumption $c_T^i(a, \theta)$ is then determined as follows:

$$\begin{aligned} c_T^i(a, \theta) &= a[y_T^i(a, \theta)] + y_T^i(a, \theta) \\ &= a[y_T^*(b, \hat{\theta})] + y_T^*(b, \hat{\theta}) \\ &= b(\hat{\theta}) + y_T^*(b, \hat{\theta}) \\ &= c_T^*(b, \hat{\theta}), \end{aligned}$$

where the first equality follows the budget constraint (22), second one from the properties of by y_T^i, third one from the assumptions of the definition (13) and the last one from the ex-post resource constraint (9). But since z_T^* satisfies the incentive compatibility constraint (10), choosing $z_T^*(b, \hat{\theta})$ yields lower utility than $z_T^*(b, \theta)$, a contradiction. Hence $z_T^i(a, \theta) = z_T^*(b, \theta)$ and, in addition, the following equality of the value functions holds for $t = T$:

$$\sum_{\theta \in \Theta} W_t^i(a, \theta) \pi(\theta | \theta_-) = V_t(b, \theta_-) \tag{38}$$

Thus, the indirect competitive equilibrium is efficient in period T.

Now suppose that the indirect competitive equilibrium is efficient in all periods greater than $t \leq T-1$. Then (38) holds in period $t+1$. Suppose that the agent enters period t with assets a satisfying $b(\cdot) = a[y_t^*(b, \cdot)]$. It follows from the properties of the asset portfolio a that $y_t^i(a, \theta) = y_t^*(b, \hat{\theta})$ for some $\hat{\theta} \in \Theta$. Suppose by contradiction that $\hat{\theta} \neq \theta$. The efficient external monitoring implies that, in such case, the asset portfolio chosen must be $a_{t+1}^i[a, \theta, y_{t+1}^*(b_{t+1}^*, \cdot)] = b_{t+1}^*(b, \hat{\theta}, \cdot)$. The consumption

$c_t^i(a,\theta)$ is determined as follows:

$$\begin{aligned}
c_t^i(a,\theta) &= a[y_t^i(a,\theta)] + y_t^i(a,\theta) - P_t[a, y_t^i(a,\theta); a_{t+1}^i] \\
&= a[y_t^i(a,\theta)] + y_t^i(a,\theta) - q_t \sum_{\theta' \in \Theta} a_{t+1}^i[y_{t+1}^i(a_{t+1}^i, \theta')]\pi(\theta'|\hat{\theta}) \\
&= a[y_t^i(a,\theta)] + y_t^i(a,\theta) - q_t \sum_{\theta' \in \Theta} a_{t+1}^i[y_{t+1}^*(a_{t+1}^i, \theta')]\pi(\theta'|\hat{\theta}) \\
&= b(\hat{\theta}) + y_t^*(b,\hat{\theta}) - q_t \sum_{\theta' \in \Theta} b_{t+1}^*(b,\hat{\theta},\theta')\pi(\theta'|\hat{\theta}). \\
&= c_t^*(b,\hat{\theta}),
\end{aligned}$$

where the first equality follows from the budget constraint (22) and the second one from the properties of y_t^i, lemma (11) and the fact that, since $y_t^i(a,\theta) = y_t^*(b,\hat{\theta})$ is chosen, the market mistakenly considers the agent to have current shock $\hat{\theta}$. The third equality follows from the fact that the induction hypothesis holds in period $t+1$, the fourth one from the properties of a_{t+1}^i and the last one from the ex-post resource constraint (9).

Consider now an alternative market allocation $\tilde{z}_t^i(a,\theta) = \{c_t^*(a,\theta), z_t^*(a,\theta), z_t^*(a,\theta,\cdot)\}$. It is easy to verify that the market allocation $\tilde{z}_t^i(a,\theta)$ is also budget feasible and satisfies external monitoring. In addition, the fact that the efficient allocation rule is incentive compatible implies that for all $\hat{\theta} \in \Theta$

$$\begin{aligned}
& u[\tilde{c}_t^i(a,\theta)] - v[\frac{\tilde{y}_t^i(a,\theta)}{\theta}] + \beta \sum_{\theta' \in \Theta} W_{t+1}^i[\tilde{a}_{t+1}^i[a,\theta, y_{t+1}^i(\tilde{a}_{t+1}^i, \cdot)], \theta] \\
&= u[c_t^*(b,\theta)] - v[\frac{y_t^*(b,\theta)}{\theta}] + \beta V_{t+1}[b_{t+1}^*(b,\theta,\cdot),\theta] \\
&\geq u[c_t^*(a,\hat{\theta})] - v[\frac{y_t^*(a,\hat{\theta})}{\theta}] + \beta V_{t+1}[b_{t+1}^*(b,\hat{\theta},\cdot),\theta] \\
&= u[c_t^i(a,\theta)] - v[\frac{y_t^i(a,\theta)}{\theta}] + \beta \sum_{\theta' \in \Theta} W_{t+1}^i[a_{t+1}^i[a,\theta, y_{t+1}^i(a_{t+1}^i, \cdot)], \theta]
\end{aligned}$$

where the first and last equality follows from the definition of $\tilde{z}_t^i$ (z_t^i) the fact that the induction hypothesis holds in period $t+1$ and from (38) applied in period $t+1$ and the inequality follows from the incentive compatibility constraint (10). The market allocation $\tilde{z}_t^i$ thus yields higher utility than $\tilde{z}_t^i$ but was not chosen, a contradiction. Hence $z_t^i(a,\theta) = z_t^*(b,\theta)$. It is now easy to verify that (38) holds in period t.

Consider now an initial asset portfolio $\hat{a}_0^i$ satisfying

$$b_0^*(\bar{A}_0, \theta_{-1}, \cdot) = a_0^i[y_0^*(b_0^*(\bar{A}_0, \theta_{-1}, \cdot), \cdot)].$$

Choosing the initial asset portfolio $\hat{a}_0^i$ is feasible, as follows from the social planner's ex-ante budget constraint (13), the initial budget constraint (23), lemma (24) and the fact that the induction hypothesis holds at time zero. It also delivers the same lifetime utility as the social planner's problem:

$$\begin{aligned}\sum_{\theta \in \Theta} W_0^i(a_0^i, \theta)\pi(\theta|\theta_{-1}) &= V_0[b_0^*(\bar{A}_0, \theta_{-1}), \theta_{-1}] \\ &= V^*(\bar{A}_0, \theta_{-1}),\end{aligned}$$

where the first equality follows from the fact that (38) holds in period zero and the second one follows from the definition of V^*. There is no other initial portfolio that can deliver higher lifetime utility: if there were one, it would be chosen by the social planner. Thus, the agent chooses $\hat{a}_0^i$ at the beginning of time zero. ■